ON COALITIONAL RATIONALITY IN A THREE-PERSON GAME

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Abstract. To determine the solution of any game in mathematical game theory, it is necessary to establish what behavior of the players should be considered optimal. In noncooperative games (games without coalitions), the concept of optimality is related, e.g., to the concepts of Nash and Berge equilibria. Optimality in the theory of cooperative games is characterized by the conditions of individual and collective rationality. This paper considers a three-person cooperative game in normal form. For this game, the concept of coalitional rationality is introduced by embracing the conditions of individual and collective rationality with some combination of the concepts of Nash and Berge equilibria. Sufficient conditions are established under which the game has a coalitional equilibrium of this type. In addition, the existence of such a solution in mixed strategies is proved in the case of continuous payoff functions and compact strategy sets of players.

Keywords: maximin, Pareto maximum, Slater maximum, coalitional rationality, Germeier convolution, mixed strategies.

INTRODUCTION

Consider a three-person game described by an ordered triple

$$\Gamma = \left\langle \{1,\, 2,\, 3\},\, \{X_i\}_{i=1,2,3},\, \{f_i(x)\}_{i=1,2,3} \right\rangle.$$

In the game Γ, $\{1, 2, 3\}$ is the set of players; each player chooses his/her *strategy* $x_i \in X_i \subset \mathbb{R}^{n_i}$ ($i=1, 2, 3$), which results in a *strategy profile* $x=(x_1, x_2, x_3)\in X = \prod_{i=1}^{3} X_i \subset \mathbb{R}^n$, $n=\sum_{i=1}^{3} n_i$. On the set X of all strategy profiles, a *payoff function* $f_i(x)$ of each player i $(i=1, 2, 3)$ is defined, and its value is called the *payoff* of player *i*. The game under study is restricted to three persons as it suffices to illustrate the main idea behind the conceptual solution defined below. Moreover, considering the game of four or more persons would lead to a large variety of coalitional structures and, consequently, to more cumbersome formulas.

Conflicts mathematically modeled, in particular, by the three-person game Γ, are usually investigated from the normative point of view, establishing what behavior of the players should be considered optimal (reasonable, expedient). The key features of optimality in mathematical game theory are intuitive beliefs about *profitability, stability, and fairness* [1]. The concept of Nash equilibrium (NE) [2, 3], dominating in noncooperative games, as well as Berge equilibrium (BE), active equilibrium, and equilibrium in threats and counterthreats, which appeared under the former's direct influence [4], is based on the property of stability. These and some other notions of optimality [5] exist in the theory of noncooperative games. In such games, each player usually pursues his/her individual goals; moreover, each player cannot join other players in a coalition to choose coordinated strategies. The antipode to this setup is cooperative games [6]: any unions of players are allowed in order to "struggle" for common interests, and unlimited negotiations are possible between players to choose and use a joint strategy profile. Of course, by the natural assumption, all agreements will be respected by the players. Optimality in cooperative game theory is characterized by the conditions of *individual* [6] and *collective* [6] *rationality*. Individual rationality means that each player's payoff is not smaller than his/her guaranteed payoff reached by acting independently (i.e., using his/her maximin strategy). Collective rationality is ensured by an appropriate vector maximum (in the Slater, Pareto, Geoffrion, Borwein, or any other sense), arising when all players create the grand coalition.

In this paper, an important notion is the *coalitional structure* of a game (the partition of players into pairwise disjoint subsets). For the three-person game Γ, there exist five possible coalitional structures: $\mathfrak{P}_1 = \{\{1\}, \{2\}, \{3\}\}$, $\mathfrak{P}_2 = \{\{1, 2\}, \{3\}\}$, $\mathfrak{P}_3 = \{\{1, 3\}, \{2\}\}$, $\mathfrak{P}_4 = \{\{1\}, \{2, 3\}\}$, and $\mathfrak{P}_5 = \{\{1, 2, 3\}\}$. Here, the structure $\mathfrak{P}_1$ corresponds to the noncooperative "character" of the game whereas the structure $\mathfrak{P}_5$ to the cooperative one. Let us formulate the conditions of individual rationality for the coalitional structure $\mathfrak{P}_1$. Hereinafter, we adopt the short notation $-i = \{\{1, 2, 3\} \setminus \{i\}\}\ \forall i \in \{1, 2, 3\}$.

For a strategy profile $x^* = (x_1^*, x_2^*, x_3^*) \in X$, the condition of individual rationality means

$$f_i^0 = \max_{x_i \in X_i} \min_{x_{-i} \in X_{-i}} f_i(x_i, x_{-i}) = \min_{x_{-i} \in X_{-i}} f_i(x_i^0, x_{-i}) \le f_i(x^*),\ i = 1, 2, 3. \tag{1}$$

In other words, under the maximin strategy x_i^0, we have the inequalities

$$f_i^0 \le f_i(x^*),\ i = 1, 2, 3. \tag{2}$$

For the coalitional structure $\mathfrak{P}_5$ in the game Γ, the condition of collective rationality will be ensured by Pareto maximality. More precisely, on the set $X^* \subset X$ of strategy profiles, a strategy profile $x^* \in X^* \subset X$ is *Pareto maximal* in the tri-criteria problem $\Gamma_{X^*} = \langle X^*, \{f_i(x)\}_{i=1,2,3} \rangle$ if $\forall x \in X^*$ the system of inequalities $f_i(x) \ge f_i(x^*),\ i = 1, 2, 3,$ is inconsistent, with at least one inequality being strict. According to Karlin's lemma [7], if

$$\sum_{i=1}^{3} f_i(x) \le \sum_{i=1}^{3} f_i(x^*)\ \ \forall x \in X^*, \tag{3}$$

then the strategy profile x^* is Pareto maximal in the problem Γ_{X^*}.

1. THE CONDITION OF COALITIONAL RATIONALITY

Based on a suitable combination of the concepts of NE and BE, we will formalize this condition for the coalitional structures $\mathfrak{P}_2, \mathfrak{P}_3$, and $\mathfrak{P}_4$.

For the coalitional structure $\mathfrak{P}_2$, the condition of a coalitional equilibrium means the four inequalities

$$f_1(x_1^*, x_2^*, x_3) \le f_1(x^*)\ \forall x_3 \in X_3, \tag{4}$$

$$f_2(x_1^*, x_2^*, x_3) \le f_2(x^*)\ \forall x_3 \in X_3, \tag{5}$$

$$f_1(x_1, x_2, x_3^*) \le f_1(x^*)\ \forall x_j \in X_j,\ j = 1, 2, \tag{6}$$

$$f_2(x_1, x_2, x_3^*) \le f_2(x^*)\ \forall x_j \in X_j,\ j = 1, 2; \tag{7}$$

for the structure $\mathfrak{P}_3$, the four inequalities

$$f_1(x_1^*, x_2, x_3^*) \le f_1(x^*)\ \forall x_2 \in X_2, \tag{8}$$

$$f_3(x_1^*, x_2, x_3^*) \le f_3(x^*)\ \forall x_2 \in X_2, \tag{9}$$

$$f_1(x_1, x_2^*, x_3) \le f_1(x^*)\ \forall x_k \in X_k,\ k = 1, 3, \tag{10}$$

$$f_3(x_1, x_2^*, x_3) \le f_3(x^*)\ \forall x_k \in X_k,\ k = 1, 3; \tag{11}$$

finally, for the structure $\mathfrak{P}_4$, the four inequalities

$$f_2(x_1, x_2^*, x_3^*) \le f_2(x^*)\ \forall x_1 \in X_1, \tag{12}$$

$$f_3(x_1, x_2^*, x_3^*) \le f_3(x^*)\ \forall x_1 \in X_1, \tag{13}$$

$$f_2(x_1^*, x_2, x_3) \le f_2(x^*)\ \forall x_l \in X_l,\ l = 2, 3, \tag{14}$$

$$f_3(x_1^*, x_2, x_3) \le f_3(x^*)\ \forall x_l \in X_l,\ l = 2, 3. \tag{15}$$

A strategy profile $x^* \in X$ satisfying all these 12 requirements will be called *coalitionally rational* in the game Γ. Let X^* denote the set of all such strategy profiles; obviously, $X^* \subset X$.

When determining the optimal solution of the game Γ, we will use not all the sixteen inequalities (the three (2), the one (3), and the twelve (4)–(15)) but only seven of them: they are the implications of the others, see the two lemmas below.

Lemma 1. *If inequalities* (6), (14), *and* (15) *are valid, they imply, respectively,*

$$f_i(x^*) \ge f_i^0 = \max_{x_i \in X_i} \min_{x_{-i} \in X_{-i}} f_i(x_i, x_{-i}) = \min_{x_{-i} \in X_{-i}} f_i(x_i^0, x_{-i}),\ i = 1, 2, 3.$$

P r o o f. Indeed, due to inequality (6), we have $f_1(x_1, x_2, x_3^*) \le f_1(x^*)\ \ \forall x_j \in X_j,\ j = 1, 2.$ Given the strategy $x_1 = x_1^0$ of player 1, the latter inequality leads to

$$f_1(x^*) \ge f_1(x_1^0, x_2, x_3^*) \ge \min_{x_2, x_3} f_1(x_1^0, x_2, x_3) = \max_{x_1} \min_{x_2, x_3} f_1(x_1, x_2, x_3) = f_1^0.$$

Similar statements are established for players $i = 2, 3$ from inequalities (14) and (15), respectively. ♦

Lemma 2. *The following obvious implications are true:* (10)→(4), (14)→(5), (6)→(8), (15)→(9), (7)→(12), and (11)→(13).

Remark 1. According to Lemmas 1 and 2, when determining the optimal solution of the game Γ based on the conditions of individual, collective, and coalitional rationality, it suffices to use only the seven requirements (3), (6), (7), (10), (11), (14), and (15) instead of all the sixteen ones (2)–(15). ♦

Thus, we arrive at the following notion of an optimal solution of the game Γ with $f=(f_1, f_2, f_3)\in\mathbb{R}^3$.

Definition. A pair $(x^*, f(x^*))\in X\times\mathbb{R}^3$ is called a coalitional equilibrium (CE) of the game Γ if:

– The six equalities hold:

$$\begin{aligned}
&\max_{x_1,x_2} f_j(x_1, x_2, x_3^*) = f_j(x^*),\ j=1, 2,\\
&\max_{x_1,x_3} f_k(x_1, x_2^*, x_3) = f_k(x^*),\ k=1, 3,\\
&\max_{x_2,x_3} f_l(x_1^*, x_2, x_3) = f_l(x^*),\ l=2, 3.
\end{aligned}\tag{16}$$

– The strategy profile $x^*\in X$ is Pareto maximal on the set of all coalitional equilibria X^* of the game Γ.

Remark 2. As an optimal solution of the game Γ, we take the pair composed of a strategy profile x^* and the corresponding payoff vector $f(x^*)=(f_1(x^*), f_2(x^*), f_3(x^*))$. Indeed, the existence of a pair $(x^*, f(x^*))$ provides an immediate answer to the two questions of mathematical game theory:

– What should players do in the game Γ?

– What will they receive as a result?

The answer is: the players should follow the corresponding strategies x_i^* from the strategy profile $x^*=(x_1^*, x_2^*, x_3^*)$.

Remark 3. We enumerate the advantages of CE as a solution of the game Γ:

- According to Lemma 1, applying x^* ensures the conditions of individual rationality.
- The strategy profile x^* brings all players to the highest payoffs (Pareto maximal with respect to the other CE in the game Γ). As we believe, this fact is an analog of the condition of collective rationality from the theory of cooperative games.
- The requirements (4)–(15) mean, e.g., the dual-purpose allocation of the resources of player 1. That is:

– Without forgetting his/her individual interests, player 1 strives to help, as much as possible, player 2 in the union {1, 2} as a member of the coalitional structure $\mathfrak{P}_2$ (see the requirements (6) and (7)).

– Without forgetting his/her interests, player 1 also helps player 3 as a member of the union {1, 3} of the coalitional structure $\mathfrak{P}_3$ (see the requirements (10) and (11)).

As we believe, formalizing these two requirements in the first and second rows of the expression (16) is a modification of the concept of NE to the case of a bi-criteria payoff function of the players; the third row of the expression (16) can be understood as a realization of the concept of BE for the same bi-criteria setup. Similar considerations concern players 2 and 3.

Finally, CE also involves the *principle of stability*: due to condition (16), an arbitrary unilateral deviation of any coalitions (composed of one or two players) from x^* cannot improve the payoff of the deviated coalition in the game Γ as compared to $f_i(x^*)$, $i = 1, 2, 3$.

Remark 4. Once an optimal solution is determined, mathematical game theory recommends settling two issues:

– Does such a solution exist?

– How can it be found? ♦

The answers are provided in the next section.

2. SUFFICIENT CONDITIONS

Let us proceed to the key result of this paper. We introduce the two n-vectors $x=(x_1, x_2, x_3)\in X\subset\mathbb{R}^n$, $n=\sum_{i=1}^{3} n_i$, and $z=(z_1, z_2, z_3)\in X$, as well as the seven scalar functions

$$\begin{aligned}
&\varphi_1(x, z) = f_1(x_1, x_2, z_3) - f_1(z),\\
&\varphi_2(x, z) = f_2(x_1, x_2, z_3) - f_2(z),\\
&\varphi_3(x, z) = f_1(x_1, z_2, x_3) - f_1(z),\\
&\varphi_4(x, z) = f_3(x_1, z_2, x_3) - f_3(z),\\
&\varphi_5(x, z) = f_2(z_1, x_2, x_3) - f_2(z),\\
&\varphi_6(x, z) = f_3(z_1, x_2, x_3) - f_3(z),\\
&\varphi_7(x, z) = \sum_{i=1}^{3} f_i(x) - \sum_{i=1}^{3} f_i(z).
\end{aligned}\tag{17}$$

Using the payoff functions of the players in the game Γ, we construct the Germeier convolution of the seven functions:

$$\varphi(x, z) = \max_{k=1,\ldots,7}\varphi_k(x, z),\tag{18}$$

defined on the set $X\times(Z=X)\subset\mathbb{R}^{2n}$, where $X=\prod_{i=1}^{3} X_i$ is the set of all strategy profiles in the game Γ.

A saddle point $(\bar{x}, z^*) \in X \times Z$ of the scalar function $\varphi(x, z)$ (17), (18) in the zero-sum two-person game

$$\Gamma^{\alpha} = \langle X, Z = X, \varphi(x, z) \rangle \quad (19)$$

is defined by the chain of inequalities

$$\varphi(x, z^*) \le \varphi(\bar{x}, z^*) \le \varphi(\bar{x}, z) \ \forall x, z \in X, \quad (20)$$

with $z^* \in X^*$ representing the minimax strategy, i.e., $\min_{z \in X} \max_{x \in X} \varphi(x, z) = \max_{x \in X} \varphi(x, z^*)$.

Proposition. *If there exists a saddle point $(\bar{x}, z^*)$ in the game Γ^{α}, then the minimax strategy $z^* \in X$ of this game is a CE of the original game Γ_3.*

P r o o f. With the strategy profile $z = \bar{x}$ substituted into inequalities (20), from formula (17) we obtain $\varphi(\bar{x}, \bar{x}) = 0$, $k = 1,..., 7$, for all $\varphi_k(\bar{x}, \bar{x}) = 0$, $k = 1,..., 7$. Then, due to inequalities (20) and the transitivity property,

$$\varphi(x, z^*) = \max\{f_1(x_1, x_2, z_3^*) - f_1(z^*),$$
$$f_2(x_1, x_2, z_3^*) - f_2(z^*),\ f_1(x_1, z_2^*, x_3) - f_1(z^*),$$
$$f_3(x_1, z_2^*, x_3) - f_3(z^*),\ f_2(z_1^*, x_2, x_3) - f_2(z^*),$$
$$f_3(z_1^*, x_2, x_3) - f_3(z^*),\ \sum_{i=1}^{3} f_1(x) - \sum_{i=1}^{3} f_i(z^*)\} \le 0$$
$$\forall x_i \in X_i,\ i = 1, 2, 3.$$

Consequently,

$$\begin{aligned} & f_j\left(x_1, x_2, z_3^*\right) \le f_j(z^*) \quad \forall x_j,\ j = 1, 2, \\ & f_k\left(x_1, z_2^*, x_3\right) \le f_k(z^*) \quad \forall x_k,\ k = 1, 3, \\ & f_l\left(z_1^*, x_2, x_3\right) \le f_l(z^*) \quad \forall x_l,\ l = 2, 3, \\ & \sum_{r=1}^{3} f_r(x) \le \sum_{r=1}^{3} f_r(z^*) \quad \forall x \in X^* \subset X. \end{aligned} \quad (21)$$

By the first three inequalities of (21) and the requirement (16), the strategy profile $z^* \in X$ is coalitionally rational in the game Γ. The last inequality of (21) and the inclusion $X^* \subset X$ ensure [7, *p. 71*] the Pareto maximality of the strategy profile x^* in the tri-criteria problem $\Gamma_{X^*} = \left\langle X^*, \{f_i(x)\}_{i=1,2,3} \right\rangle$. ♦

Remark 5. The above proposition provides the following constructive method for calculating a coalitional equilibrium of the game Γ:

– Construct the function $\varphi(x, z)$ by formulas (17) and (18).

– Find the saddle point $(\bar{x}, z)$ of the function $\varphi(x, z)$ from the chain of inequalities (20) [8].

– Find the values of the three functions $f_i(z^*)$, $i = 1, 2, 3$.

Then the pair $(z^*, f(z^*) = (f_1(z^*), f_2(z^*), f_3(z^*))) \in X \times \mathbb{R}^3$, represents a coalitional equilibrium of the game Γ.

Remark 6. If the (N + 1) scalar functions $\varphi_j(x, z)$, $j = 1,\ldots,7$, are continuous on the set $X \times Z$, and the sets $X, Z \in \operatorname{comp} \mathbb{R}^n$, then the function $\varphi(x, z) = \min_{j=1,\ldots,N+1} \varphi_j(z, z)$ is also continuous on the set $X \times Z$.

The proof of an even more general result is available in many textbooks on operations research; for example, see the book [9, *p. 54*]. It has even appeared in textbooks on convex analysis [10, *p. 146*]. ♦

Finally, the following theorem is crucial in this paper.

Theorem (existence in mixed strategies). *If the game Γ includes strategy sets $X_i \in \operatorname{comp} \mathbb{R}^{n_i}$ and payoff functions $f_i(\cdot) \in C(X)$, $i = 1, 2, 3$, then there exists a coalitional equilibrium in mixed strategies in this game.*

CONCLUSIONS

First of all, let us emphasize the *new* results of *cooperative game theory* obtained in this paper.

- The notion of a coalitional equilibrium (CE) has been formalized by considering the interests of any coalition in the game Γ.
- A constructive method for calculating CE has been provided. This method reduces to finding the minimax strategy for a special Germeier convolution, effectively constructed using the payoff functions of the players.
- The existence of CE in mixed strategies has been proved under standard mathematical programming conditions (the continuous payoff functions and compact strategy sets of the players).

As we believe, the *new qualitative results* following from this paper are also important:

– The results can be extended to cooperative games with any finite number of players (more than three). In these games, NE (BE) corresponds to Nash equilibrium (Berge equilibrium), respectively.

– CE ensures the stability of a coalitional structure to an arbitrary unilateral deviation of any coalitions.

– CE is applicable even if coalitional structures change during the game or even if all coalitions remain in force.

– CE can be used to create stable unions (alliances) of players.

And these are not all the advantages of CE.

Note another positive property as well. So far, the theory of cooperative games has been focused on the conditions of individual and collective rationality. Meanwhile, the individual interests of the players correspond to the concept of NE with its "selfish" character; collective rationality matches the concept of BE with its "altruism." However, such "forgetfulness" is not inherent to the human nature of players. These drawbacks of both concepts are leveled by coalitional rationality. Indeed, under the conditions of coalitional rationality, player 1 helps player 2 as a member of the coalition {1, 2} of the coalitional structure $\mathfrak{P}_2$ and player 3 as a member of the coalition {1, 3} of the coalitional structure $\mathfrak{P}_3$, not forgetting about him-/herself in both roles. And the other players act similarly. Thus, CE fills the gap between NE and BE by adding "care of others" to NE and self-care to BE.

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