DOI: http://doi.org/10.25728/cs.2023.2.2

INTERVAL OBSERVER DESIGN FOR DISCRETE LINEAR TIME-INVARIANT SYSTEMS WITH UNCERTAINTIES[1]

A.N. Zhirabok[2], A.V. Zuev[3], and C.I. Kim[4]

[2,3] Institute of Marine Technology Problems, Russian Academy of Sciences, Far Eastern Branch, Vladivostok, Russia
[2-4] Far Eastern Federal University, Vladivostok, Russia

[2]✉ zhirabok@mail.ru, [3]✉ alvzuev@yandex.ru, [4]✉ kim.ci@dvfu.ru

Abstract. This paper considers the problem of constructing an interval observer for systems described by discrete-time linear models under uncertainties in the form of exogenous disturbances and measurement noise (unknown bounded functions). Such an observer is designed using the minimal-dimension model of the original system invariant with respect to the disturbances. The dynamic matrix of this model is defined in the identification canonical form. We present relations to design an interval observer of minimal complexity for estimating the set of admissible values of a given linear function of the state vector. If the observer invariant with respect to the disturbances does not exist, we suggest a method to construct an observer with minimal sensitivity to them based on the singular value decomposition of system matrices. Theoretical results are illustrated by an example.

Keywords: linear systems, uncertainties, models, interval observers.

INTRODUCTION

This paper is a logical continuation of the research work [1], which considered the design of interval observers for systems described by linear models with continuous time.

In recent years, numerous studies have been devoted to the design of interval observers; for a survey, see the publications [2, 3]. The papers [4–10] presented the solution of this problem for different classes of systems as well as practical applications. As a rule, the cited authors estimated the set of admissible values of the full state vector. However, in many cases, it is of interest to estimate only a given linear function of this vector. The corresponding interval observer turns out to be significantly simpler than the full-order counterpart, and the class of systems for which such an observer can be designed is wider. In addition, when estimating a given linear function, the observer dynamics can be represented in a canonical form, which simplifies the solution procedure and extends the class of systems with interval observers.

In what follows, we state and solve the interval observer design problem for time-invariant systems described by discrete linear dynamic models with exogenous disturbances and measurement noise. The resulting interval observer estimates the set of admissible values for a given linear function of the system's state vector. This paper therefore differs from [2–10], where interval observers were designed to estimate the full state vector.

1. BASIC MODELS AND PROBLEM STATEMENT

We consider a system described by the discrete linear model

$$\begin{aligned} x(t+1) &= Fx(t) + Gu(t) + L\rho(t), \\ y(t) &= Hx(t) + v(t), \end{aligned} \tag{1}$$

with the following notations: $x(t) \in R^n$, $u(t) \in R^m$, and $y(t) \in R^l$ are the state, control, and output vectors,

[1] This work was supported by the Russian Science Foundation, project no. 23-29-000191, https://rscf.ru/project/23-19-00191/

respectively; F, G, and H are constant matrices of dimensions $n\times n$, $n\times m$, and $l\times n$, respectively; L is a known matrix of dimensions $n\times q$; $\rho(t)\in R^q$ is an unknown bounded time-varying function that describes the disturbances affecting the system, and $\|\rho(t)\|\le \rho_*$ for all $t\ge 0$; finally, $v(t)\in R^l$ is an unknown bounded time-varying function that describes measurement noise, and $\|v(t)\|\le v_*$ for all $t\ge 0$. (The symbol $\|\cdot\|$ indicates the Euclidean norm.)

According to (1), the uncertainties in the problem are represented by the measurement noise $v(t)$ and the exogenous disturbance $\rho(t)$ with the upper bounds v_* and ρ_* of their amplitudes, respectively, for all $t\ge 0$.

It is required to design a minimal-order interval observer producing the lower $\underline{z}(t)$ and upper $\bar{z}(t)$ estimates of the linear function $z(t)=Mx(t)\in R^p$ of the state vector with a given matrix M so that inequality $\underline{z}(t)\le z(t)\le \bar{z}(t)$ will hold componentwise for all $t\ge 0$.

As was demonstrated in the paper [1], for continuous-time systems, an interval observer can be designed based on the minimal-dimension model by two methods. In the first method, the matrices describing this model are found in the identification canonical form (ICF); the observer's stability is ensured using feedback, and the observer is then reduced to the Jordan canonical form to provide the Metzler property of the matrix reflecting its dynamics. In the second method, this matrix is immediately found in the Jordan form, which considerably simplifies the problem: stability and the Metzler property directly follow from the Jordan form.

In the discrete-time case, the Metzler property is not required: the matrix under consideration has to be stable and nonnegative. The ICF satisfies these two requirements and is therefore is preferable here. In addition, the feedback may not be used for the observer's stability: the ICF has zero eigenvalues, ensuring stability in the discrete-time case.

The solution is based on a minimal-dimension model insensitive to the disturbance:

$$\begin{aligned} x_*(t+1) &= F_*x_*(t)+J_*Hx(t)+G_*u(t),\\ z(t) &= H_z x_*(t)+Qy_0(t). \end{aligned} \tag{2}$$

This model estimates the variable $z(t)$ and has the following notations: $x_*\in R^k$ is the observer's state vector; k is the model dimension; F_*, J_*, G_*, H_z, and Q are matrices to be determined; finally, $y_0(t)=N_2y(t)$ for some matrix N_2 defined below. The vector $x(t)$ and the unknown vector $x_*(t)$ are related by

$$x_*(t)=\Phi x(t),$$

where the matrix Φ has to be determined. The term $J_*Hx(t)$ in formula (2) can be explained as follows. Being a reduced part of system (1), model (2) does not include the output vector $y(t)$. Hence, this vector appears as the term $J_*y(t)$ in the observer (12). Such an approach allows considering the measurement noise.

The solution of equation (2) insensitive to the disturbance $\rho(t)$ is the best in terms of the interval $\underline{z}(t)\le z(t)\le \bar{z}(t)$. As is known [11], it satisfies the condition $\Phi L=0$. To make the estimated variable $z(t)$ in model (2) insensitive to the disturbance, the variable $y_0(t)$ in this equation must be formed as follows.

Let us introduce a matrix L_0 of maximal rank such that $L_0L=0$. Then $\Phi=NL_0$ for some matrix N. Since the vector $x'(t)=L_0x(t)$ is insensitive to the disturbance, $y_0(t)=N_1x'(t)$ for some matrix N_1. On the other hand, $y_0(t)$ is part of the output vector $y(t)$, i.e., $y_0(t)=N_2y(t)$ for some matrix N_2. Then the matrices N_1 and N_2 satisfy the equation $N_1L_0=N_2H$. It has a nontrivial solution if

$$\operatorname{rank}\begin{pmatrix} H\\ L_0\end{pmatrix} < \operatorname{rank}(L_0)+\operatorname{rank}(H).$$

Under this condition, the matrices N_1 and N_2 are determined from the equation

$$(N_1 \vdots\ -N_2)\begin{pmatrix} L_0\\ H\end{pmatrix}=0, \tag{3}$$

where the symbol $\vdots$ separates two matrices. Otherwise, we should use $y(t)$ instead of $y_0(t)$ in model (2). As a result, the interval $(\underline{z}(t),\bar{z}(t))$ will be extended.

According to [11, 12], the matrices describing the model satisfy the equations

$$\Phi F=F_*\Phi+J_*H,\quad G_*=\Phi G,\quad \Phi L=0. \tag{4}$$

An additional condition is due to the second equation in model (2). With $z(t)=Mx(t)$, we write it as

$$Mx(t)=H_z\Phi x(t)+QN_2Hx(t),$$

arriving at the equation

$$M = H_z\Phi + QN_2H = (H_z \vdots \;\; Q)\begin{pmatrix}\Phi \\ N_2H\end{pmatrix}. \quad (5)$$

It has a solution if

$$\text{rank}\begin{pmatrix}\Phi \\ N_2H\end{pmatrix} = \text{rank}\begin{pmatrix}\Phi \\ N_2H \\ M\end{pmatrix}. \quad (6)$$

Based on equations (4) and (5), we can obtain relations to analyze the existence of such a solution in several cases. The first of such conditions has the form [11]

$$\text{rank}\begin{pmatrix}L_0F \\ H \\ L_0\end{pmatrix} < \text{rank}\begin{pmatrix}H \\ L_0\end{pmatrix} + \text{rank}(L_0F). \quad (7)$$

To derive the second one, let us replace the matrix Φ in equation (5) with NL_0. Then obvious transformations yield

$$M = (H_zN \vdots \;\; Q)\begin{pmatrix}L_0 \\ N_2H\end{pmatrix}.$$

The resulting equation is resolvable if

$$\text{rank}\begin{pmatrix}L_0 \\ N_2H\end{pmatrix} = \text{rank}\begin{pmatrix}L_0 \\ N_2H \\ M\end{pmatrix}. \quad (8)$$

An algorithm to check these conditions includes the following steps:

1. Determine the matrix L_0 and find the matrix N_2 from equation (3).

2. Check conditions (7) and (8). If they are true, find the matrices H_z and Q from equation (5) and construct the model and observer insensitive to the disturbance.

3. If just condition (8) fails, examine each row M_i, $i = 1, 2,..., p$, of the matrix M by replacing M in condition (8) with M_i. Take the rows satisfying this condition to form the matrix M_0 and then design an interval observer insensitive to the disturbance to estimate the variable $z_0(t) = M_0x(t)$. For the other rows of the matrix M, combined into the matrix M_*, find the robust solution described in Section 4 and then design a second observer to estimate the variable $z_*(t) = M_*x(t)$. This observer will have minimal sensitivity to the disturbance.

4. If just condition (7) fails, a robust solution is only possible. Find it by the methods described in Section 4. In this case, the interval observer will be minimally sensitive to the disturbance.

5. If conditions (7) and (8) fail both, the robust solution (see Section 4) is only possible as well. If the resulting matrix Φ satisfies condition (6), then the variable $y_0(t)$ (the undisturbed part of the vector $y(t)$) can be found. Otherwise, it is impossible, and the interval $(\underline{z}(t), \overline{z}(t))$ is further extended due to the term $Qy(t)$ corrupted by the disturbance.

2. MODEL CONSTRUCTION

The matrix F_* is found in the ICF:

$$F_* = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

As is well known, the model is stable if the eigenvalues of the matrix F_* do not exceed 1 by magnitude. For the ICF under consideration, they equal 0.

The problem is solved based on the equation [11]

$$(\Phi_1 \vdots \; -J_{*1} \vdots \; \ldots \vdots \; -J_{*k})(V^{(k)} \vdots \; L^{(k)}) = 0, \quad (9)$$

where

$$V^{(k)} = \begin{pmatrix} F^k \\ HF^{k-1} \\ \vdots \\ H \end{pmatrix},$$

$$L^{(k)} = \begin{pmatrix} L & FL & \ldots & F^{k-1}L \\ 0 & HL & \ldots & HF^{k-2}L \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 \end{pmatrix}, \; k = 1, 2, \ldots,$$

and Φ_i and J_{*i} indicate the rows of the matrices Φ and J_*, respectively. Note that the matrix $V^{(k)}$ serves to construct model (2) whereas the matrix $L^{(k)}$ to ensure its insensitivity to the disturbances. Equation (9) has a nontrivial solution if

$$\text{rank}(V^{(k)} \vdots \; L^{(k)}) < lk + n. \quad (10)$$

To design the model, we determine the minimum k from inequality (10) and the row $(\Phi_1 \vdots \; -J_{*1} \vdots \; \ldots \vdots \; -J_{*k})$ from equation (9). Then, based on the relations

$$\Phi_iF = \Phi_{i+1} + J_{*i}H, \; i = \overline{1, k-1}, \; \Phi_kF = J_{*k}H, \quad (11)$$

obtained from equations (4) and the ICF [11], we construct the matrix Φ. After that, condition (6) is verified. If it holds, the matrix M can be expressed through $(\Phi^{\mathrm{T}}\ H^{\mathrm{T}})^{\mathrm{T}}$, and the designed linear model will estimate the desired variable $z(t) = Mx(t)$; the matrices H_z and Q are determined from the algebraic equation (5) and the matrix G_* from equations (4). If condition (6) fails, another solution of equation (9) should be found for the same or increased dimension of the model. If it fails for all $k \le n$, the robust solution should be used; see Section 4.

3. INTERVAL OBSERVER DESIGN

The observer is found in the form

$$\begin{gathered} \underline{x}_*(t+1) = F_*\underline{x}_*(t) + J_*y(t) + G_*u(t) - |J_*|E_k v_*, \\ \overline{x}_*(t+1) = F_*\overline{x}_*(t) + J_*y(t) + G_*u(t) + |J_*|E_k v_*, \\ \underline{z}(t) = H_z\underline{x}_*(t) + Qy_0(t), \\ \overline{z}(t) = H_z\overline{x}_*(t) + Qy_0(t), \\ \underline{x}_*(0) = \underline{x}_{*0}, \quad \overline{x}_*(0) = \overline{x}_{*0}, \end{gathered} \tag{12}$$

where the matrix E_k of dimensions $k \times 1$ is composed of unities and the matrix $|J_*|$ is composed of the absolute values of the corresponding elements of the matrix J_*. By assumption, $x_*(0) \in [\underline{x}_{*0}, \overline{x}_{*0}]$ for some known vectors $\underline{x}_{*0}, \overline{x}_{*0} \in R^k$.

Theorem. *Let* $\underline{x}_*(0) \le x_*(0) \le \overline{x}_*(0)$. *Then the interval observer* (12) *satisfies the relations*

$$\underline{x}_*(t) \le x_*(t) \le \overline{x}_*(t) \ \text{and} \ \underline{z}(t) \le z(t) \le \overline{z}(t)$$

for all $t \ge 0$, *where*

$$\begin{gathered} \underline{z}(t) = H_z\underline{x}_*(t) + Qy_0(t), \\ \overline{z}(t) = H_z\overline{x}_*(t) + Qy_0(t) \end{gathered} \tag{13}$$

for $H_z \ge 0$ *and*

$$\begin{gathered} \underline{z}(t) = H_z\overline{x}_*(t) + Qy_0(t), \\ \overline{z}(t) = H_z\underline{x}_*(t) + Qy_0(t) \end{gathered} \tag{14}$$

for $H_z \le 0$.

P r o o f. By analogy with [2], we introduce the estimation errors

$$\begin{gathered} \underline{e}_*(t) = x_*(t) - \underline{x}_*(t),\ \overline{e}_*(t) = \overline{x}_*(t) - x_*(t), \\ \underline{e}_z(t) = z(t) - \underline{z}(t),\ \overline{e}_z(t) = \overline{z}(t) - z(t). \end{gathered} \tag{15}$$

In view of (2) and (12), it is possible to obtain the difference equations

$$\begin{aligned} \underline{e}_*(t+1) &= F_*\underline{e}_*(t) + J_*(Hx(t) - y(t)) + |J_*|E_k v_* \\ &= F_*\underline{e}_*(t) - J_*v(t) + |J_*|E_k v_*, \\ \overline{e}_*(t+1) &= F_*\overline{e}_*(t) + J_*(y(t) - Hx(t)) + |J_*|E_k v_* \\ &= F_*\overline{e}_*(t) + J_*v(t) + |J_*|E_k v_*. \end{aligned} \tag{16}$$

Since $\underline{x}_*(0) \le x_*(0) \le \overline{x}_*(0)$, formula (15) implies $\underline{e}_*(0) \ge 0$ and $\overline{e}_*(0) \ge 0$. Note that in system (16), $|J_*|E_k v_* \pm J_*v(t) \ge 0$ for all $t \ge 0$ and $F_* \ge 0$. Given $\underline{e}_*(0) \ge 0$ and $\overline{e}_*(0) \ge 0$, its solutions will be elementwise nonnegative: $\underline{e}_*(t) \ge 0$ and $\overline{e}_*(t) \ge 0$ for all $t \ge 0$ [2]. Due to $z(t) = H_z x_*(t) + Qy_0(t)$, for $H_z \ge 0$, from (13) and (15) we have

$$\begin{gathered} \underline{e}_z(t) = z(t) - \underline{z}(t) \\ = H_z x_*(t) + Qy_0(t) - (H_z\underline{x}_*(t) + Qy_0(t)) = H_z\underline{e}_*(t), \\ \overline{e}_z(t) = \overline{z}(t) - z(t) \\ = H_z\overline{x}_*(t) + Qy_0(t) - (H_z x_*(t) + Qy_0(t)) = H_z\overline{e}_*(t). \end{gathered}$$

Considering the inequalities $\underline{e}_*(t) \ge 0$, $\overline{e}_*(t) \ge 0$, and $H_z \ge 0$, we obtain $\underline{e}_z(t) \ge 0$ and $\overline{e}_z(t) \ge 0$, which is equivalent to $\underline{z}(t) \le z(t) \le \overline{z}(t)$. In the case $H_z \le 0$, from (14) and (15) it follows that

$$\begin{gathered} \underline{e}_z(t) = z(t) - \underline{z}(t) \\ = H_z x_*(t) + Qy_0(t) - (H_z\overline{x}_*(t) + Qy_0(t)) = -H_z\overline{e}_*(t), \\ \overline{e}_z(t) = \overline{z}(t) - z(t) \\ = H_z\underline{x}_*(t) + Qy_0(t) - (H_z x_*(t) + Qy_0(t)) = -H_z\underline{e}_*(t). \end{gathered}$$

In view of $\underline{e}_*(t) \ge 0$, $\overline{e}_*(t) \ge 0$, and $H_z \le 0$, we finally arrive at $\underline{e}_z(t) \ge 0$ and $\overline{e}_z(t) \ge 0$. The proof of this theorem is complete. ♦

Remark 1. If the matrix H_z is indefinite, the final result remains the same, but the formulas for calculating the upper and lower bounds become more complicated. We consider two cases as follows.

- Let H_z be a row; without loss of generality, assume that its first p components are positive and the rest are negative: $H_z = (H_z^+ \vdots\ H_z^-)$. We define

$$\underline{z}(t) = H_z^+\underline{x}_{*(p)}(t) + H_z^-\overline{x}_*^{(k-p)}(t) + Qy_0(t),$$

where $\underline{x}_{*(p)}$ and $\overline{x}_*^{(k-p)}$ are the subvectors of the state vectors $\underline{x}_*$ and $\overline{x}_*$ containing the first p and the last $(k-p)$ components, respectively. Then

$$\underline{e}_z = z - \underline{z} = H_z^+ x_{*(p)} + H_z^- x_*^{(k-p)} + Qy_0$$
$$-(H_z^+ \underline{x}_{*(p)} + H_z^- \overline{x}_*^{(k-p)} + Qy_0)$$
$$= H_z^+ \underline{e}_{*(p)} - H_z^- \overline{e}_*^{(k-p)},$$

and $\underline{e}_z(t) \geq 0$ due to $H_z^+ \geq 0$ and $H_z^- \leq 0$. By analogy, it is demonstrated that $\overline{e}_z(t) \geq 0$ for

$$\overline{z}(t) = H_z^+ \overline{x}_{*(p)}(t) + H_z^- \underline{x}_*^{(k-p)}(t) + Qy_0(t).$$

- Let the matrix H_z have several rows: $H_z = \begin{pmatrix} H_z^+ \\ H_z^- \end{pmatrix}$, where H_z^+ and H_z^- are submatrices such that $H_z^+ \geq 0$ and $H_z^- \leq 0$. We define

$$\underline{z}(t) = \begin{pmatrix} H_z^+ \underline{x}_*(t) \\ H_z^- \overline{x}_*(t) \end{pmatrix} + Qy_0(t);$$

then, obviously,

$$\underline{e}_z = \begin{pmatrix} H_z^+ \\ H_z^- \end{pmatrix} x_* + Qy_0 - \left(\begin{pmatrix} H_z^+ \underline{x}_* \\ H_z^- \overline{x}_* \end{pmatrix} + Qy_0 \right)$$
$$= \begin{pmatrix} H_z^+ \underline{e}_* \\ -H_z^- \overline{e}_* \end{pmatrix} \geq 0.$$

A more complex case is when H_z has several rows, each of the structure $(H_z^+ \vdots\ H_z^-)$. It reduces to a combination of the two cases considered.

Remark 2. The condition $\underline{x}_*(0) \leq x_*(0) \leq \overline{x}_*(0)$ is crucial in the theorem. For the positive system (14), it gives $\underline{e}_*(t) \geq 0$ and $\overline{e}_*(t) \geq 0$ for all $t \geq 0$. Due to no feedback in the observer and the stability of the matrix F_*, these inequalities will hold for some $t > 0$ even without the condition $\underline{x}_*(0) \leq x_*(0) \leq \overline{x}_*(0)$: the initial conditions are "forgotten" for $t \geq k$.

Indeed, let us denote $v_0(t) = |J_*| E_k v_* - J_* v(t) \geq 0$ and consider the first equation in system (14). According to [12], its solution can be represented as

$$\underline{e}_*(t) = F_*^t \underline{e}_*(0) + \sum_{i=0}^{t-1} F_*^{t-i-1} v_0(i). \qquad (17)$$

It is easy to check that $F_*^k = 0$. Then, for $t \geq k$, the value $\underline{e}_*(t)$ will be determined by the second term on the right-hand side of equality (17). By construction, this term is nonnegative, so $\underline{e}_*(t) \geq 0$ for all $t \geq k$. Similarly, we can show that $\overline{e}_*(t) \geq 0$ for all $t \geq k$.

4. ROBUST SOLUTION

If condition (10) does not hold for all $k < n$, we find a robust solution minimizing the contribution of the disturbance to the model. It is almost identical to the solution proposed in [1], except for minimizing the norm $\| (\Phi_1 \vdots\ -J_1 \vdots\ \ldots \vdots\ -J_k) L^{(k)} \|_F$ subject to the condition

$$(\Phi_1 \vdots\ -J_1 \vdots\ -J_2 \vdots\ \ldots \vdots\ -J_k) V^{(k)} = 0. \qquad (18)$$

In other words, the matrix R_* from the paper [1] is replaced by Φ_1. We can say that the problem is to determine a solution $(\Phi_1 \vdots\ -J_1 \vdots\ \ldots \vdots\ -J_k)$ with "maximal orthogonality" to the columns of the matrix $L^{(k)}$.

Following [1], based on all linearly independent solutions of equation (18), for some fixed dimension k we construct the matrix

$$W = \begin{pmatrix} \Phi_1^{(1)} & -J_1^{(1)} & -J_2^{(1)} & \ldots & -J_k^{(1)} \\ & & \cdots & & \\ \Phi_1^{(N)} & -J_1^{(N)} & -J_2^{(N)} & \ldots & -J_k^{(N)} \end{pmatrix}$$

and find the singular value decomposition $WL^{(k)} = U_L \Sigma_L V_L$. We choose the first transposed column of the matrix U_L as the vector of weight coefficients $w = (w_1, \ldots, w_N)$ and let $(\Phi_1 \vdots\ -J_1 \vdots\ \ldots \vdots\ -J_k) = wW$. Finally, we determine the rows of the matrix Φ from formula (11) and the matrices $G_* = \Phi G$ and $L_* = \Phi L$ to design model (2) with minimal sensitivity to the disturbances.

Due to the additional term $L_* \rho(t)$ in model (2), the dynamics of the interval observer for $v \neq 0$ are corrected as follows:

$$\underline{x}_*^+ = F_* \underline{x}_* + J_* y + G_* u - |J_*| E_k v_* - |L_*| E_k \rho_*,$$
$$\overline{x}_*^+ = F_* \overline{x}_* + J_* y + G_* u + |J_*| E_k v_* + |L_*| E_k \rho_*.$$

The expressions (14) for the estimation errors are modified appropriately:

$$\underline{e}_*^+ = F_*\underline{e}_* - J_*v + |J_*|E_k v_* + L_*\rho + |L_*|E_k\rho_*,$$
$$\bar{e}_*^+ = F_*\bar{e}_* + J_*v + |J_*|E_k v_* - L_*\rho + |L_*|E_k\rho_*.$$

Clearly, the desired result is immediate from the proof of the theorem and the obvious additional inequality $|L_*|E_k\rho_* \pm L_*\rho(t) \geq 0$.

5. INTERVAL ESTIMATION OF THE FULL STATE VECTOR

In several cases, this interval estimation approach for the variable $z(t) = Mx(t)$ can be applied for the full state vector $x(t)$ as follows. Without loss of generality, assume that the matrix H has the maximal rank and

$$H = (H_0 \quad 0),\ y(t) = H_0 x^{(1)}(t) + v(t),$$
$$x(t) = \begin{pmatrix} x^{(1)}(t) \\ x^{(2)}(t) \end{pmatrix},$$

where H_0 is a nonsingular matrix. Let us define

$$\underline{y}(t) = y(t) - E_l v_*,\ \ \bar{y}(t) = y(t) + E_l v_*,$$
$$\underline{x}^{(1)}(t) = H_0^{-1}\underline{y}(t),\ \ \bar{x}^{(1)}(t) = H_0^{-1}\bar{y}(t). \tag{19}$$

Then

$$\underline{e}^{(1)}(t) = x^{(1)}(t) - \underline{x}^{(1)}(t)$$
$$= H_0^{-1}(y(t) - v(t)) - H_0^{-1}\underline{y}(t) = H_0^{-1}(E_l v_* - v(t)),$$

$$\bar{e}^{(1)}(t) = \bar{x}^{(1)}(t) - x^{(1)}(t)$$
$$= H_0^{-1}\bar{y}(t) - H_0^{-1}(y(t) - v(t)) = H_0^{-1}(E_l v_* + v(t)).$$

Under the assumption $H_0^{-1} \geq 0$, from $E_l v_* \pm v(t) \geq 0$ we obtain $\underline{e}^{(1)}(t) \geq 0$ and $\bar{e}^{(1)}(t) \geq 0$ and, consequently, $\underline{x}^{(1)}(t) \leq x^{(1)}(t) \leq \bar{x}^{(1)}(t)$. Thus, the variable $x^{(1)}(t)$ given the condition $H_0^{-1} \geq 0$ is estimated based on the expression (19), and the disturbance $\rho(t)$ has no effect on this estimate.

Remark 3. The condition $H_0^{-1} \geq 0$ obviously holds in application-relevant cases when the components of the vector $x^{(1)}(t)$ are measured by separate sensors and $H_0 = H_0^{-1} = I_l$.

The variable $x^{(2)}(t)$ can be assigned an interval estimate using the observer (12). Assuming $z(t) = x^{(2)}(t) = M^{(2)}x(t)$ for some matrix $M^{(2)}$ and using the criterion (7) with the matrix $M^{(2)}$ instead of M, we check the possibility of designing an observer insensitive to the disturbance. Then, depending on the check results, we design an observer of the form (12) or the robust one.

Remark 4. The condition $\underline{x}_*(0) \leq x_*(0) \leq \bar{x}_*(0)$ of the theorem follows from $\underline{x}(0) \leq x(0) \leq \bar{x}(0)$ only when $\Phi \geq 0$. Indeed, for $\Phi \geq 0$ and $x(0) - \underline{x}(0) \geq 0$, we obtain $\Phi(x(0) - \underline{x}(0)) \geq 0$ and, consequently, $x_*(0) = \Phi x(0) \geq \Phi\underline{x}(0) = \underline{x}_*(0)$. The inequality $x_*(0) \leq \bar{x}_*(0)$ is established by analogy. According to Remark 1, however, this is not critical since the relation $\underline{z}(t) \leq z(t) \leq \bar{z}(t)$ will necessarily hold for $t \geq k$.

6. AN ILLUSTRATIVE EXAMPLE

We consider a discretized model of an electric drive

$$x_1(t+1) = k_1 x_2(t) + x_1(t),$$
$$x_2(t+1) = k_2 x_3(t) + x_2(t) + \rho(t),$$
$$x_3(t+1) = k_3 x_2(t) + k_4 x_3(t) + k_5 u(t), \tag{20}$$
$$y_1(t) = x_1(t) + v_1(t),\ \ y_2(t) = x_3(t) + v_2(t),$$

with the following notations: the coefficients $k_1,...,k_5$ are some drive parameters depending on the sampling interval; the disturbance $\rho(t)$ is due to an external load torque applied to the motor shaft. The model under consideration is described by the matrices

$$F = \begin{pmatrix} 1 & k_1 & 0 \\ 0 & 1 & k_2 \\ 0 & k_3 & k_4 \end{pmatrix},\ G = \begin{pmatrix} 0 \\ 0 \\ k_5 \end{pmatrix},$$

$$H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ L = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Choosing $M = (0 \quad 1 \quad 0)$, we design an interval observer for the variable $x_2(t)$. Since the disturbance enters the equation of this variable, the model will be sensitive to it. Therefore, we construct the model by letting $L = 0$. Equation (9) with $L^{(k)} = 0$, $k = 1$, takes the form

$$(\Phi \quad -J_*)\begin{pmatrix} 1 & k_1 & 0 \\ 0 & 1 & k_2 \\ 0 & k_3 & k_4 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 0\,.$$

The solution is $\Phi = (1/k_1 \quad -1 \quad 0)$ and $J_* = (1/k_1 \quad -k_2)$, yielding $G_* = 0$ and $L_* = -1$. As is easily verified, condition (5) holds and $H_z = -1$ and $Q = (1/k_1 \quad 0)$. The desired model has the form

$$x_*(t+1) = (1/k_1)H_1 x(t) - k_2 H_2 x(t) - \rho(t),$$
$$z(t) = -x_*(t) + (1/k_1) y_1(t).$$

Based on this model, considering $H_z = -1$, we design the following interval observer for the variable $z(t) = x_2(t)$:

$$\begin{aligned} \underline{x}_*(t+1) &= (1/k_1)y_1(t) - k_2 y_2(t) \\ &\quad -(1/k_1)v_{*1} - k_2 v_{*2} - \rho_*, \\ \overline{x}_*(t+1) &= (1/k_1)y_1(t) - k_2 y_2(t) \\ &\quad +(1/k_1)v_{*1} + k_2 v_{*2} + \rho_*, \\ \underline{z}(t) &= -\overline{x}_*(t) + (1/k_1)y_1(t), \\ \overline{z}(t) &= -\underline{x}_*(t) + (1/k_1)y_1(t). \end{aligned} \tag{21}$$

The variables $x_1(t)$ and $x_3(t)$ can be estimated using the expression (19):

$$\underline{x}_1(t) = y_1(t) - v_{*1}, \; \underline{x}_3(t) = y_2(t) - v_{*2},$$
$$\overline{x}_1(t) = y_1(t) + v_{*1}, \; \overline{x}_3(t) = y_2(t) + v_{*2}.$$

Comparing these estimates with those from [2] and similar papers, we can conclude the following: the proposed approach provides a simpler observer and smaller intervals because, in particular, the intervals for the variables $x_1(t)$ and $x_3(t)$ do not contain the disturbance $\rho(t)$.

For numerical simulation, we selected system (20) and the observer (21) with $u(t) = 0.2\sin(t/100)$ and the noises $v_1(t)$, $v_2(t)$, and $\rho(t)$ described by random processes with a variance of 0.5. For simplicity, we assumed that $k_1 = k_2 = k_5 = 1$ and $k_3 = k_4 = -1$. The simulation results are shown in Figs. 1 and 2, i.e., the variable $x_2(t)$ and its lower and upper bounds $\underline{x}_*(t)$ and $\overline{x}_*(t)$ for the initial conditions $x(0) = 0$, $\underline{x}(0) = -0.05$, and $\overline{x}(0) = 0.05$ and $x(0) = 0$, $\underline{x}(0) = 0.05$, and $\overline{x}(0) = -0.05$, respectively. As has been emphasized in Remark 1, the initial conditions affect only the estimates at the initial time instants.

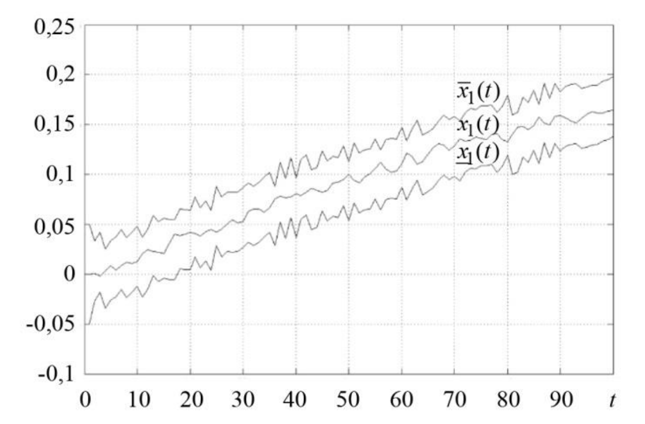

Fig. 1. The variable $x_1(t)$ and its lower $\underline{x}_1(t)$ and upper $\overline{x}_1(t)$ bounds for the initial conditions $x(0) = 0$, $\underline{x}(0) = -0.05$, and $\overline{x}(0) = 0.05$.

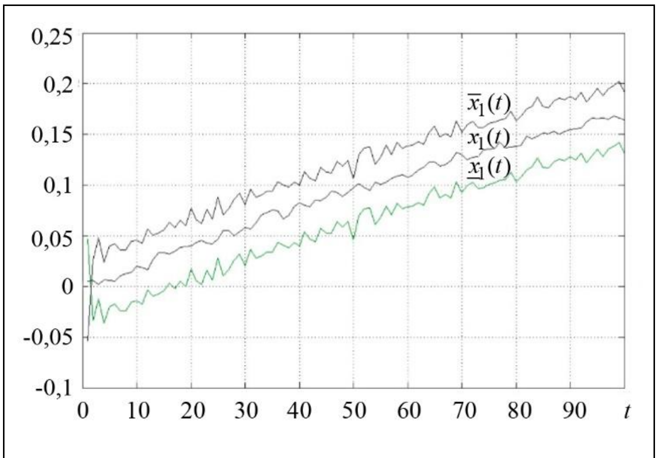

Fig. 2. The variable $x_1(t)$ and its lower $\underline{x}_1(t)$ and upper $\overline{x}_1(t)$ bounds for the initial conditions $x(0) = 0$, $\underline{x}(0) = 0.05$, and $\overline{x}(0) = -0.05$.

CONCLUSIONS

In this paper, we have designed interval observers for linear dynamic systems described by discrete-time models under exogenous disturbances and measurement noise. The relations based on the identification canonical form have been obtained to design a minimal-order interval observer estimating the set of admissible values for a given linear function of the system's state vector. A robust approach to solving the design problem has been considered as well. It has been demonstrated that the proposed solution can be used to estimate the full state vector. The theoretical results have been illustrated by a numerical example.

REFERENCES

1. Zhirabok, A., Zuev, A., and Kim, C.I., Method to Design Interval Observers for Linear Time-Invariant Systems, *Journal of Computer and Systems Sciences International*, 2022, vol. 61, no. 5, pp. 485–495.
2. Efimov, D. and Raissi, T., Interval Observer Design for Dynamic Systems with Uncertainties, *Automation and Remote Control*, 2016, vol. 77, no. 2, pp. 191–225.
3. Khan, A., Xie, W, Zhang, L., and Liu, L., Design and Applications of Interval Observers for Uncertain Dynamical Systems, *IET Circuits Devices Syst.*, 2020, vol. 14, pp. 721–740.
4. Kolesov, N., Gruzlikov, A., and Lukoyanov, E., Using Fuzzy Interacting Observers for Fault Diagnosis in Systems with Parametric Uncertainty, *Proc. of 12th International Symposium "Intelligent Systems" (INTELS'16)*, Moscow, Russia, 2016, pp. 499–504.
5. Kremlev, A. and Chebotarev, S., Synthesis of Interval Observer for Linear System with Variable Parameters, *Journal of Instrument Engineering*, 2013, vol. 56, no. 4, pp. 42–46. (In Russian.)
6. Chebotarev, S., Efimov, D., Raissi, T., and Zolghadri, A., Interval Observers for Continuous-Time LPV Systems with L_1/L_2 Performance, *Automatica*, 2015, vol. 51, pp. 82–89.
7. Mazenc, F. and Bernard, O., Asymptotically Stable Interval Observers for Planar Systems with Complex Poles, *IEEE Trans. Automatic Control*, 2010, vol. 55, no. 2, pp. 523–527.
8. Blesa, J., Puig, V., and Bolea, Y., Fault Detection Using Interval LPV Models in an Open-Flow Canal, *Control Engineering Practice*, 2010, vol. 18, pp. 460–470.
9. Zheng, G., Efimov, D., and Perruquetti, W., Interval State Estimation for Uncertain Nonlinear Systems, *Proc. IFAC Nolcos*, Toulouse, 2013.
10. Zhang, K., Jiang, B., Yan, X., and Edwards, C., Interval Sliding Mode Based Fault Accommodation for Non-minimal Phase LPV Systems with Online Control Application, *Intern. J. Control*, 2020, vol. 93, no. 11, pp. 2675–2689. DOI: 10.1080/00207179.2019.1687932.
11. Zhirabok, A., Zuev, A., and Shumsky, A.E., Methods of Diagnosis in Linear Systems Based on Sliding Mode Observers, *Journal of Computer and Systems Sciences International*, 2019, vol. 58, no. 6, pp. 898–914.
12. Kwarernaak, C. and Sivan, R., *Linear Optimal Control Systems*, New York: Wiley, 1972.

This paper was recommended for publication by S.A. Krasnova, a member of the Editorial Board.

Received January 22, 2023, and revised March 14, 2023. Accepted March 14, 2023.

Author information

Zhirabok, Aleksei Nilovich. Dr. Sci. (Eng.), Far Eastern Federal University, Vladivostok, Russia; Institute of Marine Technology Problems, Russian Academy of Sciences, Far Eastern Branch, Vladivostok, Russia
✉ zhirabok@mail.ru

Zuev, Aleksandr Valer'evich. Cand. Sci. (Eng.), Institute of Marine Technology Problems, Russian Academy of Sciences, Far Eastern Branch, Vladivostok, Russia; Far Eastern Federal University, Vladivostok, Russia
✉ alvzuev@yandex.ru

Kim, Chung Il. Postgraduate, Far Eastern Federal University, Vladivostok, Russia
✉ kim.ci@dvfu.ru

Cite this paper
Zhirabok, A.N., Zuev, A.V., and Kim, C.I., Interval Observer Design for Discrete Linear Time-Invariant Systems with Uncertainties. *Control Sciences* **2**, 15–22 (2023). http://doi.org/10.25728/cs.2023.2.2

Translated into English by *Alexander Yu. Mazurov*,
Cand. Sci. (Phys.–Math.),
Trapeznikov Institute of Control Sciences,
Russian Academy of Sciences, Moscow, Russia
✉ alexander.mazurov08@gmail.com