DOI: http://doi.org/10.25728/cs.2023.2.3

TERMINAL CONTROL OF MOVING OBJECTS IN THE CLASSES OF PIECEWISE CONSTANT AND PIECEWISE CONTINUOUS FUNCTIONS

V.K. Zavadsky[1], V.P. Ivanov[2], E.B. Kablova[3], L.G. Klenovaya[4], and V.Yu. Rutkovskii

Trapeznikov Institute of Control Sciences, Russian Academy of Sciences, Moscow, Russia

[1–4]✉ vladguc@ipu.ru

Abstract. This paper presents a terminal control problem with the separation of object's state coordinates into two types: the slowly changing coordinates figuring in boundary conditions and the coordinates of the stabilization loop. A predictive model of the object is introduced to design the control action. A differential equation is derived for predicted mismatches in the boundary conditions. The original system is discretized in time based on this equation. This problem is solved step-by-step in the classes of piecewise constant and piecewise continuous control actions. As an illustrative example, the problem of controlling the fuel consumption of a stage of a liquid-propellant launch vehicle is considered. The class of control actions is extended from piecewise constant to piecewise continuous functions in order to cover additional requirements for the control process. The continuous functions on intervals between control jumps are chosen using the local boundary conditions obtained during the terminal control design in the class of piecewise constant functions.

Keywords: terminal control, model predictive control (MPC), fuel consumption control for launch vehicles.

INTRODUCTION

Terminal control problems arise in many areas of engineering. In rocket dynamics, some examples of such problems include insertion into Earth orbit, fuel consumption until complete exhaustion from the tanks, rendezvous of spacecraft, etc. In these examples, the control problem is to bring an object (often called a plant in control theory) to given final states under known initial conditions. Terminal conditions can be defined as the required values of the object's state coordinates or in a more complex form, e.g., as some functions of the state coordinates.

The modern terminal control concept for the objects of rocket and space technology was thoroughly described in the monograph [1]. A fundamental element of terminal control is predicting the object's final state in the form of given boundary conditions.

Predictive methods in the field of rocket dynamics were considered in [2–4]. The application of modern model predictive control (MPC) methods to nonlinear systems was discussed in detail in the book [5]. The general idea of the MPC approach consists in constructing a predictive model of the object in order to find the optimal control actions at the current and subsequent time instants. Note that only the current control action is implemented and the optimization procedure is repeated at the next time instant. In the paper [6], for a given control action in a predictive model, the derivative of the predicted coordinate values over time was determined. MPC methods also evolved towards applying real-time optimization [7] and making the closed-loop system robust and adaptive [8, 12]. To reduce the computational burden, the prediction procedure is performed for a limited number of time instants.

In the rocket dynamics problems mentioned, terminal control is formed as part of the general problem of controlling an object by separating its relatively slow physical processes that determine the motion to a given target. In this case, the general control action is decomposed into the terminal control action and the problem of stabilizing the object with respect to its motion to a given target. One example is angular position stabilization for a launch vehicle with respect to the pitch angle program during its orbital insertion control. Note that the control action is directly applied to the object's dynamic part belonging to the stabilization loop. Terminal control design is usually considered independently of the stabilization loop. The object's coordinates outputted by the stabilization loop are taken as the terminal control action.

The authors [9] studied the terminal control problem with the decomposition of the general problem considering the formal description of the plant's dynamics, including the stabilization loop. Such an approach takes into account the dynamics of the transient response of the stabilization loop to the control action instead of the operating errors of this loop. In this case, the idea of predicting the object's final state is implemented for all dynamic channels of the system: from the application point of control actions to mismatches in the boundary conditions. The described approach was considered in [9–11] for control actions in the class of piecewise constant functions. The paper "On a Terminal Control Problem with Prediction of Mismatches in the Boundary Conditions" by D.D. Tabalin was mentioned in the conference chronicle [11]. This paper solves the terminal control problem in the class of piecewise continuous functions. This problem is closely related to the approach presented in [9]. Therefore, we provide the main control design results in the class of piecewise continuous functions. The class of control actions is extended from piecewise constant to piecewise continuous functions in order to cover additional requirements for the control process. The continuous functions on intervals between control jumps are chosen using the local boundary conditions obtained during the terminal control design in the class of piecewise constant functions.

1. PROBLEM STATEMENT

To understand the essence of control processes in a system bringing an object to a given final state, it is useful to separate two interconnected systems of object's equations differing in transients.

We consider a dynamic system of the form

$$\frac{dx_1}{dt} = f_1(x_1(t), x_2(t), t),$$
$$\frac{dx_2}{dt} = f_2(x_2(t), u(t), t), \qquad (1)$$
$$x(t_0) = x_0,$$

where $x_1 \in R^{n_1}$, $x_2 \in R^{n_2}$, $x = (x_1, x_2)$, $x_0 \in X_0 \subset R^{n_1+n_2}$, u denotes the control action, $u \in U \subset R^m$, and $t \in [t_0, T]$.

Here, the first system of equations for the coordinates $x_1(t)$ describes the object's motion to a given target. The objects controlled by terminal systems are very inertial in terms of transients to a given final state. (As a rule, they are integrating links.)

Their transients are controlled through other object's coordinates $x_2(t)$ with rapidly decaying transient dynamics. The essence of such control is to set the values of these coordinates. Control in the traditional sense (i.e., the position of different actuators such as drives, rudders, etc.) stabilizes the object's coordinates with respect to the set values. The operation of the closed stabilization loop is described by a system of equations for the coordinates $x_2(t)$. The control action $u(t)$ on the right-hand side of these equations is the settings for the coordinates $x_2(t)$.

One example is angular position stabilization for a launch vehicle with respect to the pitch angle program during its orbital insertion control. Thus, the terminal control action is directly applied to the dynamic part of the object belonging to the stabilization loop. In mathematical formulations of optimal control problems, terminal control design is usually considered independently of the stabilization loop.

Assume that there exists a unique solution of system (1) under any initial conditions.

The terminal control problem is to transfer the system to a desired final state x that satisfies the following boundary conditions at a time instant T:

$$\psi(x_1) : \psi_i(x_{1i}(T)) = 0, \ i \in L \subset \overline{1,\ell},$$

where ℓ is the number of these conditions. The boundary conditions are imposed only on the coordinates x_1, representing given condition vectors for individual components x_{1i}. Let the function ψ be differentiable. Note that in some terminal control problems, boundary conditions may be imposed on part of the coordinates x_2.

The time instant $T > t_0$ is either fixed or determined by the first instant of satisfying the pth boundary condition $\psi_p(x_{1p}) = 0$.

The object's coordinates x_2 are outputted by the stabilizing loop of the control system. In this case, the operation of the loop is considered only in terms of the transient response to changing the control action. Assume that the transients terminate on an interval significantly smaller than the terminal control horizon. This approach somewhat restricts generality since the dynamics of the coordinates x_2 and x_1 are supposed mutually independent.

With system (1) we associate a predictive model of the form

$$\frac{d\hat{x}_1}{d\tau} = f_1(\hat{x}_1(\tau), \hat{x}_2(\tau), \tau), \ \tau \in [t, T],$$
$$\frac{d\hat{x}_2}{d\tau} = 0, \qquad (2)$$
$$\hat{x}(t) = x(t).$$

The function

$$z(t) \equiv \psi(\hat{x}_1(T|t)),$$

where the time instant T is calculated for system (2) by analogy with the original system (1), will be called the predicted mismatch in the boundary conditions due to the predictive model (2). Therefore, the time instant T for system (2) is either fixed or determined by the first instant of satisfying the boundary condition ψ_p.

We denote by $\hat{x}_1(T|t)$ the value of $\hat{x}_1(T)$ due to the system of equations (2) that is predicted at a time instant t.

Being a function of $x(t)$ and t, $z(t)$ satisfies the differential equation

$$\frac{dz(t)}{dt} = \frac{\partial\psi}{\partial\hat{x}_1(T|t)}\left[\frac{\partial\hat{x}_1(T|t)}{\partial x_2(t)} f_2(x_2(t), u(t), t) + \frac{dT}{dt} f_1(\hat{x}_1(T|t), x_2(t), T)\right].$$

For details, see the paper [9].

From this point onwards, let T = const. In this case,

$$\frac{dz(t)}{dt} = \frac{\partial\psi}{\partial\hat{x}_1(T|t)} \frac{\partial\hat{x}_1(T|t)}{\partial x_2(t)} f_2(x_2(t), u(t), t). \qquad (3)$$

We introduce the notation dz/dx_2 for a matrix by which the function f_2 is multiplied in the expression for dz/dt. Then formula (3) can be written as

$$\frac{dz}{dt} = \frac{\partial z}{\partial x_2}(x(t), t) f_2(x_2(t), u(t), t). \qquad (4)$$

We will solve the terminal control action for object (1) by designing a feedback control action as a function of the predicted mismatches in the boundary conditions described by the differential equation (3). The control action will be chosen in the class of piecewise continuous functions.

The idea consists in a two-step control design. In the first step, the control action is constructed in the class of piecewise constant functions. Local boundary conditions are formed by solving this problem. Fulfilling the local boundary conditions in aggregate allows solving the original terminal control problem. In the second step, we extend the class of control actions: on the intervals between control jumps, the control action is assumed to be a continuous function. Such a control action will be designed considering the local boundary conditions obtained with piecewise constant control.

2. LOCAL BOUNDARY CONDITIONS AT TERMINAL CONTROL JUMPS

Assume that $u(t)$ is a piecewise constant function and t_j, j = 1, 2,..., k – 1, are the time instants of its jumps, $t_k = T$. In this case, $z(t_j)$ satisfies a difference equation, an analog of the differential equation (4). To obtain this equation, we use the results from [9].

Integrating equation (4) on a small interval $[t_j, t_j + \delta t]$ yields

$$z(t_j + \delta t)$$
$$= z(t_j) + \int_{t_j}^{t_j+\delta t} \frac{\partial z}{\partial x_2}(x(\tau), \tau) f_2(x_2(\tau), u(\tau), \tau) d\tau$$
$$= z(t_j) + \frac{\partial z}{\partial x_2}(x(t_j), t_j)\Delta x_2 + o(\delta t).$$

Here, δt is the interval of the transient on the coordinate x_2 of the object (1) during a jump of the control action from $u(t_j)$ to $u(t_{j+1})$ at the time instant t_j. For $\tau \in [t_j + \delta t, t_{j+1}]$, $f_2(x_2(\tau), u(\tau), \tau) = 0$.

For small δt, we pass to the discrete system

$$z(t_{j+1}) = z(t_j) + \frac{\partial z}{\partial x_2}(x(t_j), t_j)\Delta x_2(t_j),$$
$$j = 0, 1, 2, \ldots, k-1, \ z(t_k) = z(T), \qquad (5)$$

where

$$\Delta x_2(t_j) = \int_{t_j}^{t_j+\delta t} f_2(x_2(\tau), u(\tau), \tau) d\tau.$$

Let us reformulate the original terminal control problem as follows. We will find a discrete sequence of the increments $\Delta x_2(t_j)$ of the coordinate $x_2(t)$ at the time instants t_j, j = 0, 1, 2,..., k – 1, instead of the control action $u(t)$ in the class of piecewise constant functions.

We consider the case of no restrictions on Δx_2 and solve the terminal control problem

$$\{\Delta x_2(t_j)\} : z(t_k) = 0. \qquad (6)$$

Assume that $\frac{\partial z}{\partial x_2}(x(t_j), t_j) = \frac{\partial z}{\partial x_2}(t_j)$ in formula (5).

For the system of equations (5), problem (6) is solved backwards from the time instant t_{k-1}. For example, some condition is introduced for choosing

$\Delta x_2(t_{k-1})$ uniquely for the time instant t_{k-1}; simultaneously, a boundary condition is introduced for all control actions preceding t_{k-1}. These conditions are formulated by specifying linear operators with respect to the mismatches $z(t_k)$ and $z(t_{k-1})$. The resulting system of equations with respect to the mismatch $z(t_k)$ has the unique trivial solution. When passing to t_{k-2}, the solution for $\Delta x_2(t_{k-1})$ is taken into account.

As a result, the original control problem (6) with the right-end boundary conditions is reduced to an equivalent set of local control problems for a finite number of discrete time instants:

$$\begin{gathered}\{\Delta x_2(t_{j-1})\}: \\ \frac{\partial z^{\mathrm{T}}}{\partial x_2}(t_{j-1})K(t_j)z(t_j)=0 \quad \forall j\in\overline{p,k}, \\ \{\Delta x_2(t_1),\ \Delta x_2(t_2),\dots,\ \Delta x_2(t_{j-2}), K(t_{j-1})\}: \\ K(t_{j-1})z(t_{j-1})=0.\end{gathered} \tag{7}$$

Note that the first equation in (7) is intended to choose the current control action $\Delta x_2(t_{j-1})$. Multiplication by $\frac{\partial z^{\mathrm{T}}}{\partial x_2}(t_{j-1})$ gives the mismatches, whose number coincides with the dimension of the control vector.

The second equation specifies the local boundary conditions for choosing control actions at the time instants before t_{j-1}.

The matrix $K(t_{j-1})$ is given by the recurrence relation

$$\begin{cases} K(t_k)=\mathrm{E}, \\ K(t_{j-1})=\left(\mathrm{E}-K(t_j)\frac{\partial z}{\partial x_2}\left(\frac{\partial z^{\mathrm{T}}}{\partial x_2}K(t_j)\frac{\partial z}{\partial x_2}\right)^{-1}\right. \\ \left.\times\frac{\partial z^{\mathrm{T}}}{\partial x_2}K(t_j)\right)K(t_j),\ \forall j\in\overline{p,k}, \end{cases} \tag{8}$$

where E is an identity matrix of compatible dimensions and

$$\frac{\partial z}{\partial x_2}=\frac{\partial z}{\partial x_2}(t_{j-1}).$$

We denote $z_j(t_j)=K(t_j)z(t_j)$. The vector $z_j(t_j)$ consists of the mismatches $z(t_k)$ under the control actions $\Delta x_2(t_{j-1}),\ \Delta x_2(t_j),\dots,\ \Delta x_2(t_{k-1})$ (7).

Let the values $\Delta x_2(t_{j-1})\ \forall j\in\overline{p,k}$ be obtained from formula (7):

$$\begin{gathered}\Delta x_2(t_{j-1})=-\left(\frac{\partial z^{\mathrm{T}}}{\partial x_2}(t_{j-1})K(t_j)\frac{\partial z}{\partial x_2}(t_{j-1})\right)^{-1} \\ \times\frac{\partial z^{\mathrm{T}}}{\partial x_2}(t_{j-1})K(t_j)z(t_{j-1}) \quad \forall j\in\overline{p,k}.\end{gathered} \tag{9}$$

We determine the corresponding mismatches $z_j(t_j)$. In this case, $z_j(t_j)=z(t_k)$, and the first equation in (7) can be written as the system of linear equations for $z(t_k)$:

$$\begin{gathered}\frac{\partial z^{\mathrm{T}}}{\partial x_2}(t_{j-1})K(t_j)z(t_j) \\ =\frac{\partial z^{\mathrm{T}}}{\partial x_2}(t_{j-1})z(t_k)=0 \quad \forall j\in\overline{p,\ k}.\end{gathered} \tag{10}$$

According to [9], the system of equations (10) has the unique trivial solution $z(t_k)=0$ if the rank of the matrix $\left[\frac{\partial z}{\partial x_2}(t_{k-1})\ \ \frac{\partial z}{\partial x_2}(t_{k-2})\ \dots\ \frac{\partial z}{\partial x_2}(t_{p-1})\right]$ is equal to the dimension of the vector $z(t_k)$. In this case, the control action (9) is a solution of the original terminal control problem.

The control strategy corresponding to (7) imposes constraints on the object's trajectory for the values of $z(t)$ at a finite number of control jump instants. On the intervals between these instants, control actions may vary for the coordinates x_2. These variations can be chosen considering desired performance criteria for the state and control coordinates. Thus, it is possible to extend the class of control actions (functions) in order to optimize nonterminal criteria.

3. TERMINAL CONTROL DESIGN IN THE CLASS OF PIECEWISE CONTINUOUS FUNCTIONS

We formulate the terminal control problem in the class of piecewise continuous functions by supplementing the problem statement from Section 1.

As before, the controlled object is described by equation (1) and the predictive model by equation (2); the control action $u(t)$ has jumps at discrete time instants t_j, j = 0, 1, 2,..., k – 1. On the intervals $[t_j, t_{j+1}]$, $u(t)$ is a continuous function. The predicted mismatch vector $z(t)$ satisfies the differential equation (4) with $\frac{\partial z}{\partial x_2}(x(t),\ t)=\frac{\partial z}{\partial x_2}(t)$. We know the solution of the local control problems (7) in the class of piecewise constant functions: $\Delta x_2(t_{j-1})\ \ \forall j\in\overline{p,k}$ for the difference equation (5). We choose the control action $u(t)$ in the class of piecewise continuous functions under the following conditions on the intervals between time instants $t_{j-1}, t_j\ \ \forall j\in\overline{p,k}$:

$$\frac{\partial z^{\mathrm T}}{\partial x_2}(t_{j-1})K(t_j)z(t_j)=0\quad \forall j\in\overline{p,k},$$

$$K(t_{j-1})z(t_{j-1})=0,$$

where the matrix $K(t_{j-1})$ is given by (8). In this case,

$$z(t_k)=z(t_{k-1})+\int_{t_{k-1}}^{t_k}\frac{\partial z}{\partial x_2}(\tau)f_2(x_2(\tau),u(\tau),\tau)d\tau \quad (11)$$

for the piecewise continuous function $u(\tau)$, $\tau\in[t_{k-1}, t_k]$.

Assume that

$$\frac{\partial z^{\mathrm T}}{\partial x_2}(t_{k-1})z(t_k)=0\,. \qquad (12)$$

This condition narrows the class of piecewise continuous control actions under consideration. The new class, restricted by condition (12), is defined below.

Let $u(\tau)$ = const, $\tau\in(t_{k-1}, t_k)$. Due to the jump of the function $u(\tau)$ at the time instant t_{k-1}, in this case, we can write

$$z(t_k)=z(t_{k-1})+\frac{\partial z}{\partial x_2}(t_{k-1})\Delta x_2(t_{k-1})\,. \qquad (13)$$

The value $\Delta x_2(t_{k-1})$ is determined from condition (12) using the expression (9).

Suppose that on the interval $[t_{k-1}, t_k]$, the piecewise continuous function $u(\tau)$ satisfies the relation

$$\int_{t_{k-1}}^{t_k}\frac{\partial z}{\partial x_2}(\tau)f_2(x_2(\tau),u(\tau),\tau)d\tau =\frac{\partial z}{\partial x_2}(t_{k-1})\Delta x_2(t_{k-1}). \qquad (14)$$

In this case, (11) is transformed to (13) and, therefore, condition (12) holds.

Let the control actions at the time instants preceding t_{j-1} be chosen so that $K(t_{j-1})z(t_{j-1})=0$. (In other words, the boundary conditions specified by the second equation in (7) are satisfied.) Subtracting $K(t_{j-1})z(t_{j-1})$ from the right-hand side of (11) and performing trivial transformations, we obtain

$$z(t_k) =\frac{\partial z}{\partial x_2}(t_{k-1})\left(\frac{\partial z^{\mathrm T}}{\partial x_2}(t_{k-1})\frac{\partial z}{\partial x_2}(t_{k-1})\right)^{-1}\frac{\partial z^{\mathrm T}}{\partial x_2}(t_{k-1})z(t_{k-1}) +\int_{t_{k-1}}^{t_k}\frac{\partial z}{\partial x_2}(\tau)f_2(x_2(\tau),u(\tau),\tau)d\tau=0.$$

The first term on the right-hand side of this expression is $-\frac{\partial z}{\partial x_2}(t_{k-1})\Delta x_2(t_{k-1})$; see formula (9) for $j = k$. Due to the relation (14), we finally arrive at $z(t_k)=0$.

We now proceed to choosing the control action on the interval $[t_{k-2}, t_{k-1}]$. It is necessary to determine the new mismatch vector $z_{k-1}(t_{k-1})=K(t_{k-1})z(t_{k-1})$. For the vector $z_{k-1}(t_{k-1})$ and the piecewise continuous function $u(\tau)$, $\tau\in[t_{k-2}, t_{k-1}]$, we write

$$z_{k-1}(t_{k-1})=K(t_{k-1})z(t_{k-1})=K(t_{k-1}) \times\left(z(t_{k-2})+\int_{t_{k-2}}^{t_{k-1}}\frac{\partial z}{\partial x_2}(\tau)f_2(x_2(\tau),u(\tau),\tau)d\tau\right). \qquad (15)$$

Let the condition

$$\frac{\partial z^{\mathrm T}}{\partial x_2}(t_{k-1})z_{k-1}(t_{k-1})=0 \qquad (16)$$

hold for $u(\tau)$.

The control action on the interval $[t_{k-2}, t_{k-1}]$ is chosen using the same considerations as on the interval $[t_{k-1}, t_k]$. For the constant control action $u(\tau)$, $\tau\in(t_{k-2}, t_{k-1})$, we write

$$z_{k-1}(t_{k-1})=z_{k-1}(t_{k-2})+\frac{\partial z_{k-1}}{\partial x_2}(t_{k-2})\Delta x_2(t_{k-2}), \quad (17)$$

where

$$\frac{\partial z_{k-1}}{\partial x_2}(t_{k-2})=K(t_{k-1})\frac{\partial z}{\partial x_2}(t_{k-2})\,,$$

$$z_{k-1}(t_{k-2})=K(t_{k-1})z(t_{k-2})\,.$$

The value $\Delta x_2(t_{k-2})$ is determined from condition (16) using the expression (9).

Suppose that on the interval $[t_{k-2}, t_{k-1}]$, the piecewise continuous function $u(\tau)$ satisfies the relation

$$K(t_{k-1})\Big(\int_{t_{k-2}}^{t_{k-1}}\frac{\partial z}{\partial x_2}(\tau)f_2(x_2(\tau),u(\tau),\tau)d\tau -\frac{\partial z}{\partial x_2}(t_{k-2})\Delta x_2(t_{k-2})\Big)=0.$$

In this case, (15) is transformed to (17) and, therefore, condition (16) holds.

We define the new mismatch vector as $z_{k-2}(t_{k-2}) = K(t_{k-2})z(t_{k-2})$ under the condition $K(t_{k-2})z(t_{k-2}) = 0$, transforming (15) similarly to (11) on the interval $[t_{k-1}, t_k]$.

The procedure can be continued for all time instants preceding t_{k-1}.

As a result, for the time instant t_j, we have

$$z_j(t_j) = K(t_j)z(t_j) = K(t_j) \times \left(z(t_{j-1}) + \int_{t_{j-1}}^{t_j} \frac{\partial z}{\partial x_2}(\tau) f_2(x_2(\tau), u(\tau), \tau) d\tau \right), \quad (18)$$

$$K(t_j)\left(\int_{t_{j-1}}^{t_j} \frac{\partial z}{\partial x_2}(\tau) f_2(x_2(\tau), u(\tau), \tau) d\tau - \frac{\partial z}{\partial x_2}(t_{j-1})\Delta x_2(t_{j-1})\right) = 0. \quad (19)$$

Here $K(t_{j-1})$ is given by (8) and $\Delta x_2(t_{j-1})$ by (9). Due to formulas (19) and (9),

$$\frac{\partial z^{\mathrm{T}}}{\partial x_2}(t_{j-1})K(t_j)z(t_j) = 0 \quad \forall j \in \overline{p,k}\,. \quad (20)$$

Thus, equation (19) and the expression (9) for $\Delta x_2(t_{j-1})$ define conditions equivalent to the initial condition (20).

Letting $K(t_j)z(t_j) = 0$, we employ simple transformations of (18) to show that $z_j(t_j) = K(t_j)z(t_j) = 0.$

Thus, the control action chosen by (19), (9) in the class of piecewise continuous functions solves the terminal control problem: $z(t_k) = 0$. Note that on the intervals between jumps, the control action can be chosen in a sufficiently wide class of functions.

4. AN EXAMPLE

As an example, we consider the problem of controlling the fuel consumption of a liquid-propellant rocket. We restrict the further analysis to the problem of synchronizing the depletion of oxidizer and propellant by the time of turning the rocket stage's power unit off. This problem will be studied in the linear approximation. Due to the nonsynchronous depletion, the unused remainders of propellant components remain in the tanks, reducing the power characteristics of the rocket. The synchronization process is controlled by changing the ratio of component consumption in the power unit. In turn, a deviation of this parameter from the nominal optimal value causes losses of specific thrust. The losses become most tangible when approaching the end of the flight. All these considerations lead to qualitatively formulated requirements for the control process and determine the type of boundary conditions.

The controlled object is described by the equations

$$\dot{x}_1(t) = \frac{1}{T}x_2(t),\ \dot{x}_2(t) = -k(x_2(t) - u(t)),$$
$$t \in [t_0, t_k],\ T = t_k - t_0,\ x_2(t_0) = 0.$$

Here, $x_1(t)$ is the mismatch between the mass fractions of the propellant components and $x_2(t)$ is the relative deviation of the ratio of component consumption from the nominal value.

The boundary conditions are given by

$$x_1(t_k) = 0,\ x_2(t_k) = 0\,.$$

We determine the vector of the predicted mismatches $z(t)$ and $\dot{z}(t)$:

$$z(t) = \begin{vmatrix} x_1(t) + x_2(t)\dfrac{t_k - t}{T} \\ x_2(t) \end{vmatrix},$$

$$\dot{z}(t) = \begin{vmatrix} x_2(t)\dfrac{t_k - t}{T} \\ 1 \end{vmatrix} \dot{x}_2(t)\,.$$

First, let us solve the problem in the class of piecewise constant functions. In this case, it suffices to have two jumps of the control action $u(t)$ at time instants t_0, $t_0 < t_1 < T$. We determine the vector of predicted mismatches in the boundary conditions at the time instant t_1:

$$z(t_1) = \begin{vmatrix} x_1(t_1) + t_1' x_2(t_1) \\ x_2(t_1) \end{vmatrix},$$

$$t' = \frac{1}{T}(t_k - t)\,,\ t_1' = \frac{1}{T}(t_k - t_1)\,.$$

Note that under jumps of the function $u(t)$, the transient for the coordinate $x_2(t)$ terminates in a time significantly smaller than T. (The transient time is $\partial t < 0.01T$.) To reduce the system error due to the finite transient time ∂t, the partial derivative $\frac{\partial z}{\partial x_2}(t)$ in the example is taken at an intermediate time instant on the interval ∂t.

We have

$$z(t_k) = z(t_1) + \begin{vmatrix} t_1^* \\ 1 \end{vmatrix} \Delta x_2(t_1),$$

$$t_1^* \in (t_1',\ t_1' - \frac{\partial t}{T}),$$

where $\Delta x_2(t_1) = x_2(t_k) - x_2(t_1)$.

Under the jump of the function $u(t)$ at the time instant t_1, the condition $\begin{vmatrix} t_1^* \\ 1 \end{vmatrix} z(t_k) = 0$ implies

$$\Delta x_2(t_1) = -\frac{(x_1(t_1) + t_1' x_2(t_1))t_1^* + x_2(t_1)}{1 + t_1^{*2}}.$$

The matrix $K(t_1)$ takes the form

$$K(t_1) = \begin{vmatrix} 1 & -t_1^* \\ -t_1^* & t_1^{*2} \end{vmatrix} \frac{1}{1+t_1^{*2}}.$$

The new mismatch vector for choosing the control actions $\Delta x_2(t_0) = x_2(t_1) - x_2(t_0)$ is written in the form $z_1(t_1) = K(t_1)z(t_1)$,

where $z(t_1) = z(t_0) + \begin{vmatrix} t_0^* \\ 1 \end{vmatrix} \Delta x_2(t_0),\ t_0^* \in \left(t_0',\ t_0' - \frac{\partial t}{T} \right)$.

After trivial transformations we obtain the following condition for determining the control action $\Delta x_2(t_0)$:

$$z_1(t_1) = \begin{vmatrix} 1 \\ -t_1^* \end{vmatrix} (x_1(t_0) + (t_0^* - t_1^*)\Delta x_2(t_0)) \frac{1}{1+t_1^{*2}} = 0.$$

Hence, $\Delta x_2(t_0) = -\frac{x_1(t_0)}{t_0^* - t_1^*}$.

Note the possibility $z(t_1) \neq 0, x_1(t_1) \neq 0$.

We now solve the terminal control problem in the class of piecewise continuous functions. On the interval $[t_1, t_k]$ the control action $u(\tau)$ must satisfy the condition

$$\int_{t_1}^{t_k} \begin{vmatrix} \frac{t_k - \tau}{T} \\ 1 \end{vmatrix} \dot{x}_2(\tau) d\tau = \begin{vmatrix} t_1^* \\ 1 \end{vmatrix} \Delta x_2(t_1).$$

What is important, the function $u(\tau)$ can have a jump at the time instant t_1.

When choosing the continuous control function $u(\tau)$, $\tau \in [t_0, t_1]$, we must satisfy the condition

$$K(t_1) \int_{t_0}^{t_1} \left(\frac{t_k - \tau}{T} \right) \dot{x}_2(\tau) d\tau = K(t_1) \begin{vmatrix} t_0^* \\ 1 \end{vmatrix} \Delta x_2(t_0).$$

After trivial transformations, it takes the form

$$\int_{t_0}^{t_1} \left(\frac{t_k - \tau}{T} - t_1^* \right) \dot{x}_2(\tau) d\tau = (t_0^* - t_1^*) \Delta x_2(t_0).$$

Using integration by parts for the left-hand side, we obtain

$$x_1(t_0) + \frac{1}{T} \int_{t_0}^{t_1} x_2(\tau) d\tau = -x_2(t_1)(t'_1 - t_1^*).$$

This expression can be written as the boundary condition $x_1(t_1) = -x_2(t_1)(t'_1 - t_1^*)$.

Under this condition, continuous control actions can be chosen in a sufficiently wide class of functions. Considering the requirement for $x_2(t)$ (the deviation of the ratio of component consumption), $x_2(\tau)$ and the control action $u(\tau)$ on the interval $\tau \in [t_0, t_1]$ can be chosen as descending exponential functions.

Letting $t_0 = 0$, we set $u(\tau) = Be^{-r\tau}$, $\tau \in [t_0, t_1]$. In this case, $x_2(\tau) = \frac{kB}{k-r}(-e^{-k\tau} + e^{-r\tau})$, where $k >> r$. The parameters B and r are determined based on the initial and final conditions $x_1(t_0)$ and $x_2(t_0)$: $x_1(t_1) = -x_2(t_1)(t_1' - t_1^*)$. On the interval $\tau \in [t_1, t_k]$, we can take $u(\tau) = 0$. In this case, $x_2(\tau) = 0$ on the interval $\tau \in [t_1 + \partial t, t_k]$. We integrate the equation for $\dot{x}_1(t)$ on the interval $[t_1, t_1 + \delta t]$ to find $x_1(t_k) = x_1(t_1) + \frac{1}{kT} x_2(t_1)$. In view of the expression for $x_1(t_1)$, we obtain $x_1(t_k) = -x_2(t_1)(t_1' - t_1^* - \frac{1}{kT})$.

CONCLUSIONS

This paper has considered a terminal control problem in two statements: in the classes of piecewise constant and piecewise continuous functions. As has been shown, these statements are interconnected, and it is reasonable to consider them step-by-step.

During the design of piecewise constant control actions (the first step), local conditions are obtained for choosing control actions on each interval between control jumps. Fulfilling the local boundary conditions allows solving the original terminal control problem.

In the second step, control actions in the class of piecewise continuous functions are designed using the piecewise constant control actions constructed earlier. The local conditions yielded by the first step are used as boundary conditions for choosing control actions in the class of piecewise continuous functions on the intervals between control jumps. Note that under the local conditions, the continuous control actions can be chosen in a sufficiently wide class of functions.

REFERENCES

1. Sikharulidze, Yu.G., *Ballistika i navedenie letatel'nykh apparatov* (Ballistics and Guidance of Aircraft), Moscow: Binom. Laboratoriya Znanii, 2011. (In Russian.)
2. Haeussermann, W., Description and Performance of the Saturn Launch Vehicle's Navigation, Guidance, and Control System, *Proc. 3rd Int. IFAC Conf. on Automatic Control in Space*, Toulouse, France, 1970, vol. 3, pp. 275–312.
3. Petrov, B.N., Portnov-Sokolov, Yu.P., Andrienko, A.Ya., and Ivanov, V.P., *Bortovye terminal'nye sistemy upravleniya* (Onboard Terminal Control Systems), Moscow: Mashinostroenie, 1983. (In Russian.)
4. Veremei, E.I. and Eremeev, V.V., Introduction to Prediction-Based Control Problems, *Trudy Vserossiiskoi nauchnoi konferentsii "Proektirovanie nauchnykh i inzhenernykh prilozhenii v srede MATLAB"* (Proceedings of the All-Russian Scientific Conference "Development of Scientific and Engineering Applications in MATLAB"), Moscow, 2004, pp. 98–115. (In Russian.)

5. Grüne, L. and Pannek, J., *Nonlinear Model Predictive Control. Theory and Algorithms,* Springer, 2011.
6. Gul'ko, F.B. and Novosel'tseva, Zh.A., Application of Prediction Methods in the Design Problems of Automatic Control Systems, *Trudy VIII Vsesoyuznogo soveshchaniya po problemam upravleniya* (Proceedings of the 8th All-Union Meeting on Control Problems), Moscow, 1980, vol. 1, pp. 32–34. (In Russian.)
7. Klaučo, M., Kalúz, M., and Kvasnica, M., Real-time Implementation of an Explicit MPC based Reference Governor for Control of a Magnetic Levitation System, *Control Engineering Practice*, 2017, vol. 60, no. 3, pp. 99–105.
8. Langson, W., Chryssochoos, I., Rakovic, S.V., and Mayne, D.Q., Robust Model Predictive Control Using Tubes, *Automatica*, 2004, vol. 40, no. 1, pp. 125–133.
9. Ivanov, V.P. and Tabalin, D.D., On a Deterministic Terminal Control Method with Predictive Forecasting of Mismatches in the Boundary Conditions, *Automation and Remote Control*, 2022, vol. 83, no. 1, pp. 62–77.
10. Tabalin, D.D., Deterministic Design of Terminal Control Algorithms with Prediction of Mismatches in the Boundary Conditions, *Tezisy mezhdunarodnogo molodezhnogo nauchnogo foruma "LOMONOSOV-2020"* (Abstracts of the International Youth Scientific Forum "LOMONOSOV-2020"), Moscow, 2020. (In Russian.)
11. Seminar on Problems of Nonlinear Dynamics and Control in Moscow State University, *Diff. Uravnen.*, 2020, vol. 56, no. 8, pp. 1134–1144. (In Russian.)
12. Chai, J., Medagoda, E., and Kayacan, E., Adaptive and Efficient Model Predictive Control for Booster Reentry, *Journal of Guidance, Control, and Dynamics*, 2020, vol. 43(12), pp. 2372–2382. DOI: https://doi.org/10.2514/1.G004925.

This paper was recommended for publication by B.V. Pavlov, a member of the Editorial Board.

Received October 11, 2022,
and revised January 16, 2023.
Accepted February 14, 2023.

Author information

Zavadsky, Vladimir Konstantinovich. Cand. Sci. (Eng.), Trapeznikov Institute of Control Sciences, Russian Academy of Sciences, Moscow, Russia
✉ vladguc@ipu.ru

Ivanov, Vladimir Petrovich. Dr. Sci. (Eng.), Trapeznikov Institute of Control Sciences, Russian Academy of Sciences, Moscow, Russia
✉ vladguc@ipu.ru

Kablova, Elena Borisovna. Researcher, Trapeznikov Institute of Control Sciences, Russian Academy of Sciences, Moscow, Russia
✉ vladguc@ipu.ru

Klenovaya, Lyudmila Grigor'evna. Researcher, Trapeznikov Institute of Control Sciences, Russian Academy of Sciences, Moscow, Russia
✉ vladguc@ipu.ru

Rutkovskii, Vladislav Yul'evich. Dr. Sci. (Eng.), Trapeznikov Institute of Control Sciences, Russian Academy of Sciences, Moscow, Russia

Cite this paper
Zavadsky, V.K., Ivanov, V.P., Kablova, E.B., Klenovaya, L.G., and Rutkovskii, V.Yu., Terminal Control of Moving Objects in the Class of Piecewise Constant and Piecewise Continuous Functions. *Control Sciences* **2**, 23–30 (2023). http://doi.org/10.25728/cs.2023.2.3

Original Russian Text © Zavadsky, V.K., Ivanov, V.P., Kablova, E.B., Klenovaya, L.G., Rutkovskii, V.Yu., 2023, published in *Problemy Upravleniya*, 2023, no. 2, pp. 28–36.

Translated into English by *Alexander Yu. Mazurov*,
Cand. Sci. (Phys.–Math.),
Trapeznikov Institute of Control Sciences,
Russian Academy of Sciences, Moscow, Russia
✉ alexander.mazurov08@gmail.com