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AN ANISOTROPY-BASED BOUNDEDNESS CRITERION FOR TIME-INVARIANT SYSTEMS WITH MULTIPLICATIVE NOISES

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Abstract. This paper presents an anisotropy-based analysis of linear time-invariant systems with multiplicative noises. The system dynamics are described in the state space. The external disturbance belongs to the set of stationary sequences of random vectors with bounded mean anisotropy. The multiplicative noises are centered and have unit variance; the external disturbance and noises are mutually independent. We derive a boundedness criterion for the anisotropic norm in terms of Riccati-like inequalities using the bounded real lemma of the anisotropy-based theory. With a special change of variables, we reduce the analysis problem to a convex optimization problem with additional constraints. The existence of the latter's solution implies the bounded anisotropic norm of the system with multiplicative noises, and the minimal upper bound of the anisotropic norm can be obtained by solving this convex optimization problem.

Keywords: anisotropy-based theory, anisotropic norm, multiplicative noises, time-invariant systems, bounded real lemma.

INTRODUCTION

The attenuation of external disturbances is still one of the most topical problems in control theory [1, 2]. First appeared in the 1950s, when the growing complexity of technical systems required high accuracy as one priority, this research area has gradually formed an entire branch in modern control theory with many applications for different systems; for example, see [3–5]. In the problems of motion along a given trajectory, the control object is often subjected to disturbances whose stochastic characteristics significantly affect the choice of the control law. Some ways to reject external disturbances of bounded energy were considered in [6, 7]. Ensuring optimal control, the approach presented therein still suffers from a drawback: the resulting controllers have a high dimension. The technical implementation peculiarities of optimal control laws for continuous-time systems with bounded disturbances were analyzed in [8].

Note that such control problems were solved not only in the case of bounded disturbances. For example, in $\mathcal{H}_2$ control theory, random disturbances with known stochastic characteristics were considered; for $\mathcal{H}_\infty$ control laws, square integrable and square summable disturbances were selected for continuous-time and discrete-time systems, respectively, [9]. The choice of an appropriate optimality criterion largely depends on the type of disturbances: $\mathcal{H}_\infty$-optimal controllers have an increased conservatism due to the assumption on the worst-case input of the system and give far from optimal results under weakly colored disturbances; in contrast, $\mathcal{H}_2$-optimal control laws are oriented to no uncertainty in the stochastic parameters of Gaussian disturbances.

Despite the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control statement proposed to eliminate the drawbacks of each disturbance control method mentioned, where different types of impacts on the system are separated by channels [10, 11], a stochastic approach to $\mathcal{H}_\infty$-optimization was also developed in [12–14]. This approach was introduced by I.G. Vladimirov and was called *the anisotropy-based (control) theory* of stochastic filtering and control. The *anisotropy* of a random vector is a measure of uncertainty for the distribution function of this vector. Due to this concept, the conservatism inherent in $\mathcal{H}_\infty$ control theory was reduced. *Mean anisotropy* was defined for stationary Gaussian sequences of random vectors. A performance criterion—*the anisotropic norm*—was chosen as a stochastic $\mathcal{H}_\infty$ norm of the system. Within the anisotropy-based theory, filtering and control problems (analysis and design) were solved for linear time-invariant and time-varying deterministic models. The analysis problem with random matrices in the object's state-space description was first posed in [15]; subsequently, systems with multiplicative noises were considered. Such descriptions of dynamics are typical of mechanical systems, financial models, chemical reactions [16, 17], and network systems [18, 19], arising interest in studying systems with multiplicative noises.

Within the anisotropy-based theory, the first works on control design for a system with multiplicative noises were estimative in nature: the anisotropic norm was majorized (an upper bound was constructed), and a control method for the upper bound was proposed [20]. The paper [21] considered the analysis problem, but an exact method for calculating the anisotropic norm was developed in [22] based on the approach presented in [15]. With the analysis problem solved, it became possible to construct an estimate in the case of measurement dropout correction [23] and an estimate based on a sensor network [24]. In the case of using a sensor network, one possible way to improve the efficiency of estimation is to adjust the information exchange scheme of the sensors; for details, see [25]. The above results refer to time-varying systems; for the class of time-invariant systems, the analysis problem was solved in [26]. Based on those results, below we reduce the anisotropy-based analysis problem to systems of matrix inequalities with convex constraints.

The remainder of this paper is organized as follows. Section 1 gives a brief introduction to the anisotropy-based theory. The problem under consideration is stated in Section 2. We present the main result of the paper in Section 3. Section 4 is devoted to numerical simulation.

1. PRELIMINARIES

This section provides only the basic definitions of the anisotropy-based theory for discrete time-varying systems. A more complete description can be found in [27, 28].

1.1. Mean Anisotropy and Anisotropic Norm

The mean anisotropy of a sequence of random vectors was defined in [13]. The anisotropy of a random m-dimensional vector $w \in \mathbb{R}^m$ with a probability density function (PDF) $f(x)$ is given by

$$\mathcal{A}(w) = \min_{\lambda>0} \mathcal{D}(f \,\|\, p_\lambda),$$

where the reference probability distribution $p_\lambda(x)$ is centered Gaussian with the scalar covariance matrix λI_m, i.e.,

$$p_\lambda(x) = (2\pi\lambda)^{-m/2} \exp\left(-\frac{\|x\|^2}{2\lambda}\right),$$

and $\mathcal{D}(f \,\|\, p_\lambda)$ denotes the Kullback–Leibler divergence (differential entropy) of the PDF f with respect to p_λ, i.e.,

$$\mathcal{D}(f \,\|\, p_\lambda) = E\left[\ln \frac{f}{p_\lambda}\right],$$

where $E[\cdot]$ stands for the expectation operator.

The mean anisotropy of a sequence of random vectors $W = \{w_k\}$ is the time-averaged anisotropy of an infinitely growing fragment of the sequence

$$\bar{\mathcal{A}}(W) = \lim_{N\to\infty} \frac{\mathcal{A}(W_{0:N-1})}{N},$$

where $W_{0:N-1} = \left(w_0^{\mathrm{T}}, \ldots, w_{N-1}^{\mathrm{T}}\right)^{\mathrm{T}}$ is the extended vector. The definition and properties of mean anisotropy were discussed in detail in [28].

Consider a linear system F with input $W \in \mathbb{L}_2^m$ and output $Z \in \mathbb{L}_2^p$ sequences. If the sequence W is obtained by a linear filter G from a white-noise Gaussian sequence V, then each random vector w_j of the former sequence can be written as

$$w_j = \sum_{k=0}^{\infty} g_k v_{j-k}, \quad j \in \mathbb{Z},$$

where $g_k \in \mathbb{R}^{m\times m}$, $k \ge 0$, denotes the impulse function. The generating filter G and its transfer function $G(z)$ have the relation

$$G(z) = \sum_{k=0}^{\infty} g_k z^k$$

for $|z|<1$, $z\in\mathbb{C}$. The finite $\mathcal{H}_2$-norm $\|G\|_2$ of the transfer function $G(z)$ can be calculated as

$$\|G\|_2=\left(\sum_{k=0}^{\infty}\mathrm{tr}\left(g_k g_k^{\mathrm{T}}\right)\right)^{1/2}.$$

We denote by $F(z)$ the transfer function of a linear system F with a finite $\mathcal{H}_\infty$-norm

$$\|F\|_\infty=\sup_{|z|<1}\bar{\sigma}\left(F(z)\right)=\operatorname*{ess\,sup}_{-\pi\le\omega\le\pi}\bar{\sigma}\left(\hat{F}(\omega)\right),$$

where $\bar{\sigma}(\cdot)$ is the maximum singular value of a corresponding matrix and $\hat{F}(\omega)=\lim_{\rho\to1-}F\left(\rho e^{i\omega}\right)$.

In the anisotropy-based theory, the set of linear filters that generate sequences with a bounded mean anisotropy is denoted by

$$\mathcal{G}_a=\left\{G\in\mathcal{H}_2^{m\times m}:\ W=GV,\ \bar{\mathcal{A}}\left(W\right)\le a\right\},$$

where $\mathcal{H}_2^{m\times m}$ stands for the Hardy space of complex-valued matrix functions analytic inside the unit circle and $V=\left\{v_k\right\}_{k\in\mathbb{Z}}$ is a centered Gaussian sequence with the unit covariance matrix [12]. The anisotropic norm of a linear time-invariant system F with the input W generated by a filter G has the form

$$|||\,F\,|||_a=\sup\left\{\frac{\|FG\|_2}{\|G\|_2}:G\in\mathcal{G}_a\right\}.$$

For a causal system $F\in\mathcal{H}_\infty^{p\times m}$ satisfying the condition $\frac{\|F\|_2}{\sqrt{m}}<\|F\|_\infty$, the anisotropic norm always takes an intermediate value:

$$\frac{1}{\sqrt{m}}\|F\|_2=\lim_{a\to0}|||\,F\,|||_a\le|||\,F\,|||_a\le\lim_{a\to\infty}|||\,F\,|||_a=\|F\|_\infty.$$

Due to this property, the anisotropy-based theory generalizes $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control theories: in the limiting cases (when mean anisotropy is zero or tends to infinity), we obtain either the scaled $\mathcal{H}_2$ norm or $\mathcal{H}_\infty$ norm of the linear time-invariant system. If mean anisotropy takes intermediate values, the anisotropic norm can be called a compromise between these norms.

For disturbing sequences with nonzero mean anisotropy, the anisotropic norm is a stochastic analog of the $\mathcal{H}_\infty$ norm; hence, an information criterion on the non-uniform distribution of the external disturbance can be used to reduce the conservatism of the classical $\mathcal{H}_\infty$ norm calculated for the “worst” case.

1.2. Calculation of the Anisotropic Norm

Consider the state-space description of the system:

$$F:\begin{cases}x_{k+1}=Ax_k+Bw_k,\\ z_k=Cx_k+Dw_k,\end{cases}\tag{1}$$

where $x_k\in\mathbb{R}^{n_x}$, $w_k\in\mathbb{R}^{m_w}$, and $z_k\in\mathbb{R}^{p_z}$ are the state vector, the external disturbance, and the system output, respectively. The system matrices $A, B, C,$ and D are constant and have compatible dimensions. The system is stable if the spectral radius of the matrix A satisfies the inequality $\rho(A)<1$. The external disturbance is a colored sequence obtained by a generating filter from a white-noise sequence V. The state w_k of the filter G is a linear combination of the state vector of system (1) and the corresponding element of the Gaussian sequence V:

$$w_k=Lx_k+\Sigma^{1/2}v_k,$$

where $\Sigma\in\mathbb{R}^{m_w\times m_w}$ is a symmetric positive definite matrix and $L\in\mathbb{R}^{m_w\times n_x}$ is a matrix ensuring the asymptotic stability of ($A+BL$). There exists a parameter $q\in\left[0,\|F\|_\infty^{-2}\right)$ that relates via a special equation the mean anisotropy a and the anisotropic norm of the linear time-invariant system to the solutions of the Riccati and Lyapunov equations expressed through the state-space matrices [14]:

$$|||\,F\,|||_a=\left(\frac{1}{q}\left(1-\frac{m_w}{\mathrm{tr}\left(LPL^{\mathrm{T}}+\Sigma\right)}\right)\right)^2.$$

Moreover, the generating filter G ensures the mean anisotropy

$$a=-\frac{1}{2}\ln\det\left(\frac{m_w\Sigma}{\mathrm{tr}\left(LPL^{\mathrm{T}}+\Sigma\right)}\right),\tag{2}$$

where Σ is the covariance matrix, $P\in\mathbb{R}^{n_x\times n_x}$ is the solution of the Lyapunov equation

$$P=\left(A+BL\right)P\left(A+BL\right)^{\mathrm{T}}+B\Sigma B^{\mathrm{T}},\tag{3}$$

and the parameters $q, L,$ and Σ parameters are related to the solution of the Riccati equation:

$$
\begin{gathered}
R = A^{\mathrm{T}} R A + q C^{\mathrm{T}} C + L^{\mathrm{T}} \Sigma^{-1} L, \\
\Sigma = \left(I_{m_w} - q D^{\mathrm{T}} D - B^{\mathrm{T}} R B \right)^{-1}, \\
L = \Sigma \left(B^{\mathrm{T}} R A + q D^{\mathrm{T}} C \right).
\end{gathered} \tag{4}
$$

Analysis issues in the anisotropy-based theory were described in detail in [13, 14]. The concepts mentioned above refer to linear time-invariant systems only, just one class of models considered in this theory.

1.3. The Suboptimal Problem

The system of coupled matrix equations (2)–(4) is nonlinear, which complicates numerical solution. In the anisotropy-based theory, optimal problems are therefore often replaced by suboptimal ones, for which an efficient numerical solution method has been developed. This method involves convex optimization to find an upper bound γ on the anisotropic norm $|||F|||_a$ of system (1). See the papers [29, 30] for numerical methods for solving suboptimal problems in the anisotropy-based theory.

The anisotropic norm of the linear system (1) is bounded above by a given threshold γ if the inequalities

$$
\eta - \left(\exp(-2a) \det \Xi \right)^{1/m_w} < \gamma^2, \tag{5}
$$

$$
\begin{bmatrix} \Xi - \eta I_{m_w} & * & * \\ B & -\Theta & * \\ D & 0 & -I_{p_z} \end{bmatrix} \prec 0, \tag{6}
$$

$$
\begin{bmatrix} -\Theta & * & * & * \\ 0 & -\eta I_{m_w} & * & * \\ A & B & -\Theta & * \\ C & D & 0 & -I_{p_z} \end{bmatrix} \prec 0 \tag{7}
$$

have positive definite solutions $\Xi \in \mathbb{R}^{m_w \times m_w}$ and $\Theta \in \mathbb{R}^{p_z \times p_z}$ with a parameter $\eta > 0$. (The symbol $*$ indicates symmetric blocks with respect to the main diagonal.) The sufficient conditions (5)–(7) of anisotropic norm boundedness can be obtained from equations (2)–(4) by passing to inequalities using the Schur complement lemma, the appropriate changes of variables, and the properties of solutions of the Riccati equations and inequalities [29]. Note that inequalities like (6) and (7) are understood in the sense of positive or negative definiteness.

The linear matrix inequalities (LMIs) (6) and (7) are obtained by congruent transformations: after applying the Schur complement lemma, these inequalities should be multiplied by the matrices $\operatorname{blockdiag}\left(I_{m_w}, \Theta, I_{p_z}\right)$ and $\operatorname{blockdiag}\left(I_{p_z}, I_{m_w}, \Theta, I_{p_z}\right)$ on the left and right, respectively.

Remark 1. This method of passing to matrix inequalities is not the only way to obtain a suboptimal solution based on the original optimal problem. Using the change of variables $\Theta^{-1} = \Pi$, we can introduce the inequality

$$
\begin{bmatrix} \Theta & I_{p_z} \\ I_{p_z} & \Pi \end{bmatrix} \succ 0 \tag{8}
$$

to eliminate the nonlinearity in inequalities (6) and (7) and use the algorithm for calculating the mutually inverse (reciprocal) matrices Θ and Π [31, 32].

The corresponding optimization problem has the form

$$
\gamma^2 \underset{\Theta, \Xi, \eta, \gamma^2}{\rightarrow} \min
$$

subject to the constraints (5)–(8). The minimum value γ^2 can be found using standard optimization procedures in applied software packages.

2. PROBLEM STATEMENT

Consider a linear discrete time-invariant system F with the state-space realization (1), where $w_k \in \mathbb{R}^{m_w}$ is a disturbance with a given upper bound a on its mean anisotropy. Let the free dynamics matrix A be represented as a linear combination of known matrices with random coefficients:

$$
A = A_0 + \sum_{i=1}^{n} \xi_{i,k} A_i, \tag{9}
$$

where the random variables $\xi_{i,k}$, $i = \{1, \ldots, n\}$, have zero mean and unit covariance. The existence of the first two moments of these variables is sufficient to apply the anisotropy-based theory methods. The matrices A_i, $i = \{0, \ldots, n\}, B, C,$ and D are known and have compatible dimensions. An additional condition, an analog of the Hurwitz property in the classical case of discrete time-invariant systems, has the form

$$\lim_{k\to\infty}\rho\left(\left(E\left[A^k\right]\right)^{\frac{1}{k}}\right)<1, \tag{10}$$

where $\rho(\cdot)$ is the spectral radius.

The problem is to find a condition on matrices of system (1), (9) under which its anisotropic norm will not exceed a given threshold γ:

$$|||F|||_a \le \gamma.$$

3. THE MAIN RESULT

In the general case, all matrices of system (1) may contain multiplicative noises, but we will focus on the problem statement above: for such models, one application is sensor systems with random dropouts in which the closed loop system contains multiplicative noises only in the matrix A.

The following lemma will serve for deriving a boundedness condition for the anisotropic norm of system (1) with the free dynamics matrix (9).

Lemma [26]. *The anisotropic norm* $|||F|||_a$ *of system* (1) *with the additional conditions* (9) *and* (10) *is bounded above by a positive number* γ *if there exist positive definite matrices* $R_1, R_2 \in \mathbb{R}^{n_x\times n_x}$ *and a parameter* $q\in\left[0, \|F\|_\infty^{-2}\right)$ *satisfying the system of modified Riccati-like equations*

$$\begin{gathered} R_1=\sum_{i=0}^{n} A_i^{\mathrm{T}} R_1 A_i + qC^{\mathrm{T}}C, \\ R_2 = A_0^{\mathrm{T}} R_2 A_0 + L^{\mathrm{T}} S^{-1} L, \\ S=\left(I_{m_w} - qD^{\mathrm{T}}D - B^{\mathrm{T}}R_1 B - B^{\mathrm{T}} R_2 B\right)^{-1}, \\ L = S\left(qD^{\mathrm{T}}C + B^{\mathrm{T}} R_1 A_0 + B^{\mathrm{T}} R_2 A_0\right) \end{gathered} \tag{11}$$

and the special inequality

$$-\frac{1}{2}\ln\det\left(\left(1-q\gamma^2\right)S\right)\ge a, \tag{12}$$

where $a > 0$ *is the mean anisotropy bound for the input sequence of random vectors* $\{w_k\}$.

This lemma is a modified analog of the bounded real lemma for time-invariant systems within the anisotropy-based theory [33]. Formulas (11) and (12) contain nonlinearities, which may complicate finding the solution. Therefore, it is necessary to reduce the equations to LMIs with an additional convex constraint. Before formulating this result as a theorem, we prove another assertion.

Theorem 1. *Let the mean anisotropy of the disturbance* $\{w_k\}$ *of system* (1) *with the additional condition* (10) *be bounded above by a number* $a \ge 0$. *If the inequality*

$$\begin{gathered} \tilde{R} \succ \sum_{i=0}^{n} A_i^{\mathrm{T}} R A_i + qC^{\mathrm{T}}C + L^{\mathrm{T}}S^{-1}L, \\ S=\left(I_{m_w} - qD^{\mathrm{T}}D - B^{\mathrm{T}}\tilde{R}B\right)^{-1}, \\ L=S\left(B^{\mathrm{T}}\tilde{R}A_0 + qD^{\mathrm{T}}C\right), \end{gathered} \tag{13}$$

jointly with the special inequality

$$-\frac{1}{2}\ln\det\left(\left(1-q\gamma^2\right)S\right)\ge a$$

has a solution $\tilde{R}\succ 0$, $S\succ 0$, $q\in\left[0, \|F\|_\infty^{-2}\right)$, *then the anisotropic norm of system* (1) *with* (9) *is bounded above by* $\gamma>0$.

P r o o f of Theorem 1. We introduce a new matrix variable of the form

$$R = R_1 + R_2.$$

It satisfies an equation similar to the Riccati equation

$$\begin{gathered} R=\sum_{i=0}^{n} A_i^{\mathrm{T}} R A_i + qC^{\mathrm{T}}C + L^{\mathrm{T}}S^{-1}L - \sum_{i=1}^{n} A_i^{\mathrm{T}} R_2 A_i, \\ S=\left(I_{m_w} - qD^{\mathrm{T}}D - B^{\mathrm{T}}RB\right)^{-1}, \\ L=S\left(qD^{\mathrm{T}}C + B^{\mathrm{T}}RA_0\right), \end{gathered}$$

obtained using formula (11) and the variable R. According to the properties of the solutions of Riccati equations and inequalities [34], there exists a matrix $\tilde{R}=\tilde{R}^{\mathrm{T}}\succ 0$ satisfying inequality (13). ♦

Theorem 1 provides sufficient boundedness conditions for the anisotropic norm of the system with multiplicative noises. However, their verification is difficult due to the nonlinearity contained in formulas (12) and (13). The next result expresses a boundedness condition for the anisotropic norm in terms of LMIs with a convex constraint.

Theorem 2. *Consider the system with multiplicative noises* (1) *and the additional conditions* (9) *and* (10) *and let the mean anisotropy of the external disturbance be bounded above by a given number* $a \ge 0$. *The anisotropic norm of the system will not exceed a given threshold* γ,

$$|||F|||_a \le \gamma,$$

if the inequalities

$$\begin{bmatrix} \sum_{i=0}^{n} A_i^{\mathrm{T}} R A_i - R + C^{\mathrm{T}} C & * \\ B^{\mathrm{T}} R A_0 + D^{\mathrm{T}} C & \eta I_{m_w} - D^{\mathrm{T}} D - B^{\mathrm{T}} R B \end{bmatrix} \prec 0, \quad (14)$$

$$\begin{bmatrix} \eta I_{m_w} - \Psi - D^{\mathrm{T}} D & * \\ RB & R \end{bmatrix} \succ 0, \qquad (15)$$

$$\ln\det\Psi \geq 2a + \ln\left(\eta - \gamma^2\right) \qquad (16)$$

have solutions $R = R^{\mathrm{T}} \succ 0$, $\Psi = \Psi^{\mathrm{T}} \succ 0$, *and* $\eta > 0$.

P r o o f of Theorem 2. We choose $\eta = q^{-1}$ as a new variable. With the change $R = q^{-1}\tilde{R}$, inequality (14) can be obtained from inequality (13) by applying the Schur complement lemma. Next, we introduce a matrix Θ satisfying the relation $0 \prec \Theta \prec S^{-1}$. Then the matrix $\Psi = q^{-1}\Theta$ will satisfy inequality (15) after applying the Schur complement lemma. The convex constraint (16) is the special inequality (14) written in terms of the new variables. ♦

Obviously, system (14)–(16) is convex in the variable γ^2. Hence, we can formulate the convex optimization problem

$$\gamma^2 \underset{R,\Psi,\eta,\gamma^2}{\rightarrow} \min$$

under the existence of solutions of the LMIs (14) and (15) with the convex constraint (16). This convex optimization problem can be solved using standard semidefinite programming tools.

4. NUMERICAL SIMULATION

As an illustrative example, we consider a two-mass oscillating system described in [35]. The system was closed by a standard linear-quadratic controller and discretized. Its state-space implementation has the form

$$x_{k+1} = \left(A_0 + \xi_1 A_1 + \xi_2 A_2\right) x_k + B w_k,$$
$$z_k = C x_k + D w_k,$$

where the mean anisotropy of the external disturbance (a sequence of random vectors $\{w_k\}$) is bounded above by a given number a and the random variables ξ_1, ξ_2 are centered and have unit variance. The numerical matrices are known:

$$A_0 = \begin{bmatrix} 0.9918 & 0.0444 & 0.0031 & -0.0043 \\ -0.3177 & 0.7829 & 0.1190 & -0.1651 \\ 0.0012 & 0.0000 & 0.9988 & 0.0500 \\ 0.0498 & 0.0012 & -0.0499 & 0.9987 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} 0.0992 & 0.0044 & 0.0003 & -0.0004 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0001 & 0.0000 & 0.1492 & 0.0075 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 \end{bmatrix},$$

$$C = \begin{bmatrix} 1.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 1.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 1.0000 & 0.0000 \\ -6.0545 & -4.7020 & 1.5823 & -2.7857 \end{bmatrix},$$

$$B = \begin{bmatrix} -0.0001 \\ -0.0043 \\ 0.0012 \\ 0.0507 \end{bmatrix}, \quad D = 0.$$

The table below combines the upper bounds (thresholds) γ of the anisotropic norm $|||F|||_a$ calculated for different mean anisotropies a of the external disturbance.

Note that the $\mathcal{H}_\infty$ norm of the system is 3.3244, i.e., the anisotropic estimator provides a much better quality of estimation in terms of the root-mean-square gain.

The anisotropic norm threshold depending on the mean anisotropy

The mean anisotropy a	0.0	0.01	0.05	0.10	0.20	0.50	1.00	1.50	2.00	3.00
The anisotropic norm threshold γ	0.3035	0.3048	0.3124	0.3211	0.3363	0.3655	0.3737	0.4299	0.9739	2.9180

5. CONCLUSIONS

This paper has presented an anisotropy-based analysis of linear discrete time-invariant systems with multiplicative noises. The bounded real lemma and a special change of variables have been adopted to establish a boundedness condition for the anisotropic norm of the system in terms of state-space matrices. Moreover, the upper bound on the anisotropic norm can be numerically minimized by standard semidefinite programming tools. As an illustrative example, the upper bound has been calculated for the anisotropic norm of an oscillating system. As demonstrated above, anisotropy-based estimation can significantly improve the quality of estimation under a priori information (the bounded mean anisotropy of the external disturbance), especially in the cases of weakly colored disturbances.

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