DYNAMIC ANISOTROPY-BASED CONTROLLER DESIGN FOR TIME-INVARIANT SYSTEMS WITH MULTIPLICATIVE NOISE

A. V. Yurchenkov

Trapeznikov Institute of Control Sciences, Russian Academy of Sciences, Moscow, Russia

✉ alexander.yurchenkov@yandex.ru

Abstract. This paper considers a linear discrete time-invariant system with multiplicative noise and a control input under an external disturbance from a special class. The plant's dynamics are described in the state space. The class of external disturbances contains a set of stationary Gaussian sequences with a bounded mean anisotropy. The anisotropic norm of the closed-loop control system is chosen as the performance criterion. It is required to design a dynamic link-based control scheme under which the anisotropic norm of the closed-loop control system will be bounded by the minimum possible threshold. At the first stage of solving this problem, the controller's dynamics are written out and the plant under consideration is augmented. The boundedness criterion of the anisotropic norm in terms of matrix inequalities is used to derive sufficient conditions for the existence of a solution of a convex optimization problem to minimize the upper bound of the anisotropic norm. A special change of variables is performed in the resulting inequalities to eliminate the nonlinear dependence on the unknown controller matrices. After a linearizing inversible change of variables, the optimization problem is solved numerically using standard methods. At the last stage, the desired controller matrices are calculated in the state space to ensure the bounded anisotropic norm of the closed-loop control system.

Keywords: linear discrete time-invariant systems, anisotropy-based theory, dynamic control, LMI, convex optimization.

INTRODUCTION

The active development of automatic control theory in the 20th century caused the creation of tools for attenuating external disturbances, which has become one of the most important problems in this theory. Since the approach to attenuating Gaussian disturbances with the linear quadratic performance criterion had been pioneered [1], many methods for dealing with external disturbances were proposed. Some of them are tuned to the case of known stochastic characteristics of input signals. On the other hand, the $\mathcal{H}_\infty$-optimization approach [2] offers a way to parry the "worst-case" disturbance. However, $\mathcal{H}_\infty$ control has excessive conservatism, and optimal $\mathcal{H}_2$ controllers are sensitive to small parameter variations. Therefore, they turn out non-robust, and control optimality is violated accordingly. Despite the fundamental differences between $\mathcal{H}_2$- and $\mathcal{H}_\infty$-theories, some studies combining the two methods were published [3–6].

One branch of control theory, investigating ways to attenuate external disturbances, was developed by I.G. Vladimirov about thirty years ago [7, 8]; it covers both the $\mathcal{H}_2$- and $\mathcal{H}_\infty$-optimal control theories as limiting cases. This theory, called the anisotropy-based (control) theory by the author, offers a stochastic approach to $\mathcal{H}_\infty$ control and, at the same time, has a close terminological connection to information theory. The central concept of the anisotropy-based theory is the anisotropy of a random vector, which originally corresponded to the relative entropy of the normalized distribution function of a random vector on the unit sphere with respect to the uniform distribution. Thus, for the uniform distribution, the anisotropy value is zero, and the denser the distribution becomes along certain axes, the higher the anisotropy value will be, up to infinity. This concept was later modified [9]. Recently, the anisotropy of a random vector is understood as the Kullback–Leibler divergence between two probability density functions (PDFs), one belonging to a fixed random vector and the other to a Gaussian family of random vectors with zero mean and a scalar covariance matrix. With this definition, one can give a simple geometric interpretation of anisotropy: it measures the difference (distance) between a random vector and a set of centered Gaussian vectors with a scalar covariance matrix. The performance criterion in the anisotropy-based theory is related to the anisotropic norm, a stochastic analog of the $\mathcal{H}_\infty$ norm of a dynamic system.

Within the theoretical framework based on the anisotropy of random vectors, many analysis and design problems were solved for both time-varying [9] and time-invariant systems [10]. However, until recently, only linear and deterministic plants were considered. The first attempt to study stochastic plants from the anisotropic point of view was undertaken in [11]. The analysis therein replaced the approach proposed in [12, 13], which involved the majorants of norms for systems with multiplicative noise.

Systems with multiplicative noise are an important example of stochastic systems. They describe mechanical, hybrid, and biological systems, financial models, and many other objects and processes [14, 15]. An anisotropy-based robust performance analysis of time-varying systems was presented in [16], and time-invariant systems were considered in [17]. The problem of constructing an output estimator for a time-varying system was successfully solved in [18], and the adjacency matrix of a sensor network with dropouts was tuned in [19]. In view of the results obtained within the framework of anisotropic analysis for time-invariant systems [17], the control design problem can also be posed and solved. Below, we consider a dynamic controller and formulate a convex optimization problem to calculate its gain matrices in the state space. The matrix inequalities are linearized using the procedure described in [20]. The developed controller can be applied to the automatic control of any moving objects. Section 1 provides a summary of the anisotropy-based theory. In Section 2, we describe the system and problem statement; in Section 3, the solution of this problem. The results of numerical simulation are demonstrated in Section 4.

1. THEORETICAL BACKGROUND

This section recalls the basic concepts of the anisotropy-based theory for time-invariant systems. More detailed information can be found in [10, 21–23].

The anisotropy of a random vector W from the space $\mathbb{R}^m$ with a PDF f is given by

$$\mathbf{A}(W) = \min_{\lambda>0} \mathbf{D}(f \,||\, p_\lambda),$$

where

$$\mathbf{D}(f \,||\, p_\lambda) = \mathbf{E}\left[\ln \frac{f}{p_\lambda}\right]$$

represents the relative entropy (or the Kullback–Leibler information divergence) with respect to a reference PDF of the form

$$p_\lambda(x) = (2\pi\lambda)^{-m/2} \exp\left(-\frac{||x||^2}{2\lambda}\right),$$

which is chosen Gaussian with zero mean and a scalar covariance matrix λI_m, where I_m denotes an identity matrix of order m. From this point onwards, the notation $\mathbf{E}[\cdot]$ corresponds to the expectation operator and $\|\cdot\|$ is the Euclidean vector norm. The anisotropy of a random vector is not a norm due to violating the axioms of norms. At the same time, anisotropy is a measure of the closeness of a random vector to vectors obeying the standard Gaussian distribution.

Consider the extended vector composed of elements of a random vector sequence $\{w_k\}$:

$$W_{s:t} = (w_s^{\mathrm{T}}, w_{s+1}^{\mathrm{T}}, \ldots, w_t^{\mathrm{T}})^{\mathrm{T}}, \; s \le t.$$

For the extended vector $W_{s:t}$, the limit

$$\overline{\mathbf{A}}(W) = \lim_{N\to\infty} \frac{\mathbf{A}(W_{0:N-1})}{N}$$

is called the mean anisotropy of the sequence $\{w_k\}$ [10]. The anisotropy-based theory introduces a particular performance criterion, known as the anisotropic norm. First, we consider the mean-square gain

$$Q(Z,W)=\sqrt{\frac{\mathbf{E}(|Z|^2)}{\mathbf{E}(|W|^2)}},$$

where Z and W are the output and input of a linear system with a transfer matrix $F\in\mathbb{C}^{p\times m}$, respectively. The expression

$$\sup_{W\in\mathcal{L}_2^m} Q(Z,W)=\sqrt{\max_{1\le k\le m}\lambda_k(F^{\mathrm{T}}F)}=\|F\|_\infty$$

is the definition of the $\mathcal{H}_\infty$ norm, where $\mathcal{L}_2^m$ corresponds to square summable signals and λ_k is the k th eigenvalue. If the input signal of the system F has mean anisotropy with an upper bound a, the anisotropic norm can be defined as

$$\sup_{\bar{\mathbf{A}}(W)\le a} Q(Z,W)=|||F|||_a .$$

For a nonspherical system (i.e., the one whose scaled $\mathcal{H}_2$ norm is smaller than the $\mathcal{H}_\infty$ norm), the anisotropic norm has a remarkable property: either the scaled $\mathcal{H}_2$ or $\mathcal{H}_\infty$ norm can be obtained as the limiting cases:

$$\frac{1}{\sqrt{m}}\|F\|_2\le|||F|||_a\le\|F\|_\infty .$$

Note that the left bound is reached under zero mean anisotropy; the right bound, as mean anisotropy tends to infinity (when the sequence loses randomness).

2. PROBLEM STATEMENT

In this paper, we describe dynamic objects (both the controlled plant and the controller) in the time domain using the state-space representation. Consider a linear discrete time-invariant system with multiplicative noise of the form

$$\begin{aligned}x(k+1)&=Ax(k)+B_1w(k)+B_2u(k),\\ z(k)&=C_1x(k)+D_{12}u(k),\\ y(k)&=C_2x(k)+D_{21}w(k)\end{aligned}\tag{1}$$

with the zero initial condition $x(0)=0$. Here, $x(k)\in\mathbb{R}^{n_x}$ is the state vector; $\{w(k)\}_{k\ge 0}$, $w(k)\in\mathbb{R}^{m_w}$, is a colored sequence with a known upper bound a on mean anisotropy; $u(k)\in\mathbb{R}^{m_u}$ is the control input; $z(k)\in\mathbb{R}^{p_z}$ is the controlled output; $y(k)\in\mathbb{R}^{p_y}$ is the observed output. All matrices in system (1) have compatible dimensions. By assumption, system (1) is controllable. Unlike the system considered in [13], where multiplicative noises were included in the control coefficient, the system matrix A in the current problem statement is represented as

$$A=A_0+\sum_{i=1}^{n}\mu_i(k)A_i,$$

where the matrices A_i are known and have appropriate dimensions. The random variables $\mu_i(k)$, $i=1,\ldots,n$, obey the standard Gaussian distribution with zero mean and unit covariance, are mutually independent of each other and of the external disturbance vectors $w(t)$ for all time instants k and t.

The problem is to find matrices A_c, B_c, C_c, and D_c of the state-space realization of a full-order dynamic controller

$$\begin{aligned}\xi(k+1)&=A_c\xi(k)+B_cy(k),\\ u(k)&=C_c\xi(k)+D_cy(k),\end{aligned}\tag{2}$$

where $\xi_i(k)\in\mathbb{R}^{n_x}$ stands for the controller's internal state, under which the anisotropic norm of the closed-loop control system would not exceed a number $\gamma>0$.

3. THE MAIN RESULT

Before proceeding to the main result, we consider a linear discrete time-invariant system F with multiplicative noise of the following form:

$$\begin{aligned}x(k+1)&=(A_0+\sum_{i=1}^{n}\mu_i(k)A_i)x(k)+Bw(k),\\ z(k)&=Cx(k)+Dw(k),\end{aligned}\tag{3}$$

where $x(k)\in\mathbb{R}^{n_x}$ denotes the state vector, $w(k)\in\mathbb{R}^{m_w}$ is a disturbance, and $z(k)\in\mathbb{R}^{p_z}$ means the system output. Real matrices have compatible dimensions. By assumption, the input sequence is random with a given upper bound a on its mean anisotropy. A boundedness condition of the anisotropy norm was derived in [17] in terms of a special system of equations and inequalities. The analysis of systems with multiplicative noise was reduced to a convex optimization problem in [24].

For system (3), the anisotropic norm will be bounded under the following conditions.

Theorem 1 [24]. *Consider system* (3) *and let the mean anisotropy of the external disturbance be bounded above by a given number* $a \geq 0$. *The anisotropic norm of the system will not exceed a given threshold* γ,

$$|||F|||_a \leq \gamma,$$

if the system of inequalities

$$\begin{bmatrix} \sum_{i=0}^{n} A_i^{\mathrm{T}} R A_i - R + C^{\mathrm{T}} C & * \\ B^{\mathrm{T}} R A_0 + D^{\mathrm{T}} C & -\eta I_{m_w} + D^{\mathrm{T}} D + B^{\mathrm{T}} R B \end{bmatrix} \prec 0, \quad (4)$$

$$\begin{bmatrix} \eta I_{m_w} - S - D^{\mathrm{T}} D & * \\ RB & R \end{bmatrix} \succ 0, \quad (5)$$

$$\ln \det S \geq 2a + m_w \ln(\eta - \gamma^2)$$

has solutions $R \succ 0$, $S \succ 0$, *and* $\eta > \gamma^2$.

In inequalities (4) and (5) and further, the expression $[\cdot] \prec 0$ should be understood in the sense of the negative definiteness of an appropriate matrix, and the asterisk $(*)$ denotes a block symmetric with respect to the principal diagonal.

The original system (1) closed with the controller (2) takes the form

$$\zeta(k+1) = \left(A_0 + \sum_{i=0}^{n} \mu_i(k) A_i\right)\zeta(k) + Bw(k),$$
$$z(k) = C\zeta(k) + Dw(k), \quad (6)$$

where $\zeta(k) \in \mathbb{R}^{2n_x}$ is the augmented state vector

$$\zeta(k) = \begin{bmatrix} x(k) \\ \xi(k) \end{bmatrix},$$

and the matrices $A_i \in \mathbb{R}^{2n_x \times 2n_x}$, $i = 0, \ldots, n$, $B \in \mathbb{R}^{2n_x \times m_w}$, $C \in \mathbb{R}^{p_z \times 2n_x}$, and $D \in \mathbb{R}^{p_z \times m_w}$ have the following block structure:

$$A_0 = \begin{bmatrix} A_0 + B_2 D_c C_2 & B_2 C_c \\ B_c C_2 & A_c \end{bmatrix},$$

$$A_i = \begin{bmatrix} A_i & 0 \\ 0 & 0 \end{bmatrix}, \; B = \begin{bmatrix} B_2 D_c D_{21} \\ B_c D_{21} \end{bmatrix}, \quad (7)$$

$$C = \begin{bmatrix} C_1 + D_{12} D_c C_2 & D_{12} C_c \end{bmatrix}, \; D = D_{12} D_c D_{21}.$$

For the closed-loop control system (6), based on the results of [20, 25], we now formulate the following statement regarding the boundedness of the anisotropic norm as a convex optimization problem.

Theorem 2. *Consider system* (3) *with multiplicative noise and let the mean anisotropy of the external disturbance be bounded above by a given number* $a \geq 0$. *For a fixed number* $\gamma > 0$, *the dynamic controller* (2) *ensures the boundedness of the anisotropic norm,* $|||F|||_a < \gamma$, *if the system of inequalities*

$$\begin{bmatrix}
-\Pi_{11} & * & * & * & * & * & * & \ldots & * & * & * \\
-I_{n_x} & -\Phi_{11} & * & * & * & * & * & \ldots & * & * & * \\
0 & 0 & -\eta I_{m_w} & * & * & * & * & \ldots & * & * & * \\
A_0\Pi_{11} + B_2\mathbf{C} & A_0 + B_2\mathbf{D}C_2 & B_1 + B_2\mathbf{D}D_{21} & -\Pi_{11} & * & * & * & \ldots & * & * & * \\
\mathbf{A} & \Phi_{11}A_0 + \mathbf{B}C_2 & \Phi_{11}B_1 + \mathbf{B}D_{21} & -I_{n_x} & -\Phi_{11} & * & * & \ldots & * & * & * \\
A_1\Pi_{11} & A_1 & 0 & 0 & 0 & -\Pi_{11} & * & \ldots & * & * & * \\
0 & 0 & 0 & 0 & 0 & -\Pi_{12}^{\mathrm{T}} & -\Pi_{22} & * & * & * & * \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
A_n\Pi_{11} & A_n & 0 & 0 & 0 & 0 & 0 & \ldots & -\Pi_{11} & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & -\Pi_{12}^{\mathrm{T}} & -\Pi_{22} & * \\
C_1\Pi_{11} + D_{12}\mathbf{C} & C_1 + D_{12}\mathbf{D}C_2 & D_{12}\mathbf{D}D_{21} & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & -I_{p_z}
\end{bmatrix} \prec 0, \quad (8)$$

$$\begin{bmatrix} \Psi-\eta I_{m_w} & * & * & * \\ B_1+B_2\mathbf{D}D_{21} & -\Pi_{11} & * & * \\ \Phi_{11}B_1+\mathbf{B}D_{21} & -I_{n_x} & -\Phi_{11} & * \\ D_{12}\mathbf{D}D_{21} & 0 & 0 & -I_{p_z} \end{bmatrix} \prec 0, \qquad (9)$$

$$\eta>\gamma^2,\ \Pi_{11}\succ 0,\ \Phi_{11}\succ 0,\ \begin{bmatrix} \Pi_{11} & I_{n_x} \\ I_{n_x} & \Phi_{11} \end{bmatrix}\succ 0, \qquad (10)$$

$$\ln\det\Psi \ge 2a+m_w\ln(\eta-\gamma^2), \qquad (11)$$

is solvable with respect to the variables $\eta>0$, $\Psi=\Psi^{\mathrm{T}}$, $\Phi_{11}=\Phi_{11}^{\mathrm{T}}$, $\Pi_{11}=\Pi_{11}^{\mathrm{T}}$, $\Pi_{22}=\Pi_{22}^{\mathrm{T}}$, Π_{12}, $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, *and* $\mathbf{D}$. *Moreover, the controller's gain matrices are related to the solution of inequalities* (8)-(11) *as follows:*

$$\begin{aligned} A_c &= \Phi_{12}^{-1}(\mathbf{A}+\Phi_{11}B_2\mathbf{D}C_2\Pi_{11}-\mathbf{B}C_2\Pi_{11} \\ &\quad -\Phi_{11}B_2\mathbf{C}-\Phi_{11}A_0\Pi_{11})\Pi_{12}^{-\mathrm{T}}, \\ B_c &= \Phi_{12}^{-1}(\mathbf{B}-\Phi_{11}B_2D_c), \\ C_c &= (\mathbf{C}-D_cC_2\Phi_{11})\Pi_{12}^{-\mathrm{T}}, \\ D_c &= \mathbf{D} \end{aligned} \qquad (12)$$

where

$$\Phi_{12}=(I_{n_x}-\Phi_{11}\Pi_{11})\Pi_{12}^{-\mathrm{T}},$$

the matrices Φ_{12} *and* Π_{12} *are nonsingular, and* $\Pi_{12}^{-\mathrm{T}}$=$(\Pi_{12}^{\mathrm{T}})^{-1}$.

The proof of this theorem is given in the Appendix.

Remark 1. An important requirement is the coincident dimensions of the state vectors of the plant $x(k)$ and the dynamic controller $\xi(k)$. In this case, the gain matrices of the controller (2) can be found unambiguously. According to [25], under the full column rank of the matrices Φ_{12} and Π_{12}, the controller's gain matrices A_c, B_c, C_c, and D_c exist but are not unique. ♦

Based on Theorem 2, it is easy to formulate another statement, which has already become classical for anisotropic problems in convex optimization terms.

Theorem 3. *The anisotropic norm of system* (6) *is bounded above by the minimum threshold* γ *if the convex optimization problem*

$$\gamma^2 \underset{(8)-(11)}{\to} \min$$

is solvable with respect to the variables γ^2, η, $\Psi=\Psi^{\mathrm{T}}\succ 0$, $\Phi_{11}=\Phi_{11}^{\mathrm{T}}\succ 0$, $\Pi_{11}=\Pi_{11}^{\mathrm{T}}\succ 0$, $\Pi_{22}=\Pi_{22}^{\mathrm{T}}\succ 0$, Π_{12}, $\mathbf{A},\mathbf{B},\mathbf{C}$, *and* $\mathbf{D}$. *The gain matrices of the dynamic controller* (2) *are given by formulas* (12).

Remark 2. Due to multiplicative noise in system (1), the convex optimization problem contains the matrix Π. This allows one not to check the existence of two matrices satisfying the equation $\Phi_{12}\Pi_{12}^{\mathrm{T}}=I_{n_x}-\Phi_{11}\Pi_{11}$, a necessary condition in the case of systems without multiplicative noise [23]. ♦

4. NUMERICAL SIMULATION

In this section, we analyze the numerical experiment carried out on the aircraft takeoff-landing model [26]. The control is implemented by changing the angle of the aircraft's rear nozzle, thrust through the rear nozzle, and thrust through the front nozzle, which has a fixed position; the aircraft's pitch and position of the center of mass, as well as the rates of their change, are chosen as the state variables. The linear discrete-time model in the state space is described by the matrices

$$A_0=\begin{bmatrix} 1.0000 & 0.0004 & -0.0536 & -0.0976 \\ -0.0007 & 0.9966 & 0.4938 & -0.0100 \\ 0.0000 & 0.0001 & 0.9980 & 0.0000 \\ 0.0000 & 0.0000 & 0.0100 & 1.0000 \end{bmatrix},$$

$$B_1=\begin{bmatrix} 0.0003 & 0.0000 & -0.0001 \\ -0.0002 & 0.0005 & 0.0008 \\ -0.0006 & 0.0000 & 0.0034 \\ 0.0000 & 0.0000 & 0.0001 \end{bmatrix},$$

$$B_2=\begin{bmatrix} 0.0021 & 0.0000 & -0.0003 \\ 0.0000 & 0.0020 & 0.0004 \\ 0.0000 & 0.0000 & 0.0012 \\ 0.0000 & 0.0000 & 0.0000 \end{bmatrix},$$

$$C_1=\begin{bmatrix}0.7071 & 0 & 0 & 0\end{bmatrix},\ D_{12}=[0\ \ 0\ \ 0.7071],$$

$$C_2=\begin{bmatrix} 0 & 0 & 57.2958 & 0 \\ 0 & 0 & 0 & 57.2958 \\ 0.1045 & -0.9945 & 0.1375 & 51.5791 \\ -0.0002 & 0.0045 & 0 & 0 \end{bmatrix},$$

$$D_{21}=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0.0212 & 0 & 0 \end{bmatrix}.$$

Multiplicative noise as a single term enters the dynamics equation with the matrix coefficient $A_1=A_0\cdot 10^{-2}$. Mean anisotropy is chosen to be 5. The following gain matrices of the dynamic anisotropy-based controller were obtained during the calculations:

$$A_c = \begin{bmatrix} 1.9773 & -2.3361 & -7.8981 & 49.2172 \\ -0.0232 & 1.7152 & -1.1628 & 4.7785 \\ -0.0368 & 0.6143 & -0.1482 & 0.1708 \\ -0.1262 & 0.4600 & 0.2936 & -2.2690 \end{bmatrix},$$

$$B_c = \begin{bmatrix} 0.0078 & 0.5486 & -1.0382 & -0.0013 \\ 0.0024 & -0.8364 & 0.08879 & 0.0003 \\ 0.0115 & -0.2878 & 0.3202 & -0.0187 \\ 0.0017 & -0.1795 & 0.2283 & -0.0028 \end{bmatrix},$$

$$C_c \cdot 10^{-4} = \begin{bmatrix} -0.0640 & -1.9133 & 1.2458 & -5.1640 \\ -0.0311 & -0.0188 & 0.1100 & -0.6393 \\ 0.0000 & -0.0068 & 0.0039 & -0.0149 \end{bmatrix},$$

$$D_c \cdot 10^{-3} = \begin{bmatrix} -0.0250 & 9.3135 & -9.8997 & -0.0111 \\ -0.0067 & -0.2866 & 0.3740 & 0.0037 \\ -0.0001 & 0.0329 & -0.0353 & 0.000 \end{bmatrix}.$$

The convex optimization problem was numerically solved using standard MATLAB tools with additional packages for semidefinite programming problems [27, 28].

Figure 1 shows the Bode diagram for two anisotropy values, 1 (Fig. 1a) and 5 (Fig. 1b). Note that for any mean anisotropy exceeding 10, the simulation gives approximately the same results characteristic of $\mathcal{H}_\infty$ control.

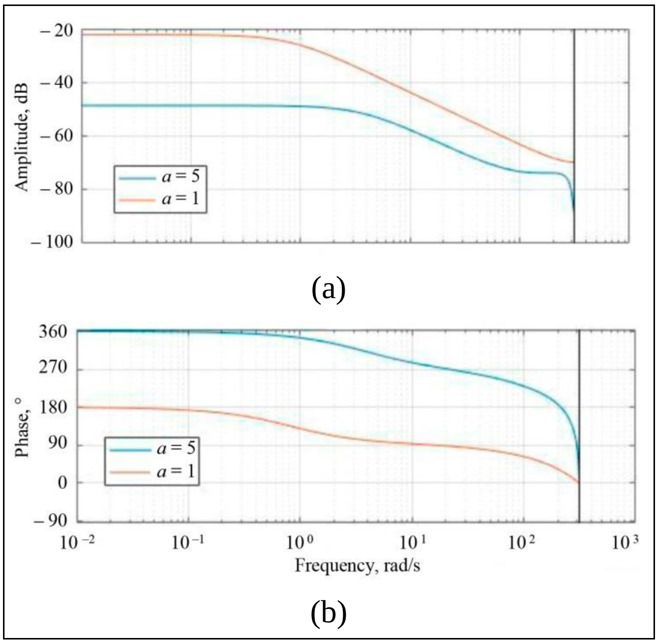

Fig. 1. The Bode diagrams of closed-loop control systems.

The table below presents the upper bounds of the anisotropic norm of the closed-loop control system calculated under different values of mean anisotropy.

The bounds of the anisotropic norm

a	0	1	5	10	15
γ	0.0012	0.2197	0.3087	0.3142	0.3142

Also, it seems interesting to compare the controlled outputs under anisotropy-based control and standard $\mathcal{H}_2$ control. Let mean anisotropy be equal to 5. We calculate the Euclidean norm for the controlled output, plotted in Fig. 2, where the application of anisotropy-based control is indicated by *AB*. As it turns out, for anisotropy-based control, this norm takes a value of 0.0158; for $\mathcal{H}_2$ approach, a value of 0.0856. In other words, with anisotropy-based control, the quadratic performance criterion can be improved by 72%.

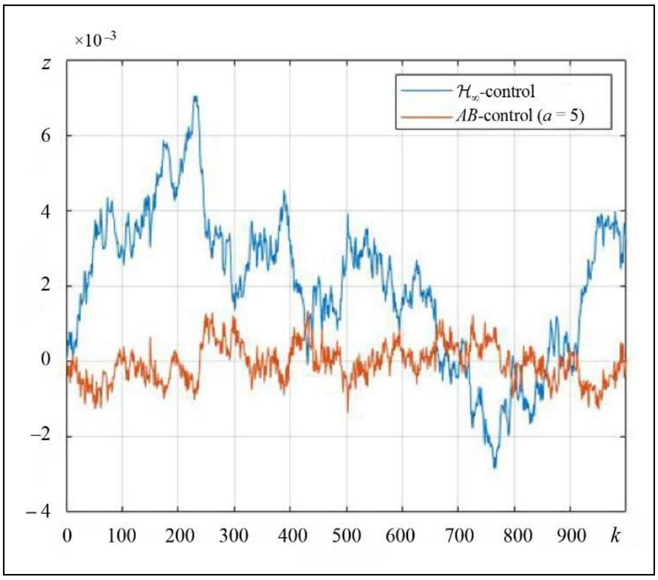

Fig. 2. The controlled outputs of the system with different control.

CONCLUSIONS

This paper has considered an algorithm for calculating the gain matrices of a dynamic anisotropy-based controller in the state space. The bounded real lemma for time-invariant systems has been used as the main tool. By assumption, the dynamic controller has full dimension, which ensures its uniqueness. Sufficient conditions for the existence of a dynamic anisotropy-based controller for the closed-loop system have been established in terms of a special system of nonlinear matrix inequalities. A linearizing inversible change of variables has been applied to reduce the boundedness conditions of the anisotropic norm of the closed-loop system to the solvability condition of the special system of inequalities. A threshold for the upper bound of the anisotropic norm of the closed-loop system has been obtained by solving a convex optimization problem.

Acknowledgments. *This work was supported by the Russian Science Foundation, project no. 24-21-20055.*

APPENDIX

P r o o f of Theorem 2. According to Theorem 1, system (6) closed by the dynamic controller (2) has a bounded anisotropic norm if the inequalities

$$\begin{bmatrix} \sum_{i=0}^{n} A_i^{\mathrm T}\Phi A_i - \Phi + C^{\mathrm T}C & * \\ B^{\mathrm T}\Phi A_0 + D^{\mathrm T}C & -\eta I_{m_w} + D^{\mathrm T}D + B^{\mathrm T}\Phi B \end{bmatrix} \prec 0, \quad (A1)$$

$$\begin{bmatrix} \eta I_{m_w} - \Psi - D^{\mathrm T}D & * \\ \Phi B & \Phi \end{bmatrix} \succ 0, \quad (A2)$$

$$\ln\det \Psi \geq 2a + m_w \ln(\eta - \gamma^2) \quad (A3)$$

have solutions $\eta > 0$, $\Psi = \Psi^{\mathrm T}$, and $\Phi = \Phi^{\mathrm T}$. The system of inequalities (A1)–(A3) is nonlinear with respect to the system matrices that depend on the controller's matrices (7). To correct this, we apply the Schur complement lemma [29] to inequality (A1):

$$\begin{bmatrix} -\Phi & * & * & * & \ldots & * & * \\ 0 & -\eta I_{m_w} & * & * & \ldots & * & * \\ A_0 & B & -\Phi^{-1} & * & \ldots & * & * \\ A_1 & 0 & 0 & -\Phi^{-1} & \ldots & * & * \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ A_n & 0 & 0 & 0 & \ldots & -\Phi^{-1} & * \\ C & D & 0 & 0 & \ldots & 0 & -I_{p_z} \end{bmatrix} \prec 0. \quad (A4)$$

Let $\Pi = \Phi^{-1}$ be the new matrix variable. Obviously, $\Phi\Pi = I_{2n_x}$, and the matrices Φ and Π have a block structure:

$$\Phi = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{12}^{\mathrm T} & \Phi_{22} \end{bmatrix}, \quad \Pi = \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^{\mathrm T} & \Pi_{22} \end{bmatrix}.$$

We introduce the matrices

$$\Phi_1 = \begin{bmatrix} I_{n_x} & \Phi_{11} \\ 0 & \Phi_{12}^{\mathrm T} \end{bmatrix}, \quad \Pi_1 = \begin{bmatrix} \Pi_{11} & I_{n_x} \\ \Pi_{12}^{\mathrm T} & 0 \end{bmatrix}.$$

It is easy to show that

$$\Pi_1^{\mathrm T}\Phi\Pi_1 = \Phi_1^{\mathrm T}\Pi_1 = \Phi_1^{\mathrm T}\Pi\Phi_1 = \Pi_1^{\mathrm T}\Phi_1 = \begin{bmatrix} \Pi_{11} & I_{n_x} \\ I_{n_x} & \Phi_{11} \end{bmatrix}. \quad (A5)$$

Applying a congruent transformation with the matrix

$$\text{block diag}\,(\Pi_1^{\mathrm T},\ I_{m_w},\ \Phi_1^{\mathrm T},\ I_{n_x}, \ldots,\ I_{n_x},\ I_{p_z})$$

to inequality (A4) yields the new inequality

$$\begin{bmatrix} -\Pi_1^{\mathrm T}\Phi\Pi_1 & * & * & * & \ldots & * & * \\ 0 & -\eta I_{m_w} & * & * & \ldots & * & * \\ \Phi_1^{\mathrm T}A_0\Pi_1 & \Phi_1^{\mathrm T}B & -\Phi_1^{\mathrm T}\Pi\Phi_1 & * & \ldots & * & * \\ A_1\Pi_1 & 0 & 0 & -\Pi & \ldots & * & * \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ A_n\Pi_1 & 0 & 0 & 0 & \ldots & -\Pi & * \\ C\Pi_1 & D & 0 & 0 & \ldots & 0 & -I_{p_z} \end{bmatrix} \prec 0. \quad (A6)$$

Inequality (A6) is still nonlinear in some matrix variables. The blocks $-\Pi_1^{\mathrm T}\Phi\Pi_1$ and $-\Phi_1^{\mathrm T}\Pi\Phi_1$ can be written according to the notation (A5). Consider the third block in the first column; it also has a block structure:

$$\Phi_1^{\mathrm T}A_0\Pi_1 = \begin{bmatrix} A_0\Pi_{11} + B_2\mathbf{C} & A + B_2\mathbf{D}C_2 \\ \mathbf{A} & \Phi_{11}A_0 + \mathbf{B}C_2 \end{bmatrix},$$

where

$$\mathbf{A} = \Phi_{12}A_c\Pi_{12}^{\mathrm T} + \Phi_{12}B_cC_2\Pi_{11}$$

$$+ \Phi_{11}B_2C_c\Pi_{12}^{\mathrm T} + \Phi_{11}(A_0 + B_2D_cC_2)\Pi_{11},$$

$$\mathbf{B} = \Phi_{12}B_c + \Phi_{11}B_2D_c,$$

$$\mathbf{C} = C_c\Pi_{12}^{\mathrm T} + D_cC_2\Phi_{11},$$

$$\mathbf{D} = D_c$$

is a linearizing change of variables similar to the one proposed in [20, 30]. The blocks $\Phi_1^{\mathrm T}B$ and $C\Pi_1$ can be represented as follows:

$$\Phi_1^{\mathrm T}B = \begin{bmatrix} B_1 + B_2\mathbf{D}D_{21} \\ \Phi_{11}B_1 + \mathbf{B}D_{21} \end{bmatrix},$$

$$C\Pi_1 = [C_1\Pi_{11} + D_{12}\mathbf{D} \quad D_{12}\mathbf{D}D_{21}].$$

Thus, we have arrived at inequality (8). Next, the Schur complement lemma can be applied to inequality (A2) to obtain

$$\begin{bmatrix} \Psi - \eta I_{m_w} & * & * \\ \Phi B & -\Phi & * \\ D & 0 & -I_{p_z} \end{bmatrix} \prec 0.$$

Now we perform a congruent transformation of the last inequality using the matrix

$$\text{block diag}\,(I_{m_w},\ \Pi_1^{\mathrm T},\ I_{p_z}).$$

Clearly, this transformation brings to inequality (9). Note that the special inequality (11) remains unchanged.

The inverse change of variables (12) is uniquely defined under the nonsingularity of the matrices Φ_{12} and Π_{12}.

REFERENCES

1. Astrom, K.J., *Introduction to Stochastic Control Theory*, Academic Press, 1970.
2. Zames, G., Feedback and Optimal Sensitivity: Model Reference Transformations, Multiplicative Seminorms, and Approximate Inverses, *IEEE Transactions on Automatic Control*, 1981, vol. 26, no. 2, pp. 301–320.
3. Haddad, W.M., Bernstein, D.S., and Mustafa, D., Mixed-Norm $H2/H_\infty$ Regulation and Estimation: the Discrete-Time Case, *Systems & Control Letters*, 1991, vol. 16, no. 4, pp. 235–247.
4. Zhou, K., Glover, K., Bodenheimer, B., and Doyle, J., Mixed H2 and H∞ Performance Objectives I: Robust Performance Analysis, *IEEE Transactions on Automatic Control*, 1994, vol. 38, pp. 1564–1574.
5. Khargonekar, P.P., Rotea, M.A., and Baeyens, E., Mixed H2/H∞ Filtering, *International Journal of Robust and Nonlinear Control*, 1996, vol. 6, pp. 313–330.
6. Scherer, C.W., *Mixed $H2/H_\infty$ Control*, Springer-Verlag, 1995.
7. Semyonov, A.V., Vladimirov, I.G., and Kurdyukov, A.P., Stochastic Approach to H_∞–optimization, *Proceedings of the 33rd Conference on Decision and Control*, 1994, pp. 2249–2250.
8. Vladimirov, I.G., Kurdjukov, A.P., and Semyonov, A.V., Anisotropy of Signals and the Entropy of Linear Stationary Systems, *Doklady Math.*, 1995, vol. 51, pp. 388–390.
9. Vladimirov, I.G., Diamond, P., and Kloeden, P., Anisotropy–based Robust Performance Analysis of Finite Horizon Linear Discrete Time Varying Systems, *Automation and Remote Control*, 2006, vol. 67, no. 7, pp. 1265–1282.
10. Vladimirov, I.G., Kurdyukov, A.P., and Semyonov, A.V., On Computing the Anisotropic Norm of Linear Discrete-time-invariant Systems, *Proceedings of 13th IFAC World Congr.*, San Francisco, 1996, pp. 179–184.
11. Kustov, A.Yu., State-Space Formulas for Anisotropic Norm of Linear Discrete Time Varying Stochastic System, *Proceedings of 15th Int. Conf. on Electr. Eng., Comp. Science and Autom. Control*, Porto, 2018, pp. 1–6.
12. Yurchenkov, A.V., Kustov, A.Yu, Kurdyukov, A.P. Anisotropy-based Bounded Real Lemma for Discrete-time Systems with Multiplicative Noise, *Doklady Math.*, vol. 93, no. 2, pp. 238–240.
13. Yurchenkov, A.V., Anisotropy-Based Controller Design for Linear Discrete-Time Systems with Multiplicative Noise, *Journal of Computer and Systems Sciences Int.*, 2018, vol. 57, no. 6, pp. 864–873.
14. Won, M. and Ranade, G., Control of Systems with Multiplicative Observation Noise, *Proceedings of the* 62nd IEEE Conference on Decision and Control, Singapore, 2023, pp. 3950–3955.
15. Lu, C., Lu, X., Wang, H., et al., Control for Multiplicative Noise Systems with Packet Dropouts and Multiple Delays, *Proceedings of the* 40th Chinese Control Conference, Shanghai, 2021, pp. 1544–1549.
16. Belov, I.R., Yurchenkov, A.V., and Kustov, A.Yu., Anisotropy-Based Bounded Real Lemma for Multiplicative Noise Systems: the Finite Horizon Case, *Proc. of the 27th Med. Conf. on Control and Automation*, Akko, 2019, pp. 148–152.
17. Kustov, A.Yu., Timin, V.N., and Yurchenkov, A.V., Boundedness Condition for Anisotropic Norm of Linear Discrete Time-invariant Systems with Multiplicative Noise, *Journal of Physics: Conference Series*, 2021, vol. 1864, art. no. 012068.
18. Yurchenkov, A.V., Kustov, A.Yu., and Timin, V.N., The Sensor Network Estimation with Dropouts: Anisotropy-based Approach, *Automatica*, 2023, vol. 151, art. no. 110924.
19. Yurchenkov, A.V., Default Sensor Network Setup based on the Anisotropic Criterion, *Herald of the Bauman Moscow State Tech. Univ. Ser. Nat. Sci.*, 2023, vol. 106, no. 1, pp. 45–63. (In Russian.)
20. Gahinet, P., Explicit Controller Formulas for LMI-based H∞ Synthesis, *Automatica*, 1996, vol. 32, pp. 1007–1014.
21. Diamond, P., Vladimirov, I.G., Kurdyukov, A.P., and Semyonov, A.V., Anisotropy-based Performance Analysis of Linear Discrete Time Invariant Control Systems, *International Journal of Control*, 2001, vol. 74, no. 1, pp. 28–42.
22. Diamond, P., Kloeden, P., and Vladimirov, I.G., Mean Anisotropy of Homogeneous Gaussian Random Fields and Anisotropic Norms of Linear Translation-Invariant Operators on Multidimensional Integer Lattices, *Journal of Applied Mathematics and Stochastic Analysis*, 2003, vol. 16, no.3, pp. 209–231.
23. Vladimirov, I.G., Kurdyukov, A.P., and Semyonov, A.V., The Stochastic Problem of H∞-Optimization, *Doklady Math.*, 1995, vol. 52, pp. 155–157.
24. Yurchenkov, A.V., An Anisotropy-Based Boundedness Criterion for Time-Invariant Systems with Multiplicative Noises, *Control Sciences*, 2022, no. 5, pp. 13–20.
25. Scherer, C.W., Gahinet, P., and Chilali, M., Multiobjective Output-feedback Control via LMI Optimization, *IEEE Transactions on Automatic Control*, 1997, vol. 42, pp. 896–911.
26. Toffner-Clausen, S., *System Identification and Robust Control: A Case Study Approach*, Berlin: Springer-Verlag, 1996.
27. Lofberg, J., YALMIP: A Toolbox for Modeling and Optimization in MATLAB, *Proceedings of the 2004 IEEE International Conference on Robotics and Automation*, New Orleans, 2004, pp. 284–289.
28. Sturm, J.F., Using SeDuMi 1.02, a MATLAB Toolbox for Optimization over Symmetric Cones, *Optimization Methods and Software*, 1999, vol. 11, pp. 625–653.
29. Boyd, S. and Vandenberghe, L., *Convex Optimization*, Cambridge: Cambridge University Press, 2004.
30. Tchaikovsky, M.M., Synthesis of Anisotropic Controllers via Convex Optimization and Semidefinite Programming, *Large-Scale Systems Control*, 2013, no. 42, pp. 100–152. (In Russian.)

This paper was recommended for publication by M.V. Khlebnikov, a member of the Editorial Board.

Received November 3, 2024,
and revised January 31, 2025.
Accepted February 6, 2025.

Author information

Yurchenkov, Alexander Viktorovich. Cand. Sci. (Phys.–Math.), Trapeznikov Institute of Control Sciences, Russian Academy of Sciences, Moscow, Russia
✉ alexander.yurchenkov@yandex.ru
ORCID iD: https://orcid.org/0000-0002-2088-5577

Cite this paper

Yurchenkov, A.V., Dynamic Anisotropy-Based Controller Design for Time-Invariant Systems with Multiplicative Noise. *Control Sciences* **1,** 25–33 (2025).

Original Russian Text © Yurchenkov, A.V., 2025, published in *Problemy Upravleniya*, 2025, no. 1, pp. 30–39.

This paper is available under the Creative Commons Attribution 4.0 Worldwide License.

Translated into English by *Alexander Yu. Mazurov*,
Cand. Sci. (Phys.–Math.),
Trapeznikov Institute of Control Sciences,
Russian Academy of Sciences, Moscow, Russia
✉ alexander.mazurov08@gmail.com