CONSTRUCTIVE D-PARTITION FOR TWO PARAMETERS ENTERING A POLYNOMIAL LINEARLY. PART I: Description of the Boundaries of D-Partition Regions

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Abstract. In the stability analysis of linear systems depending on several parameters, the D-partition method is often used, also known as the D-decomposition method in the literature. This method describes the stability region of a characteristic polynomial via the equation of its boundary. A constructive D-partition method proposed below identifies individual parts of curves and straight lines on the parameter plane that form the boundaries of the D-partition regions and, in particular, the stability region. A characteristic polynomial linearly dependent on two parameters and a stability region with a piecewise rational parametric boundary are considered. In this case, the boundary of each D-partition region is a finite set of arcs of rational curves and segments, rays, or straight lines that can be found explicitly. The rational curve arcs are parameterized on intervals whose limits are found by calculating the real roots of auxiliary polynomials. A D-partition, bounded (localized) on a compact set, consists of a finite number of segments and arcs of rational curves parameterized on the segments.

Keywords: constructive D-partition, D-decomposition, root localization, root clustering, rational curves, localized D-partition.

INTRODUCTION

Controller design for linear control systems is closely related to the analysis of the characteristic polynomial. This applies both for systems written in the input–output form using transfer functions and for systems with the state-space representation.[1] In this case, the stability of a system is determined by the roots of its characteristic polynomial. Consider a characteristic polynomial of degree n that depends on two real parameters:

$$G(s,\, k_1,\, k_2). \tag{1}$$

Here, k_1 and k_2 are parameters representing, e.g., controller coefficients (gains) or some uncertainty. All roots of the polynomial (1) of asymptotically stable continuous-time systems lie in the left complex half-plane:

$$\mathbf{D} = \{s \in \mathbb{C}\colon \operatorname{Re} s < 0\}. \tag{2}$$

(Such polynomials are called Hurwitz polynomials.)

For stable discrete-time systems, all roots of the characteristic polynomial lie inside the unit circle ($\mathbf{D} = \{s \in \mathbb{C}\colon |s| < 1\}$); such polynomials are called Schur polynomials. This case can be reduced to the analysis of a continuous polynomial using a linear-fractional (Möbius) transformation with denominator cancellation: a polynomial $G_0(z)$ is Schur if and only if the polynomial $G(s) = (1-s)^n G_0\left(\frac{1+s}{1-s}\right)$ is Hurwitz and $G_0(-1) \neq 0$ [1]. The last condition describes a special case in which the degree of the polynomial $G(s)$ may be less than that of $G_0(z)$; this occurs if G_0 has a root -1.

[1] If the system is not decomposed into several independent systems, i.e., its matrix can be reduced to the Frobenius (Brunovský canonical) form.

In general, the required localization set of the roots of a polynomial $\mathbf{D} \subset \mathbb{C}$ can be any subset of the complex plane, selected based on the requirements for the system. The behavior of a system is determined not only by the roots of its characteristic polynomial but also by its structure and initial conditions. Despite this, several important properties (besides stability) can be analyzed based on the location of the roots. For instance, it is possible to limit the degree of stability, the damping ratio, and other engineering performance criteria [2–4].

If all n roots of a polynomial lie in a set $\mathbf{D}$, it is called **D**-stable. For the sake of convenience, we will simply call it stable. By analogy, the roots of a polynomial lying in a set $\mathbf{D}$ will be called *stable*, and those lying outside $\mathbf{D}$ *unstable*. Without loss of generality, we will demonstrate the stability analysis of a continuous-time system; in this case, $\mathbf{D}$ is the open left complex half-plane, and **D**-stability means that the polynomial is Hurwitz. General requirements for the set $\mathbf{D}$ are provided in Section 1.

The roots of the polynomial (1) depend on the parameters k_1 and k_2, and the problem is to determine the parameter values under which the polynomial is stable (i.e., all its roots are stable). The results are used both to analyze the system stability with respect to these parameters (e.g., the physical parameters of the system included in the characteristic polynomial) and to design low-order controllers, such as PI, PD, or PID controllers with one fixed gain. If a system is stabilizable, then the set of such parameters is non-empty; it is called the *stability region* of the system in the parameter space. Let us denote the stability region by D_n, where the subscript corresponds to the number of stable roots.

To construct the stability region, we apply the D-partition method, originally proposed by Yu.I. Neimark. This method serves to describe a stability region using its boundary [5–9]. The idea behind the method can be outlined as follows. The boundary $\partial\mathbf{D}$ of a root localization region (the imaginary axis for continuous-time systems, the unit circle for discrete – time systems, etc.) is mapped onto the parameter plane, implicitly or explicitly. The resulting boundary divides the parameter plane into connected components, which we will call *D-partition regions*. Within each component mentioned, the number of stable roots does not change. The set of D-partition regions in which all roots are stable forms the stability region. The number of the components can be estimated for Hurwitz and Schur polynomials [10]. Thus, the boundary ∂D_n of a stability region satisfies the condition

$$\partial D_n \subset \left\{ \left(k_1, k_2\right) : G\left(s, k_1, k_2\right) = 0_{\mathbb{C}},\ s \in \partial\mathbf{D} \right\}. \quad (3)$$

The zero symbol $0_{\mathbb{C}}$ in formula (3) emphasizes that the equation is complex. Let the boundary of a root localization region be parameterized by a real parameter $w : \partial\mathbf{D} = \left\{ s(w) : w \in W \subset \mathbb{R} \right\}$. Then the D-partition is given by the so-called *main equation*

$$G\left(s(w), k_1, k_2\right) = 0_{\mathbb{C}},\ w \in W, \quad (4)$$

and the *degree drop condition* of the polynomial imposed on the coefficient at s^n:

$$G_n\left(k_1, k_2\right) = 0_{\mathbb{C}}. \quad (5)$$

Equation (5) describes the change of the total number of roots. It corresponds to a structural change in the system and can be interpreted as follows: in this case, at least one of the roots is “infinite” and unstable (by the number of consecutive leading terms equal to 0).

The features of solving the main equation are presented in Section 1. The advantage of D-partition is its simplicity and clarity. The drawbacks of the D-partition method are obvious:

- the dependence on a small number of parameters (one, two, or three; some exceptions for special polynomial structures were investigated in [11]);
- nontrivial application to polynomials that depend on parameters nonlinearly: equation (4) must be resolved with respect to the parameters;
- redundancy in the construction of boundaries, in particular, that of a stability region.

The last point is due to the fact that the mapping of the imaginary axis onto the parameter plane limits the connected components of a stability region and, in addition, separates all other regions with different numbers of stable roots. Moreover, the image of the imaginary axis can separate regions with the same number of stable roots. As a consequence of this problem, the boundary of a stability region is typically defined and analyzed in graphical and numerical form.

In this paper, we address the latter problem using an explicit and constructive description of the arcs of the stability region boundary and the algorithmization of its construction. This is achieved using a combination of algebraic and computational algorithms. The method is suitable for polynomials with the linear dependence on the parameters and arbitrary root localization regions whose boundary $\partial\mathbf{D}$ is parameterized by a piecewise rational function. The D-partition method is used in many practical problems of designing controllers with such a structure [5]. Therefore, the explicit identification (detection) of a stability region is topical.

This study further develops the results presented in [12]. The parameterization of the stability region boundary obtained below can be used for subsequent optimization within the stability region and for other tasks related to the analysis and design of controllers with desired characteristics specified by a root localization region. (These issues will be discussed in part II of the paper.)

Alternative approaches

We outline the main approaches to describing a stability region and its boundaries; in the general case, they are applicable to a larger number of parameters. Let us divide them into four groups.

The simplest and most obvious method is the brute force exploration over the parameters k, e.g., using a regular grid. At each grid node, the roots of a polynomial are calculated, and their number in a root localization region $\mathbf{D}$ is counted. Based on this parameter, the node's belonging to the corresponding region of the D-partition is determined. To implement this approach, at least an approximate localization of the stability region is required. As a rule, it is available in the form of a parallelepiped $\boldsymbol{K} = K_1 \times K_2 \times \ldots \times K_\ell = \left[\underline{k}_1, \overline{k}_1\right] \times \left[\underline{k}_2, \overline{k}_2\right] \times \ldots \times \left[\underline{k}_\ell, \overline{k}_\ell\right]$; The ranges of all ℓ parameters, $K_i = \left[\underline{k}_i, \overline{k}_i\right]$, $i = 1, \ldots, \ell$, are taken from the "physical" constraints of the analyzed system or based on other considerations. However, the number of trial points (grid nodes) is inversely proportional to the lth degree of grid (mesh) fineness, and the stability region boundary will be described approximately. For example, small stability or instability regions, as compared to the grid fineness, may be missed. This simple method quickly evaluates or localizes a stability region. Some examples of direct parameter enumeration are shown in Fig.1. Clearly, the stability region on the left includes only seven trial points; with a sparser grid, it would not have been detected.

Within the second approach, the parameter plane is divided into sets (often rectangles), and their belonging to a stability region is determined, e.g., using sufficient criteria for robust stability [13]. The algorithm has a complexity estimate with respect to the degree of the polynomial and surely yields an internal approximation of a stability set. The same approach is applicable to the analysis of systems with uncertainty [14].

The third approach is to describe a stability region implicitly as a system of equations and inequalities, e.g., matrix ones. This allows solving "higher-level" problems, such as obtaining an optimal controller in some sense, or designing a controller or observer that satisfies additional constraints, such as the H_∞ norm of transfer functions, etc., among stable controllers. The third approach is convenient due to its organic integration of the specified D-stability constraints into an optimization problem. For instance, the parameter constraints written in terms of the positive (negative) definiteness of certain matrices are well consistent with control problems in the *semidefinite programming* formulation [15]. Such formulations are natural

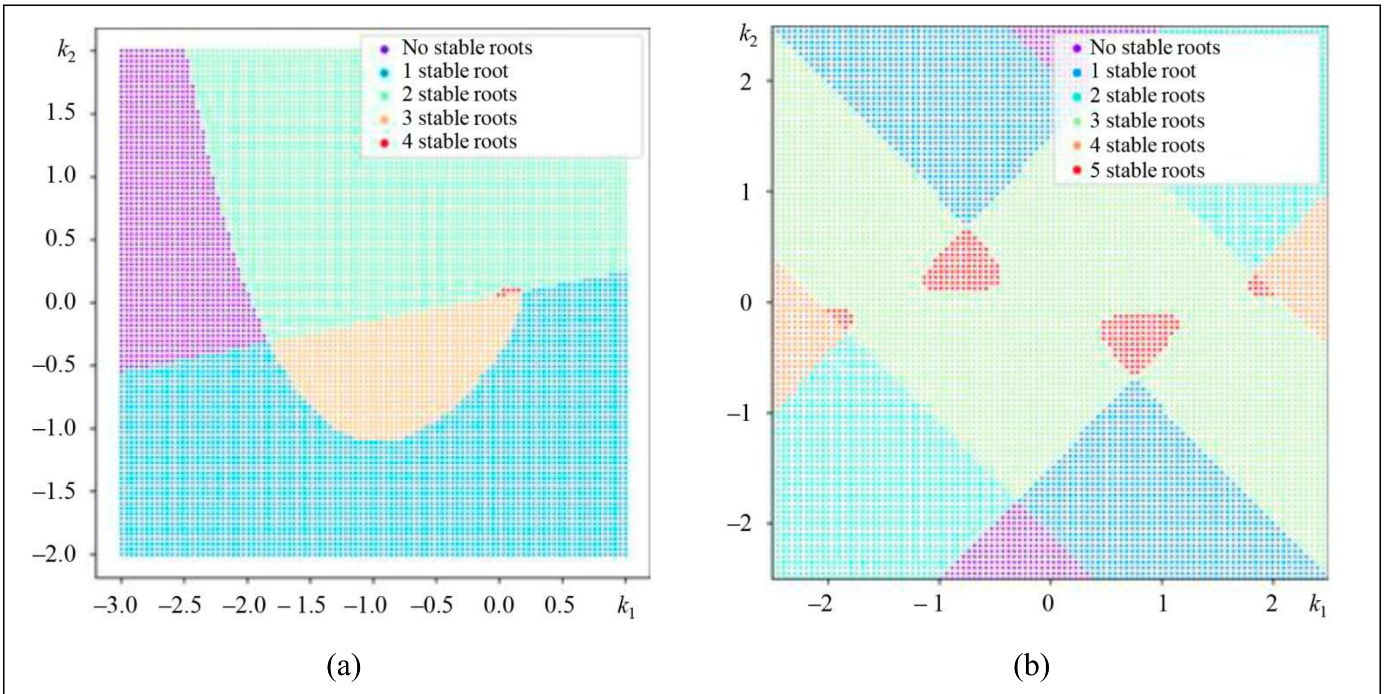

Fig.1 . Direct parameter enumeration: (a) Example 1 and (b) Example 2; see Section 5.

when the system is represented in the state space (as an ODE) using matrices where the parameters enter linearly, and they will enter the characteristic polynomial polynomially. Despite the convenience of specifying a region by formulas, in fact, a stability region can have a complex structure, be nonconvex, multiply connected, or even disconnected. As a result, the optimization problem on this set is complex in itself. Note a similar approach where the boundary of a stability region or, more generally, a region of parameters satisfying problem requirements (not necessarily in terms of polynomial's roots) is described using a certain boundary equation (*guardian map*) [16, 17]. This group also includes other approaches in which the boundary of a stability region is used implicitly. For example, in the direct search for an optimal controller, the numerical method "stays inside" a stability region due to a barrier-type function that tends to infinity when approaching the region boundary [18].

Finally, the fourth group of approaches is algebraic analysis of equation (4) as a system of two equations for a set of real parameters $w, k_1, k_2, \ldots$ As a rule, the main equation is polynomial, and its solution can be analyzed using algebraic methods [4, 20]. In particular, it is possible to eliminate the variable w, obtain the equation of the D-partition boundary, and then extract its parts corresponding to the stability region boundary. A set of points selected simultaneously allows identifying not only the arcs of the stability region boundary but also the region itself. The algorithm includes the construction of a Gröbner basis and its ideal, or a similar exclusion of the variables (which is essentially equivalent to writing the solution of the main equation (4) in analytical form) and cylindrical algebraic decomposition (CAD) for a system of polynomial equations [19, 20]. However, this method requires special software, and its numerical implementation can be time-consuming and memory-intensive; moreover, the result is very sensitive to the coefficients of the polynomial and the way they are defined.

Technically, all listed methods are reduced to analyzing a certain curve (curves, or straight lines, further called lines for brevity) on the plane that is the solution of a certain equation defining the boundaries of D-partition regions. The formulations of the corresponding problems depend on the method used to describe the curve. In this paper, we specify the process of constructing the D-partition for one and two parameters by combining algebraic and numerical methods. For this purpose, we use elementary auxiliary constructs to calculate the real roots of some polynomials. The result is an analytical description of the stability region boundary as a set of parameterized curves and lines with definite parameterization intervals. For this description of the boundary in the form of a set of curves and lines, we propose several numerical approximation methods for a stability region and/or its boundary with a desired accuracy. The methods help to solve a number of stability region-related problems easily, such as the localization of a stability region, robustness analysis, etc.

1. CONSTRUCTIVE D-PARTITION

1.1. D-Partition for Polynomials Linearly Dependent on Parameters

Consider a characteristic polynomial linearly dependent on two parameters k_1 and k_2:

$$G(s, k_1, k_2) = k_1 P(s) + k_2 Q(s) + R(s). \tag{6}$$

Let $\mathbf{D} \subset \mathbb{C}$ be a regular open[2] localization region for the roots. Further, assume that the polynomials $P(s)$, $Q(s)$, and $R(s)$ have no common roots.[3]

When varying the parameters k_1 and k_2, the roots of the polynomial change and may enter the region $\mathbf{D}$ or, conversely, leave it. Let the boundary $\partial\mathbf{D}$ of the root localization region have a real-valued parameterization by polynomials, or rational functions, or a set of such polynomials and functions.

Consider D-partition using the example of Hurwitz polynomials. Without loss of generality, we take the boundary of the asymptotic stability region (2) with the simply connected boundary $\Gamma = \{s \in \mathbb{C} : \operatorname{Re} s = 0\}$ and the parameterization

$$s(w) = jw, \ w \in (-\infty, \infty). \tag{7}$$

In the general case, assume that the boundary is described by a set of simple curves, each representing

[2] Or a regular closed region; in this case, the belonging of each *arc of the boundary* of the stability region to the *stability region* itself must be checked separately, in particular, by separately considering the intersection points and junction of the boundary arcs, while the methodology for constructing the D-partition remains the same. This technical issue also occurs for closed but irregular root localization regions, e.g., with $\mathbf{D} = \{s : \operatorname{Re} s < 0, \operatorname{Im} s = 0\}$, during aperiodicity analysis. The results of this paper are valid for open sets as well, where "internal" and "hanging" parts of the boundary of the root localization region and similar arcs of the stability region boundary are possible.

[3] Otherwise, the common root lying in the region $\mathbf{D}$ can be isolated as a common factor, which leads to a polynomial of degree $n-1$. If the common root does not belong to the region $\mathbf{D}$, then the polynomial is unstable for any values of the parameters.

a piecewise rational[4] function; see Section 2. Note that if the set $\mathbf{D}$ is symmetric with respect to the real axis and the coefficients of the polynomials are real, then it suffices to study only the "upper" part of the boundary with $\operatorname{Im} s \geq 0$, particularly during asymptotic stability analysis, $w \in [0, \infty)$. In this case, the intersection points of the boundary Γ with the real axis will be treated as the junction points of the boundary arcs.

The parameters k on the stability region boundary must satisfy the main equation (4) or the degree drop equation (5). We specify them for the polynomial (6). The leading coefficient of the polynomial $G_n(k_1, k_2)$ is linear in the parameters k_1 and k_2, and equation (5) connects the coefficients of the polynomials $P(s), Q(s)$, and $R(s)$ at s^n, denoted by $P_n = a_0$, $Q_n = b_0$, and $R_n = c_0$, respectively. Some (but not all) of them can be zero:

$$G_n(k_1, k_2) = a_0 k_1 + b_0 k_2 + c_0 = 0. \tag{8}$$

The main equation (4) of the D-partition for the boundary (7) takes the form $G(jw, k_1, k_2) = 0_{\mathbb{C}}$. It is polynomial with respect to all parameters and is equivalent to the system of two real linear equations

$$\begin{cases} \operatorname{Re} G(jw, k_1, k_2) = k_1 \operatorname{Re} P(jw) \\ \qquad + k_2 Re\ Q(jw) + \operatorname{Re} R(jw) = 0 \\ \operatorname{Im} G(jw, k_1, k_2) = k_1 \operatorname{Im} P(jw) \\ \qquad + k_2 \operatorname{Im} Q(jw) + \operatorname{Im} R(jw) = 0. \end{cases} \tag{9}$$

This is a linear vector equation with respect to the variables k_1 and k_2 with the coefficients polynomially dependent on w. Its solution can be written in explicit form. For this purpose, consider *singular (or critical) frequencies* nullifying the determinant of the matrix

$$T(w) = \begin{pmatrix} \operatorname{Re} P(jw) & \operatorname{Re} Q(jw) \\ \operatorname{Im} P(jw) & \operatorname{Im} Q(jw) \end{pmatrix}. \tag{10}$$

The critical frequencies are determined by a polynomial equation compiled for (10):

$$\det T(w) = \operatorname{Re} P(jw) \operatorname{Im} Q(jw) \\ - \operatorname{Re} Q(jw) \operatorname{Im} P(jw) = 0. \tag{11}$$

Let it have M different real roots w_i, and let at most n of them be on the interval $[0, \infty)$ for the case of asymptotic stability with the boundary (7), including the zero root (in the general case, at most $2n$ roots). For each root, the solution set of (9) is either a line, called a *singular line*, or an empty set. In addition, the rows of the augmented matrix composed of $T(w_i)$ and $(-\operatorname{Re} R(jw_i), -\operatorname{Im} R(jw_i))^{\mathrm{T}}$ are either linearly dependent or linearly independent. In the latter case, no solution exists, and the corresponding root w_i is ignored. The augmented matrix itself is nonzero by the assumption of no common roots of the polynomials P, Q, and R. Assuming the existence of a solution, we denote by a_i and b_i, the elements of the nonzero row of the matrix $T(w_i)$ and by c_i the free term. Then the solution of system (9) at $w = w_i$ is described by the line equations

$$a_i k_1 + b_i k_2 + c_i = 0,\ i = 1, \ldots, M. \tag{12}$$

The linear dependence of the rows of the augmented matrix covers the case when one row is zero; for example, for the boundary (7), this occurs when nullifying the imaginary part for $w = 0$, which corresponds to a singular line.

On the remaining intervals $(w_1 = 0, w_2)$, $(w_2, w_3), \ldots, (w_{M-1}, w_M), (w_M, \infty)$, the solutions of equation (9) define a rational curve on the plane consisting of at most $M+1$ connected arcs, according to the number of intervals where the functions $k_1(w)$ and $k_2(w)$ are continuous:

$$\begin{aligned} k_1(w) &= \frac{1}{\det T(w)} \big(\operatorname{Im} R(jw) \operatorname{Re} Q(jw) \\ &\quad - \operatorname{Im} R(jw) \operatorname{Re} Q(jw)\big), \\ k_2(w) &= \frac{1}{\det T(w)} \big(\operatorname{Re} R(jw) \operatorname{Im} P(jw) \\ &\quad - \operatorname{Im} R(jw) \operatorname{Re} P(jw)\big). \end{aligned} \tag{13}$$

Thus, the D-partition is described by the curve (13), further referred to as the main curve, and $K+1$ line equations (8) and (12).

In general, the boundary of the root localization region Γ is piecewise continuous, with different arcs having their own parameterization; as a result, the D-partition is defined by several main curves and a set of lines. Moreover, the root localization boundary may consist of several connected components [3]. A constructive D-partition algorithm in the general case is presented below. Its idea is to identify the boundaries

[4] When substituting the rational function $s(w)$, the main equation (4) reduces to a polynomial equation with respect to w; and one should separately consider the cases where the denominator of $s(w)$ is zero. If the root localization set is multiply connected, it may be necessary to analyze not only the stability region D_n but also other regions D_d, $d = 1, \ldots, n$ [3, 21]. Within the approach proposed, such situations can be considered uniformly, describing a region with an arbitrary number of stable roots.

of individual D-partition regions and, in particular, the stability region, using the one-dimensional parameterization of the boundary arcs.

1.2. Constructive D-Partition for Two Real Parameters

We propose a numerical-algebraic algorithm for describing individual D-partition regions by calculating their boundaries. Some transformations are performed analytically whereas the others numerically. All nontrivial numerical operations can be reduced to calculating the real roots of certain polynomials. The approach proposed copes with the difficulties inherent in the alternative methods, see the Introduction. In particular, it requires no special software and can be effectively implemented. On the other hand, it explicitly describes the boundaries of D-partition regions in the form of rational curve arcs and segments. Only the intervals of the parameters of these curves and segments are obtained numerically.

Algorithm 1. Constructive D-partition.

Input: a polynomial of degree n that linearly depends on the parameters, of the form (6); a root localization region $\mathbf{D}$ with a parameterized boundary

$$\Gamma=\bigcup_{\ell}\Gamma_{\ell},\ \Gamma_{\ell}=\left\{s_{\ell}(w)\in\mathbb{C}: w\in W_{\ell}\right\},\quad \ell=1,\ldots,L, \tag{14}$$

and $0\le d\le n$ as the number of stable roots sought for.

Step 1. For all arcs of the boundary Γ_{ℓ}, compile the D-partition equations in the form of a system of two equations, e.g.,

$$\begin{cases}\operatorname{Re}G\left(s_{\ell}(w),\ k_1,k_2\right)=k_1\operatorname{Re}P\left(s_{\ell}(w)\right)\\ \qquad +k_2\operatorname{Re}Q\left(s_{\ell}(w)\right)+\operatorname{Re}R\left(s_{\ell}(w)\right)=0\\ \operatorname{Im}G\left(s_{\ell}(w),\ k_1,k_2\right)=k_1\operatorname{Im}P\left(s_{\ell}(w)\right)\\ \qquad +k_2\operatorname{Im}Q\left(s_{\ell}(w)\right)+\operatorname{Im}R\left(s_{\ell}(w)\right)=0,\end{cases}\quad w\in W_{\ell}. \tag{15}$$

Step 2. Obtain the solution of (15) as a set of parametric curves $\left(k_{\ell,1}(w),\ k_{\ell,2}(w)\right)$, $w\in W_{\ell}$, and singular lines; add to this set the line from the degree reduction equation (8) if its solution set is non-empty. In this case, the intervals W_{ℓ} can be divided into intervals, half-intervals, or segments corresponding to the continuous curve arcs and separated by critical frequencies.

Step 3. Find all intersection points of the curves and lines obtained in the previous step, and the points of self-intersection of the curves $\left(k_{\ell,1}(w),\ k_{\ell,2}(w)\right)$ within the intervals W_{ℓ} on the intervals with a continuous dependence on the parameter w. When calculating the intersection points, consider the finite limit points as belonging to the corresponding curves (see Section 3).

As a result, the intervals W_{ℓ} will be divided into the subintervals $W_{\ell,i}$ corresponding to continuous curve arcs from one intersection to another (or infinite arcs without intersections). The lines are splitted into segments parameterized by the intervals $t\in T_m\subset\mathbb{R}$.

Step 4. Determine which D-partition regions lie on each side of each curve arc and segment. Sequentially group the curve arcs and segments that bound the part of the D-partition region with d roots.

Output: an ordered set of curve arcs and segments (or intervals) parameterized by the intervals $W_{\ell,i}$ and T_m that forms the boundaries of the region D_d.

In Step 3, it is essential to consider the finite limit points of the curves in order to calculate the division of singular lines into segments representing the boundaries of different D-partition regions.

The algorithm is mainly applied to find the stability region D_n, i.e., for $d=n$. Other D-partition regions may be needed when analyzing a multiply connected root localization region or if the initial root localization set is described by the union of sets (see footnote 4 on p. 5)

Algorithm 1 can be generalized to the case of a nonlinear dependence on the parameters: it is necessary to obtain an explicit solution of the main D-partition equation $G\left(s_{\ell}(w),\ k_1,\ k_2\right)=0_{\mathbb{C}}$ with respect to the parameters, depending on the value of the parameter w then. This may require re-parameterization or the introduction of extra parameters to describe the branches of curves or additional lines. Moreover, it is necessary to provide algorithms for finding intersection points of the resulting curves and lines. Apparently, to obtain such a solution even in the case of a polynomial dependence on k_1 and k_2 is a nontrivial problem. The identification of solvable cases is an interesting area of further research.

Next, we characterize the boundaries of D-partition regions for piecewise rational boundaries of the root localization region in a bounded set.

2. CONSTRUCTIVE D-PARTITION FOR A ROOT LOCALIZATION REGION WITH A RATIONAL BOUNDARY

Consider the implementation features of Algorithm 1 and its applicability. We begin with the boundary Γ (14) of the root localization set consisting of a single curve: $\Gamma=\left\{s(w)\in\mathbb{C}: w\in W\right\}$. In the general case, the results obtained are valid for each boundary arc Γ_ℓ. Without loss of generality, we will consider the stability region D_n as the desired D-partition region.

First of all, it is necessary to solve equations (15) with respect to w. If $s(w)$ can be represented as a rational complex function,

$$s(w) = s_{\text{real}}(w) + j\, s_{\text{imag}}(w)$$

with rational functions $s_{\text{real}}(w)$ and $s_{\text{imag}}(w)$, then the functions $P(s(w))$, $Q(s(w))$, and $R(s(w))$ (hence, $\operatorname{Re} P(s(w))$, etc.) are also rational. By multiplying the denominators of the functions $\operatorname{Re} P(s(w))$, $\operatorname{Re} Q(s(w))$, and $\operatorname{Re} R(s(w))$ (or $\operatorname{Im} P(s(w))$, $\operatorname{Im} Q(s(w))$, and $\operatorname{Im} R(s(w))$ for the second equation) by the least common multiple, the system of equations (15) can be reduced to the form (9), where the degrees of the polynomials P_1, P_2, Q_1, Q_2, R_1, and R_2 depend both on the degree of the original polynomial G and on the degrees of the numerator and denominator of the function $s(w)$:

$$\begin{cases} k_1 P_1(w) + k_2 Q_1(w) + R_1(w) = 0 \\ k_1 P_2(w) + k_2 Q_2(w) + R_2(w) = 0. \end{cases} \tag{16}$$

The conversion of equation (15) to a polynomial system of two equations is not unique. Similar to the conversion from Schur polynomials to Hurwitz ones, with the substitution of the rational function $s(w)$, one can first multiply the polynomial $G(s(w), k_1, k_2)$ by the denominator of the function $s(w)$ risen to power n. Thus, the denominators will be eliminated from equations (15) since the function $s(w)$ describes the parameterization of the curve and its denominator does not vanish on the interval W. Only after this transform should the resulting polynomial equation be divided into the real and imaginary parts. In both cases, the coefficients of the polynomials P_1, P_2, etc. are calculated analytically by substituting the function $s(w)$ and separating the real and imaginary parts. This method seems preferable: in the above operation of separating the real and imaginary parts of a complex polynomial (see the previous method), it is necessary to multiply the numerator by the complex conjugate denominator, which doubles the degree of the polynomials. In turn, the degree of the numerator and denominator of the complex function $G(s(w), k_1, k_2)$ is a multiple of the degree of the original system and those of the numerator and denominator of the function $s(w)$.

The main solution of system (16), like the solution of system (9) in the case $s(w) = -jw$, is described by a single rational function (on the intervals of its continuity) and, possibly, by singular lines. The singular lines correspond to the parameters w, called critical frequencies[5] by analogy to Section 1. They are determined by the equation similar to (11):

$$\det T(w) = P_1(w) Q_2(w) - Q_1(w) P_2(w) = 0. \tag{17}$$

The general solution $k(w) = (k_1(w), k_2(w))$, similar to formulas (13) , is expressed by the rational functions

$$\begin{aligned} k_1(w) &= \frac{1}{\det T(w)} \big(R_2(w)\, Q_1(w) \\ &\quad - R_1(w)\, Q_2(w) \big) = \frac{k_{1,\text{num}}(w)}{k_{1,\text{den}}(w)}, \\ k_2(w) &= \frac{1}{\det T(w)} \big(R_1(w)\, P_2(w) \\ &\quad - R_2(w)\, P_1(w) \big) = \frac{k_{2,\text{num}}(w)}{k_{2,\text{den}}(w)}, \\ & w \in W. \end{aligned} \tag{18}$$

Here, the subscripts "num" and "den" indicate the polynomials in the numerator and denominator, respectively.

Depending on the context, it is convenient to consider $k(w)$ without singular frequencies either as a single curve defined by a general expression or as a set of continuous curves defined on open intervals.

2.1. Straight Lines and Intersections with Them

Similar to equation (9) for asymptotic stability analysis, the system of equations (16) is polynomial, and its solutions may include singular lines; we denote them by analogy with the lines (12), with the coefficients a_i, b_i, c_i, $i = 0, \ldots, M$. They are supplemented by the singular line with the subscript $i = 0$, corresponding to the degree drop condition (8). The total number of roots depends on the particular functions $G(\ldots)$ and $s(w)$ and the degrees of the polynomials forming them. The critical frequencies w_i are determined numerically from equation (17); only the real

[5] Sometimes, critical frequencies are not the roots w_i themselves but the function values $s(w_i)$, also termed generalized critical frequencies.

roots belonging to the interval W are needed here. The list of critical frequencies consists of only those w_i for which the solution set of system (16) is non-empty.

The intersection of these lines with the main curve of the D-partition (18) is given by the equation

$$\begin{gathered} a_i\left(R_2(w)Q_1(w)-R_1(w)Q_2(w)\right) \\ +b_i\left(R_1(w)P_2(w)-R_2(w)P_1(w)\right) \\ +c_i\left(P_1(w)Q_2(w)-Q_1(w)P_2(w)\right)=0, \\ i=0,\ldots, K. \end{gathered} \tag{19}$$

This equation is polynomial with respect to w, and its real roots (denoted by w_m, $m=1,\ldots$) can be calculated explicitly. These roots, together with the critical frequencies w_i, divide the interval W into segments and intervals corresponding to the simple, continuous parts of the D-partition boundaries, the arcs of the main curve. These arcs form the main, non-trivial part of the D-partition. The intersection points are determined from equation (18) as $k(w_m)=(k_1(w_m),\, k_2(w_m))$. They divide the fundamental curve into arcs and, moreover, divide the singular lines into segments or infinite intervals (rays).

2.2. Selection of Segments on a Straight Line

On the plane of parameters, the intersection points of singular lines with each other and with the main curve determine the segments and intervals of each singular line. Consider a line on the plane given by equation (12), where the numbers a and b are not simultaneously zero:

$$ak_1+bk_2+c=0. \tag{20}$$

We describe the relationship between the algebraic and parametric representations of a line:

$$\begin{gathered} k(t)=\ t\ d+p,\ -\infty<t<+\infty, \\ p,\ d\in\mathbb{R}^2,\ \ d\neq 0. \end{gathered} \tag{21}$$

Equation (20) with

$$a=d_2,\ b=-d_1,\ c=d_1p_2-d_2p_1 \tag{22}$$

yields the expression (21). The above representation is unique up to a (nonzero) multiplier.

Conversely, the form (20) can be obtained from formula (21). This form is associated with any pair of a direction vector d and an origin point p satisfying the independent system of equations

$$\begin{cases} ad_1+bd_2=0 \\ ap_1+bp_2+c=0. \end{cases} \tag{23}$$

The direction vector can be taken as $d=(-b,\, a)^{\mathrm{T}}$; it is defined up to a nonzero multiplier. The second equation in system (23) coincides with equation (20), and any point on the line satisfying (20) is selected as p. (This condition is also obvious from the fact that the point $k(0)=p$ belongs to the line.) For example, one can take the intersection point of the straight line with the X axis, $(-c/a,\, 0)$, if $a\neq 0$, or with the Y axis, $(0,\, -c/b)$, if $b\neq 0$. It is possible to take the point closest to the origin:

$$p_1=-\frac{ac}{a^2+b^2},\ p_2=-\frac{bc}{a^2+b^2}. \tag{24}$$

If the point k on the straight line is known, we can immediately select $p=k$. For instance, if all intersection points of the straight line (20) with other lines or with the main curve (curves) of the D-partition are known, then the point (24) can be replaced by the one with the minimum (or maximum) abscissa or ordinate, and the direction vector d can be selected so that all segments and intervals of interest correspond to the intervals with a nonnegative value t. In this case, the direction vector d can be normalized so that the parameters of all segments of interest fall within the interval $t\in[0,\,1]$. This is convenient under the a priori localization of the stability region described in Section 3.

For the parameterization (21), it is of interest to determine the parameter t by a point on the line, the so-called *inversion* problem. In particular, for Algorithm 1, it is necessary to determine the line segments between the intersection points with other lines or with the main curve of the D-partition. Let a point $k=(k_1,\, k_2)^{\mathrm{T}}$ lie on the straight line (21). The parameter value can be obtained from the equation for one of the coordinates: $t^*=(k_1-p_1)/d_1$ if $d_1\neq 0$, or $t^*=(k_2-p_2)/d_2$ if $d_2\neq 0$; or from the expression

$$t^*=\frac{(k_1-p_1)d_1+(k_2-p_2)d_2}{d_1^2+d_2^2}=\frac{(k-p)^{\mathrm{T}}d}{\|d\|^2}. \tag{25}$$

If the point k is obtained numerically and does not lie on the line, formula (25) will correspond to the point closest to k on the line (its Euclidean projection):

$$\begin{aligned} k^* &= t^*d+p=\frac{(k-p)^{\mathrm{T}}d}{\|d\|^2}d+p \\ &= p+\frac{dd^{\mathrm{T}}}{d^{\mathrm{T}}d}(k-p)=\left(I-\frac{dd^{\mathrm{T}}}{d^{\mathrm{T}}d}\right)p+\frac{dd^{\mathrm{T}}}{d^{\mathrm{T}}d}k \\ &= k+\left(I-\frac{dd^{\mathrm{T}}}{d^{\mathrm{T}}d}\right)(p-k), \end{aligned} \tag{26}$$

where I stands for an identity matrix of dimension 2×2, and $\frac{dd^{\mathrm{T}}}{d^{\mathrm{T}}d}$ is the projector onto the line.

It remains to consider the intersection points of two lines. Let the first be given algebraically, by equation (20), and the second parametrically by formula (21). If the lines intersect and do not coincide with each other, then they are not parallel, and $(a,\, b)d = ad_1 + bd_2 \neq 0$. Here, $(a,b)d$ denotes the matrix product of the row vector $(a,\, b)$ and the column vector d, i.e., the standard inner product of the vectors $(a,\, b)^{\mathrm{T}}$ and d. Then the intersection point k^* is found by substituting formula (21) into equation (20) as follows:

$$t_2^* = -\frac{c + ap_1 + bp_2}{ad_1 + bd_2} = -\frac{c + (a,\, b)\,p}{(a,\, b)d},$$
$$k^* = t_2^* d + p = -\frac{c + (a,\, b)\,p}{(a,\, b)d}\, d + p. \tag{27}$$

Here, t_2^* refers to the parameterization of the second line. To find the value of the parameter t_1^* corresponding to the first line, we need to consider its parameterization (21), substituted into the second line equation (20).

3. LOCALIZED D-PARTITION AND LIMIT POINTS

In practice, D-partition is performed in a bounded closed region (a compact set), e.g., in a rectangle $\boldsymbol{K} = [\underline{k}_1, \overline{k}_1] \times [\underline{k}_2, \overline{k}_2]$. We will call it an (a priori) localization region and the D-partition in this region a localized D-partition. Of course, localized D-partition yields part of the stability region within the corresponding localization region. However, in many cases, the chosen localization region contains the entire stability region since the latter is often bounded and small. Sometimes it can be determined in advance that a polynomial is unstable outside a region $\boldsymbol{K}$, e.g., using necessary stability criteria. For instance, for a polynomial to be Hurwitz, all its coefficients must have the same sign (Stodola's criterion), etc. [22].

The boundaries of a rectangle $\boldsymbol{K}$ are vertical and horizontal segments; for such segments, equation (19) becomes simpler, cf. the equation for one of the components (18):

$$R_2(w)Q_1(w) - R_1(w)Q_2(w)$$
$$= x\big(P_1(w)Q_2(w) - Q_1(w)P_2(w)\big), \tag{28}$$
$$x = \underline{k}_1,\, \overline{k}_1,$$

$$R_1(w)P_2(w) - R_2(w)P_1(w)$$
$$= y\big(P_1(w)Q_2(w) - Q_1(w)P_2(w)\big), \tag{29}$$
$$y = \underline{k}_2,\, \overline{k}_2.$$

When solving each of these equations, it is necessary to check whether the other coordinate falls within the desired interval (i.e., whether the main curve intersects the segment on the line). If the roots of the first equation are w_m, one should select only those for which $k_2(w_m) \in [\underline{k}_2, \overline{k}_2]$, and vice versa. Basically, such a check has not been required to find the intersections of the main curve with singular lines.

The intersection of singular lines with the boundaries of a region $\boldsymbol{K}$ is done considering their parameterization

$$\begin{pmatrix} \overline{k}_1 - \underline{k}_1 \\ 0 \end{pmatrix} t + \begin{pmatrix} \underline{k}_1 \\ \underline{k}_2 \end{pmatrix}, \begin{pmatrix} \overline{k}_1 - \underline{k}_1 \\ 0 \end{pmatrix} t + \begin{pmatrix} \underline{k}_1 \\ \overline{k}_2 \end{pmatrix},$$
$$\begin{pmatrix} 0 \\ \overline{k}_2 - \underline{k}_2 \end{pmatrix} t + \begin{pmatrix} \underline{k}_1 \\ \underline{k}_2 \end{pmatrix}, \begin{pmatrix} 0 \\ \overline{k}_2 - \underline{k}_2 \end{pmatrix} t + \begin{pmatrix} \overline{k}_1 \\ \underline{k}_2 \end{pmatrix}, \tag{30}$$
$$t \in [0,\, 1].$$

When calculating the intersection of the segments (30) with the singular lines (20) using formula (27), one additionally checks the inclusion $t_2^* \in [0,\, 1]$; if it fails, there is no intersection of the singular line with the segment.

The D-partition localized in a bounded set is convenient because it simplifies the analysis of limit points. In what follows, except for Lemma 4, we consider a single main curve $k(w) = (k_1(w),\, k_2(w))$. Let adjacent critical frequencies w_i and w_{i+1} be found on an interval W. The curve is continuous on the open interval $(w_i,\, w_{i+1})$ between the critical frequencies. Consider the limit points of the curve on this interval and any endpoint w_c of the segment. The left endpoint $w_c = w_i$ corresponds to the right-sided limit whereas the right one $w_c = w_{i+1}$ to the left-sided limit. There are two possible cases.

Lemma 1. *If the right-sided (left-sided) limit* $\lim\limits_{w\to w_c} k(w) = k_c$ *exists, then the limit point* $k_c = (k_{c,1}, k_{c,2})$ *lies on the singular line corresponding to the critical frequency* w_c.

P r o o f. Let us directly substitute the solution (18) into system (16); within the interval (w_i, w_{i+1}), both equations turn into the identity $k_1(w)P(w) + k_2(w)Q(w) + R(w) = R(w)\frac{\det T(w)}{\det T(w)} - R(w) = 0$, $w \neq w_c$, since $\det T(w) \neq 0$.

As $w \to w_c$, the identity remains valid due to continuous dependence of (11) on k; in particular, all the terms $k_{c,1}P(w_c)$, $k_{c,2}Q(w_c)$, and $R(w_c)$ depending continuously on w will satisfy the corresponding equality. Thus, the point k_c satisfies equation (11), i.e., the solution set is non-empty at $w = w_c$, and the rank of the augmented matrix does not exceed that of the matrix $T(w_c)$. By assumption, the polynomials P, Q, and R have no common roots; therefore, the polynomials P_1, P_2, Q_1, Q_2, R_1, and R_2 also have no common roots, i.e., the matrix $T(w_c)$ is nonzero. In addition, $\det T(w_c) = 0$, meaning that the rank of the matrix $T(w_c)$ and, consequently, the rank of the augmented matrix, is 1. As a result, the solution of system (16) at $w = w_c$ is a singular line containing k_c. ♦

According to Lemma 1, the curve can be further defined at the point w_c by the value k_c, keeping the continuity of the curve and closing its definitional interval at the corresponding endpoint.

For the sake of definiteness, consider the left endpoint of the segment, $w \to w_i$, and the corresponding limit on the right as $w \to w_i +$. The following lemma characterizes the infinite arcs of curves outside the region $\boldsymbol{K}$.

Lemma 2. *If the right-sided limit* $\lim\limits_{w \to w_i +} k(w)$ *does not exist and* $k(w_0) \in \boldsymbol{K}$ *for some* $w_0 \in (w_i, w_{i+1})$, *then there exists* $w_c \in (w_i, w_0]$ *such that* $k(w_c) \in \boldsymbol{K}$ *and* $k(w) \notin \boldsymbol{K}$ *for all* $w \in (w_i, w_c)$.

P r o o f. It follows from continuity. By construction, the sign of the determinant $\det T(w)$ is constant within the interval (w_i, w_{i+1}). Let us find all intersection points of the curve $k(w)$, $w \in (w_i, w_{i+1})$, with the boundaries of the region $\boldsymbol{K}$, e.g., from equations (28) and (29); among them, we select the minimum value of the parameter, denoting it by w_c. The existence of a minimum is ensured by the compactness of the region $\boldsymbol{K}$ and the continuity of the function $k(w)$. The point $k(w_c)$ does indeed exist since $k(w_0) \in \boldsymbol{K}$ and $k(w)$ has no limit as $w \to w_i +$. The function $k(w)$ is rational; therefore, $\|k(w)\|$ increases infinitely as $w \to w_i +$, and $k(w_{i+}) \notin \boldsymbol{K}$ for a value $w_{i+} < w_c$ sufficiently close to w_i. On the interval (w_{i+}, w_c), the function $k(w)$ is continuous and does not intersect the boundary of the region $\boldsymbol{K}$, so the arc of the curve $k(w)$ also lies outside the region $\boldsymbol{K}$ for all $w \in (w_i, w_c)$. ♦

Obviously, Lemma 2 is valid for both endpoints of the segment or interval (w_i, w_{i+1}). Note that with Lemmas 1 and 2, for the D-partition inside the region $\boldsymbol{K}$, it suffices to consider the segment $[w_c, w_d] \subset (w_i, w_{i+1})$, where w_c and w_d are determined algorithmically. To check the intersection of the boundary's arc with the rectangle $\boldsymbol{K}$, it suffices to find the roots of the four polynomials (28), (29).

For completeness, we should study infinite parameterization intervals and their limit points. Assume that after the division by critical frequencies, the interval W includes the infinite one (w_i, ∞).

Lemma 3. *If the limit* $\lim\limits_{w \to \infty} k(w)$ *does not exist, then the order of the rational function's numerator is greater than the order of its denominator,* $\|k(w)\| \underset{w \to \infty}{\to} \infty$, *and an analog of Lemma 2 is valid. That is, if* $k(w_0) \in \boldsymbol{K}$ *for some* $w_0 \in (w_i, \infty)$, *then there exists* $w_c \ge w_0$ *such that* $k(w_c) \in \boldsymbol{K}$ *and* $k(w) \notin \boldsymbol{K}$ *for all* $w > w_c$.

The proof of Lemma 3 repeats that of Lemma 2.

Finally, the case of the existence of a limit on an infinite interval of the parameter is characterized by the following lemma. It considers the arc of the curve $\Gamma_i = \{s_i(w), w \in W_i\}$ on the unbounded section $(\ldots, \infty)$.

Lemma 4. *If the limit* $\lim\limits_{w \to \infty} k_i(w) = k_\infty$ *for the arc of the curve* Γ_i *exists, then either the limit* $\lim\limits_{w \to \infty} s_i(w)$ *exists and belongs to the curve* Γ *or the limit* $\lim\limits_{w \to \infty} s_i(w)$ *does not exist and the point* k_∞ *lies on the singular line satisfying the degree reduction condition* (8).

P r o o f. According to this lemma, consider two cases.

1. If the limit $s_0 = \lim\limits_{w \to \infty} s_i(w)$ exists, then it belongs to the closure of Γ_i since $s_i(w) \in \Gamma_i$. In turn, the closure of Γ_i is a subset of the entire boundary Γ because the boundary is closed.

2. If the limit $\lim\limits_{w \to \infty} s_i(w)$ does not exist, then $r(w) \underset{w \to \infty}{\to} \infty$ in the polar representation $s_i(w) = r(w) e^{j\phi(w)}$. In this case, $s(w)$ is a root of the polynomial (6). Let us divide the polynomial by s^n. As a result, the coefficients of the functions $P(s)/s^n$, etc., at the parameters k_1 and k_2 and in the free term tend to those of the polynomials P, Q, and R at s^n, and the remaining coefficients tend to zero. For each $w \in W_i$, $s_i(w)$ is a root of the equation $P(s)/s^n = 0$, and the limit point k_∞ satisfies the degree

drop equation (8). In other words, k_∞ lies on the corresponding singular line. ♦

The limit point $s_\infty = \lim_{w\to\infty} s_i(w)$ (if exists) is the junction point of the arcs of the root localization boundary, i.e., it either belongs to the curve Γ_m, $m \neq i$, or is the other endpoint of the same arc of the curve Γ_i. In addition, the limit point matches a finite value of the parameter w (possibly from another arc of the curve Γ_m and $s_\infty = s_m(w)$) since all boundary points are parameterized by definition.

Lemmas 1 and 4 characterize the finite limit points of the curve arcs. They are considered in Step 3 of Algorithm 1 when searching for the intersection points. Strictly speaking, in such cases, the curve is further defined at the finite limit point w_i (the boundary of the interval); for infinite limit points as $w \to \infty$, the parameterization is changed, see the proof of Theorem 1 below. When checking the belonging of a limit point k to a singular line, according to formula (26), one can use the distance from the point to the line $\left\| \left(I - \frac{dd^{\mathrm{T}}}{d^{\mathrm{T}} d} \right)(p-k) \right\|$. If this distance does not exceed the specified accuracy, then the limit point can be considered an intersection point. This point splits the line into two parts by the parameter t, calculated using formula (25). To check the belonging of the limit point to another main curve defined by a rational function, the algebraic form of this curve is used, together with inversion (the recovery of the parameter by a point on the curve, see [23]). The split can be obtained in another way, using the fact that the curve is defined by a system of equations. For this purpose, the values of the components of the limit point k (of the *first* main curve) are substituted into one of the equations (16) of the *second* main curve; the resulting equation is solved with respect to w. Then the roots found are substituted into the second equation of the second main curve; those are selected for which the equality holds for the same components of k. For the limit point (of the first main curve) being on the second main curve, this procedure yields the latter's parameter matching this point.

The above lemmas completely characterize the boundaries of the localized D-partition regions of the polynomial (6) with a piecewise rational boundary of the root localization region. By assumption, the boundary of the localization region $\boldsymbol{K}$ consists of a finite number of rational curve arcs.

Theorem 1. *The boundaries of the D-partition regions localized in a compact set $\boldsymbol{K}$ consist of a finite number of segments, arcs of the boundary $\partial\boldsymbol{K}$ of the localization region, and arcs of rational curves defined on finite closed intervals of parameters.*

P r o o f. For a linear dependence of a polynomial on parameters, the D-partition is described by singular lines and rational curves of the form (13) defined on open (if the endpoint of the interval corresponds to a critical frequency or is unbounded) or closed (if the curve corresponds to an arc of the boundary defined on a closed interval) numerical intervals. Each endpoint of the interval is considered independently, so the interval can be closed on one side and open on the other. Due to the compactness of the set $\boldsymbol{K}$, the parts of the singular lines belonging to $\boldsymbol{K}$ are segments, and their number is finite by the assumption of a rational boundary of $\boldsymbol{K}$. If an arc of the boundary of the root localization region consists of a single point, it is mapped either to a singular line or to a point on the parameter plane (a degenerate segment). If the simply connected component j of the region D_i (denoted by $D_{i,j}$) of the original non-localized D-partition does not lie entirely inside or outside $\boldsymbol{K}$, the localized D-partition will include the region $D_{i,j} \cap \boldsymbol{K}$, and part of the latter's boundary will, in general, be the boundary of $\boldsymbol{K}$. In this case, the number of arcs of each main curve outside and inside $\boldsymbol{K}$ is finite by the assumption of a rational boundary of $\boldsymbol{K}$. As for the remaining parts of the boundary (i.e., the fractional curve arcs defined on intervals with at least one open or unbounded endpoint), Lemma 2 or Lemma 3 will be valid for unbounded $k(w)$: the corresponding endpoint of the subinterval is closed.

Consider a rational curve arc defined on the interval W with a finite open endpoint w_2 and an existing limit $k(w)$. For the sake of definiteness, we take the right endpoint of the interval $W = (w_0, w_2)$ and the limit $\lim_{w \to w_2 -} k(w) = k_2$. There are two possible cases. If $k_2 \in \boldsymbol{K}$, then (Lemma 1) the rational curve (13) can be further defined at the point w_2 by the value k_2. If $k_2 \notin \boldsymbol{K}$, then (following the proof of Lemma 2), due to the compactness of the set $\boldsymbol{K}$, a subinterval $(w_0, w_1]$ with a closed right endpoint is selected within the interval such that $k(w_1) \in \boldsymbol{K}$ and $k(w) \notin \boldsymbol{K}$, $w \in (w_1, w_2)$. Thus, the interval's right endpoint corresponding to the arc of the curve (the boundary $k(w)$) lying in the region $\boldsymbol{K}$ is closed. The same is valid for the left open endpoint of the interval.

Finally, we need to consider the case with an unbounded interval, e.g., $[w_1, \infty)$, and an existing limit $\lim_{w\to\infty} k(w) = k_\infty$. In this case, the parameterization can be changed; for example, see [24]. Without limiting generality, the left endpoint is closed since one of the above cases is valid. An interval unbounded from below is considered by analogy. The interval $(-\infty, +\infty)$, which is unbounded on both sides, is divided into three intervals, e.g., $(-\infty, -1]$, $[-1, 1]$, $[1, \infty)$. If $k_\infty \notin \boldsymbol{K}$, then, similar to the previous case with an open endpoint of the interval, the region $\boldsymbol{K}$ contains an arc of the curve $k(w)$ with a subinterval $[w_1, w_2]$ whose right endpoint is closed. If $k_\infty \in \boldsymbol{K}$, then we choose $w_0 < w_1$, e.g., $w_0 = w_1 - 1$. With the parameter $u = 1/(w - w_0)$, $u \in (0, 1]$ introduced, the fractional curve $k(w)$ has the fractional parameterization $k_u(u) = k(w_0 + 1/u)$, which can be defined at $u = 0$ as well by the limit value k_∞. Thus, all arcs of the boundaries of the localized D-partition defined by the curve (13) are actually defined on finite closed intervals. ♦

Under the assumption of a rational boundary of the localization region $\boldsymbol{K}$, the boundaries of the localized D-partition regions are either segments or rational curve arcs defined on finite closed intervals (segments), and their number is finite.

4. CONSTRUCTIVE D-PARTITION IMPLEMENTATION

4.1. Intersection of Two Main Curves

If the boundary Γ of a root localization region is described by several functions $s_\ell(w) \in \mathbb{C}: w \in W_\ell$, $\ell = 1, 2, \ldots$, then it is necessary to find all points of their pairwise intersections. Consider the two main curves and the corresponding functions, denoted by $k^a(w)$, $w \in W$, and $k^b(u)$, $u \in U$:

$$k_1^a(w) = \frac{k_{1,\text{num}}^a(w)}{k_{1,\text{den}}^a(w)},\ k_2^a(w) = \frac{k_{2,\text{num}}^a(w)}{k_{2,\text{den}}^a(w)}$$

$$\text{and} \tag{31}$$

$$k_1^b(u) = \frac{k_{1,\text{num}}^b(u)}{k_{1,\text{den}}^b(u)},\ k_2^b(u) = \frac{k_{2,\text{num}}^b(u)}{k_{2,\text{den}}^b(u)}.$$

Their intersection is described by a system of two rational equations, which can be reduced to the system of polynomial equations

$$\begin{cases} k_{1,\text{num}}^a(w)\ k_{1,\text{den}}^b(u) = k_{1,\text{num}}^b(u)\ k_{1,\text{den}}^a(w) \\ k_{2,\text{num}}^a(w)\ k_{2,\text{den}}^b(u) = k_{2,\text{num}}^b(u)\ k_{2,\text{den}}^a(w), \end{cases}$$

$$w \in W,\ u \in U.$$

One uses only the solutions not corresponding to singular lines, i.e., $k_{1,\text{den}}^a(w) \neq 0$ and $k_{1,\text{den}}^b(u) \neq 0$. This system of two polynomial equations can be solved numerically, e.g., by Newton's method started near the solution, or by decomposing the parameter plane (k_1, k_2) [25] or the argument space [24], or by algebraic methods, e.g., by resultant approach. We emphasize that, as in the case of the intersection of the main curve and lines, only the values of the parameters w and u corresponding to the intersections are determined numerically.

Recall the idea behind application of resultants using an example of solving a system of two polynomials

$$p(w, u) = 0,\ q(w, u) = 0 \tag{32}$$

depending on two parameters; details can be found in the monograph [21]. The first parameter is taken out into monomials, and the second one remains in the coefficients of the polynomial of the first parameter. Let the maximum degree of these polynomials in u be m. In other words, at least one of the polynomials $p_m(w)$ and $q_m(w)$ is not identically equal to zero:

$$\begin{aligned} p(w, u) = p_m(w)u^m + p_{m-1}(w)u^{m-1} + \ldots \\ + p_1(w)u + p_0(w) = 0, \end{aligned} \tag{33}$$

$$\begin{aligned} q(w,u) = q_m(w)u^m + q_{m-1}(w)u^{m-1} + \ldots \\ + q_1(w)u + q_0(w) = 0. \end{aligned} \tag{34}$$

Next, the coefficients $p_i(w)$ and $q_i(w)$ are used to construct a matrix $R(w)$ of dimensions $n \times n$, called the resultant. There are different ways to construct it; the most famous ones are the Bézout and Sylvester resultants. The key property is that the two polynomials (31) have common roots (in u) if and only if the determinant of the matrix $R(w)$ is vanishing, i.e., the coefficients of the polynomials $p_i(w)$ and $q_i(w)$ satisfy a certain condition. A common root of the polynomials means the existence of a value u for which equalities (32) and (33) are satisfied. In this case, the coefficients of the polynomials correspond to a particular value of w, i.e., a solution of the system of equations is obtained. Thus, one should first solve the equation

$$\det R(w) = 0 \tag{35}$$

i.e., find the roots w_j of this polynomial, excluding the roots of the polynomial $k_{1,\text{den}}^a(w)$ among them. Next, it is necessary to substitute each of the roots found into the coefficients of the polynomials (33), (34), calculate their roots $u_{p,i}(w_j)$ and $u_{q,i}(w_j)$, and then select only those matching roots that are not roots of the polynomial $k_{1,\text{den}}^b(u)$. The resulting pairs $(u_{p,i}, w_j)$ give the solution of system (32). By construction, they are the solutions of equations (31); thus, the parameters of the curves corresponding to the intersection points have been obtained. With alternative methods used to find the intersection points (e.g., on the plane k), one faces the inversion problem—restoring the parameters w and u by the point (p, q). This can also be done using resultants [23].

The above procedure allows finding all solutions of system (31). This approach has two features as follows. The first is an increase in complexity since the polynomial (35) has a degree of order m^2. To calculate its coefficients, it is desirable to use accurate algebraic computations. The procedure can be supplemented with the refining solution of system (31) or (32) by Newton's method, using the obtained pair (u, w) as the starting point.

The second feature is that, in order to apply resultants, the polynomials must be independent. In particular, they must have no common factors, their coefficients must be linearly independent, etc. Such cases correspond to the degeneration of the resultant due to the structure of the polynomials $p()$ and $q()$ and the dependence of their coefficients on w. In these cases, for any values of w, equation (35) becomes an identity and cannot be used to find individual w. Algebraically, this situation corresponds to an infinite number of solutions of system (32), i.e., the overlapping of the rational curves $k^a(w)$ and $k^b(u)$ on each other, or to a set of solutions independent of one parameter, e.g., $(w, 0)$ for any w. The latter case corresponds to the common root of the numerator and denominator $k_1^a(w), k_2^a(w)$, which must be reduced when writing the curves. As a rule, D-partition results in the main curves of general position, and the approach involving resultants is successful. An important exception here is the problem of finding the self-intersection of the main curve: this problem is solved using different parameterizations of the same curve.

4.2. Self-Intersection of a Main Curve

The last component for describing the arcs of the D-partition boundary is the points of the self-intersection of a main curve. The rational curve (18) can intersect itself, dividing the plane into additional regions. The points of self-intersection satisfy the system of equations

$$\begin{cases} k_1(w)=k_1(u) \\ k_2(w)=k_2(u), \end{cases}$$
$$w \neq u.$$

This system reduces to the system of polynomial equations

$$\begin{cases} k_{1,\text{num}}(w)k_{1,\text{den}}(u)=k_{1,\text{num}}(u)k_{1,\text{den}}(w) \\ k_{2,\text{num}}(w)k_{2,\text{den}}(u)=k_{2,\text{num}}(u)k_{2,\text{den}}(w), \end{cases}$$
$$w \neq u.$$

A challenge arises due to the inequality condition of the parameters since the system has the trivial solution $w=u$. Hence, it is impossible to use the resultants described in subsection 4.1: they are degenerate. Following the paper [25], the trivial solution can be explicitly excluded by writing the equation of the so-called *reduced differences*:

$$\begin{cases} p(w, u)=\dfrac{\left(k_{1,\text{num}}(w)\, k_{1,\text{den}}(u)\right.}{w-u} \\ \qquad -\dfrac{\left.k_{1,\text{num}}(u)\, k_{1,\text{den}}(w)\right)}{w-u}=0 \\ q(w, u)=\dfrac{\left(k_{2,\text{num}}(w)\, k_{2,\text{den}}(u)\right.}{w-u} \\ \qquad -\dfrac{\left.k_{2,\text{num}}(u)\, k_{2,\text{den}}(w)\right)}{w-u}=0. \end{cases} \tag{36}$$

This system can be solved using the same methods as in subsection 4.1. Also, a solution method in the plane of the parameters p, q was proposed in [25]. That is, to form reduced polynomials, the original polynomials were written in the Bernstein polynomial basis, and the parameters w and u were restored by the intersection point.

Sometimes, the polynomials $k_1(w)$ and $k_2(w)$ contain only even (or only odd) degrees of w, and the reduction of system (36) is insufficient. In this case, the curve turns out to be a multiple of itself, and the system has the additional solution $w=-u$. It must also be excluded by dividing system (36) by the term $w+u$. This is equivalent to excluding the solutions $w^2=u^2$.

Situations with even and odd functions $k_1(w)$ and $k_2(w)$ are common in the stability analysis of polynomials with real coefficients and a root localization region $\mathbf{D}$ symmetric with respect to the real axis. Under these conditions, the roots of the polynomial are complex conjugate, and the point k on the boundary of the D-partition regions is associated with two roots on the boundary $\partial\mathbf{D}$. These roots correspond to two values of the parameters, say, w_1 and w_2. If the function $s(w)$ has the conjugacy property $s(-w)=\overline{s(w)}$ (see formula (7) or Example 1 below), then these parameters are explicitly related: $w_2=-w_1$. In this case, the function $k(w)$ is even, and due to its rational nature, the numerator and denominator contain terms depending on even powers of w; therefore, the calculations can be simplified by halving the degree of the polynomial using the parametric change $v=w^2$.

4.3. Detection of Adjacent D-Partition Regions

For the arcs of the stability region boundary defined by parts of the main curve or segments, in Step 4 of Algorithm 1, it is necessary to determine the D-partition regions separated by a given arc (segment). For the main curve of the D-partition, one can employ the classical hatching rule [7] since the direction of the normal to the boundary Γ inside the root localization region is known; like the mapping $w \to k$, the main curve formula (18) can be used to determine the normal to the curve toward the region with a larger number of stable roots. For segments and lines bounding the stability region, such a rule does not exist.

The idea is to numerically determine the number of stable roots on different sides of segments or curves: by stepping a small distance on both sides along the normal from an arbitrary point $k_\ell(w_0),\ w_0 \in W_\ell,$ on the segment or curve $k_\ell(w_0)$, where w_0 is an inner point of the interval W_ℓ (e.g., its middle). As a normal for a curve arc, we can take the vector $\left(k_2'(w),\ -k_1'(w)\right)$ orthogonal to the tangent $k'(w)$; as a normal for a line segment, the vector $(p_2, -p_1)=(a, b)$ orthogonal to the direction vector. It is possible to apply randomized algorithms with both the parameter $w \in W_\ell$ and the distance chosen randomly, e.g., according to the Laplace distribution.

Another alternative is to use the one-dimensional D-partition along the normal line. In this case, it suffices to check the number of roots only on the line segments adjacent to the boundary.

5. EXAMPLES

Let us demonstrate the constructive D-partition and its applications with examples.

Example 1 [27, *p. 77*]. Consider a continuous-time system with the transfer function $\dfrac{(s-1)(s-2)}{(s+1)\left(s^2+s+1\right)}$ closed by the PI controller $k_1+\dfrac{k_2}{s}$. The characteristic polynomial of the closed-loop system is

$$G(s, k_1, k_2) = k_1 s(s-1)(s-2) + k_2(s-1)(s-2) + s(s+1)\left(s^2+s+1\right). \tag{37}$$

We require the closed-loop system to have a given stability margin σ. Then the boundary of the root localization region is described by the function $s(w)=-\sigma+jw$. Let us choose $\sigma=0.2$. Since the polynomial (37) has real coefficients, it suffices to take the upper part of the boundary, $W=[0, \infty)$. The D-partition regions obtained by direct parameter enumeration are shown in Fig.1a.

The degree drop condition (at s^4) takes the form $0\cdot k_1+0\cdot k_2+1=0.$ It has no solution.

We substitute $s(w)$ into equation (37) and separate the real and imaginary parts to get the equations

$$\begin{gathered} k_1\left(3.6w^2-0.528\right)+k_2\left(-w^2+2.64\right) \\ +w^4+1.04w^2-0.1344=0, \\ k_1\left(-w^3+3.32w\right)+k_2\cdot(-3.4w)-1.2w^3+0.408w=0. \end{gathered} \tag{38}$$

Its solution determines the main curve $k(w),\ w\in[0, \infty)$, with the components

$$k_1(w)=\frac{4.6w^4-7.112w^2+0.62016}{-w^4-6.28w^2-6.9696},$$

$$k_2(w)=\frac{-w^6+8.68w^4-5.4208w^2-0.230784}{-w^4-6.28w^2-6.9696}.$$

Compared to the expressions (13), the common factor w in the numerator and denominator is eliminated here. The determinant (17) takes the form $-\left(w^4+6.28w^2+6.9696\right)w.$ It has the unique real root $w_0=0$. Note that the example satisfies the conditions specified at the end of subsection 4.2; upon separating the complex equation into two real ones, one of the resulting equations will contain only even powers of *w*, and the other will contain only odd powers of *w*. In addition, the function $k(w)$ contains only even degrees, and it can be replaced by an equivalent representation of the curve with rational functions of lower order:

$$k_1(v)=\frac{4.6v^2-7.112v+0.62016}{-v^2-6.28v-6.9696},$$

$$k_2(v)=\frac{-v^3+8.68v^2-5.4208v-0.230784}{-v^2-6.28v-6.9696},$$

$$v\in[0, \infty).$$

Next, for simplicity, consider the initial parameters w. The critical frequency $w_0=0$ is associated with the singular line corresponding to equation (38):

$$-0.528k_1+2.64k_2-0.1344=0$$

with the parameterization $p+td$, where $p=(-0.00979021; 0.04895105)^{\mathrm{T}}$ and $d=(-2.64; -0.528)^{\mathrm{T}}$. The D-partition is shown in Fig. 2a.

The main curve intersects the singular line at the two points $k_{(1)}=(0.17279287;\ 0.08546766)$ and $k_{(2)}=(-1.82840784;\ -0.31477248)$ corresponding to the parameter values $w_1=0.70951628$ and $w_2=2.70323801$, respectively; they are calculated according to subsection 2.1. There are no self-intersections of the main curve.

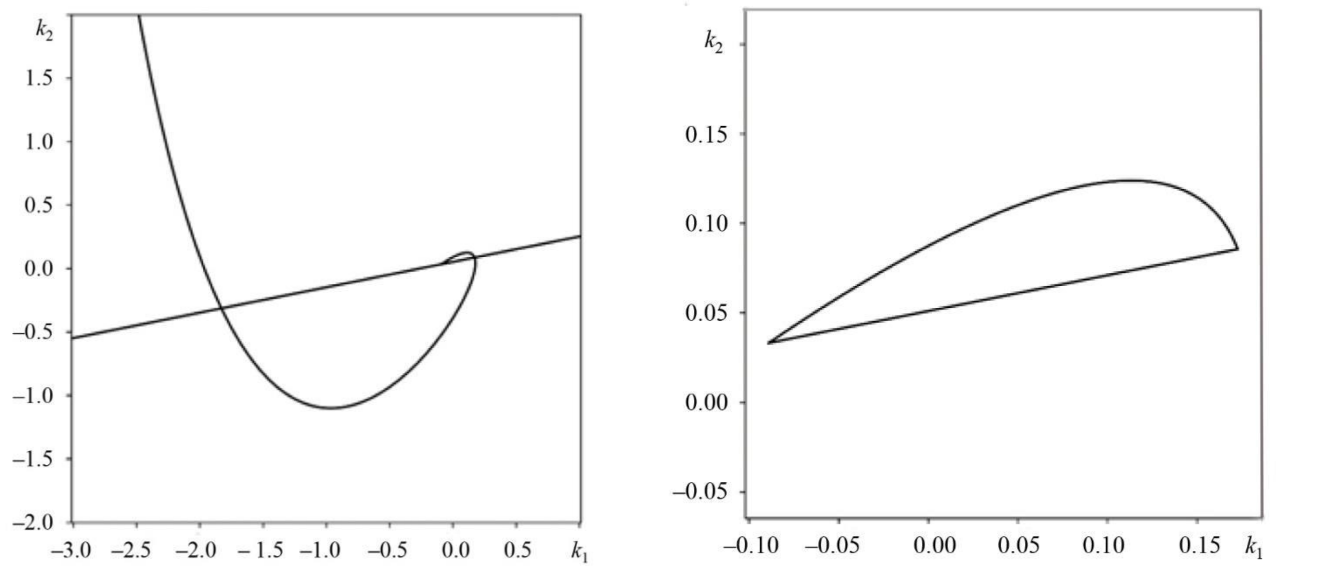

Fig. 2. (a) the D-partition for Example 1 and (b) the stability region of the polynomial.

The points w_1 and w_2 are associated with the parameter values $t_1 = -0.06916025$ and $t_2 = 0.68887031$, respectively, on the line. The main curve has the limit point $k_{(0)} = k(0)$

$$= \left(-\frac{0.62016}{-6.9696} = -0.08898072; \ \frac{-0.230784}{-6.9696} = 0.03311295\right)^{\mathrm{T}}$$

as $w \to 0$, which is associated with the point with $t_0 = 0.02999640$ on the line.

The stability region is bounded by one arc of the fundamental curve $k(w), w \in [0, w_1]$, and the segment $p + td$, $t \in [t_1, t_0]$ (see Fig. 2b).

Example 2 [10]. Consider the characteristic polynomial $G_0(z, k_1, k_2) = z^n + k_1 z^{n-1} + (1+\varepsilon) z^{n-2} + k_2$ of a discrete-time system; in the paper [10], the D-partition for this polynomial was obtained using trigonometric functions. The stability of the discrete polynomial is equivalent to the Hurwitz property of the polynomial

$$G(s, k_1, k_2) = (s+1)^5 + (1+\varepsilon)(s-1)^2 (s+1)^3 + k_1 (s-1)(s+1)^4 + k_2 (s-1)^5,$$

obtained from $G_0(z, k_1, k_2)$ by the Möbius transformation. The boundary of the root localization region (2) is described by the function $s = jw$, $w \in [0, \infty)$, considering the symmetry with respect to the real axis. Let us choose $n = 5$ and $\varepsilon = 0.1$.

Direct parameter enumeration yields the approximate boundaries of the D-partition regions presented in Fig. 1b.

The boundary of the D-partition regions consists of the only main curve

$$k_1(w) = \frac{-16.6\, w^8 + 128.8\, w^6 - 221.2\, w^4 + 128.8\, w^2 - 16.6}{8\left(w^8 - 6\, w^6 + 6\, w^2 - 1\right)},$$

$$k_2(w) = \frac{-0.2\, w^8 - 0.8\, w^6 - 1.2\, w^4 - 0.8\, w^2 - 0.2}{8\left(w^8 - 6\, w^6 + 6\, w^2 - 1\right)}$$

and two singular lines. The determinant (17) takes the form $8w\left(w^8 - 6w^6 + 6w^2 - 1\right)$. It has the four nonnegative real roots $\left\{0, -1+\sqrt{2}, 1, 1+\sqrt{2}\right\}$, which determine the continuity intervals of the main curve. Of these, only the zero root corresponds to the first singular line $-k_1 - k_2 + 2.1 = 0$ with the parameterization $dt + p$, where $p = (1.05; 1.05)$ and $d = (1; -1)$. In addition, there exists the limit point $k_{(0)} = (2.075; 0.025)$ as $w \to 0$, which lies on this line at $t = 1.025$ (see Lemma 1).

The degree drop condition is satisfied by the second singular line $k_1 + k_2 + 2.1 = 0$ with the parameterization $dt + p$, where $p = (-1.05; -1.05)$ and $d = (-1; 1)$. The main curve has the second limit point $k_{(\infty)} = (-2.075; -0.025)$ as $w \to \infty$, which lies on the above singular line (of degree drop) at $t = 1.025$ (see Lemma 4).

Let us choose the localization region $\boldsymbol{K} = [-2.5, 2.5] \times [-1.5, 1.5]$. The main curve lies in the region $\boldsymbol{K}$

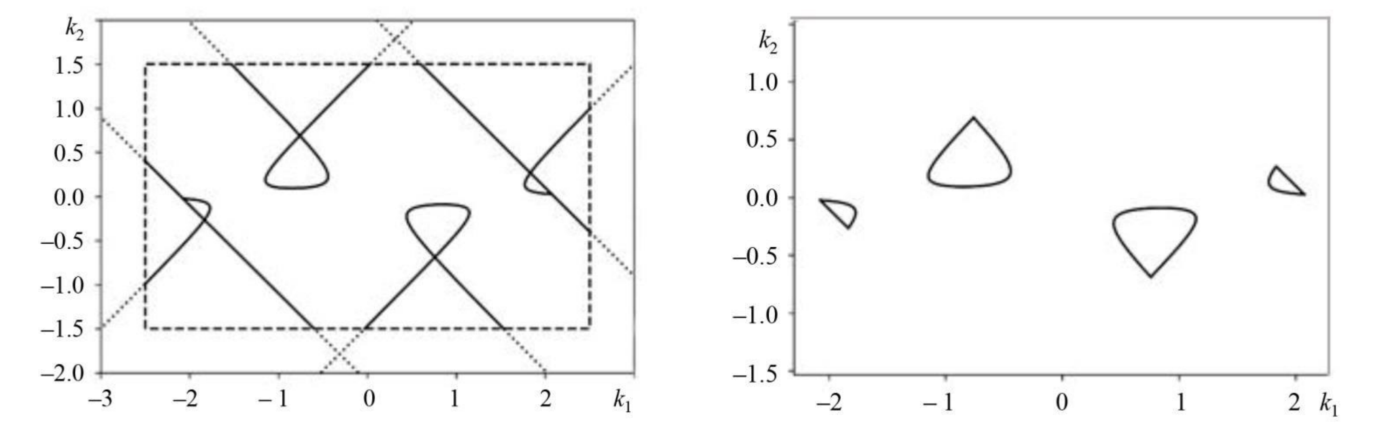

Fig.3: (a) the D-partition for Example 2 and its localization in the rectangle and (b) the components of the stability region.

for $w \in [0,\ 0.40398478]$, $[0.42121903,\ 0.98346081]$, $[1.01681733,\ 2.37406180]$, $[2.47534075,\ \infty)$. The singular lines belong to the region $\boldsymbol{K}$ for $t \in [-0.45,\ 1.45]$ and $t \in [-0.45,\ 1.45]$, respectively (the intervals are the same for both lines). The localized D-partition is presented in Fig.3a.

On the first interval $[0,\ 0.40398478]$, the main curve starts (as $w \to 0$) on the first singular line and intersects it at $w = 0.37796447$. On the last interval $[2.47534075,\ \infty)$, the main curve intersects the second singular line at $w = 2.64575131$ and ends on it as $w \to \infty$.

On the intervals $[0.42121903,\ 0.98346081]$ and $[1.01681733,\ 2.37406180]$, the curve intersects itself at $w = 0.42972375$ and 0.96431209 (the first self-intersection) as well as at $w = 1.03700867$ and 2.32707640 (the second self-intersection).

Thus, the stability region consists of four components (see Fig.3b):

1) the segment of the first singular line for $t \in [1.025,\ 1.45]$ and the arc of the main curve for $w \in [0,\ 0.37796447]$;

2) the arc of the main curve for $w \in [0.42972375,\ 0.96431209]$;

3) the arc of the main curve for $w \in [1.03700867,\ 2.32707640]$;

4) the segment of the second singular line for $t \in [1.025,\ 1.45]$ and the arc of the main curve for $w \in [2.64575131,\ \infty)$.

The last arc of the curve can be written with a modified parameterization, see the proof of Theorem 1, as $k_u(u) = k_\ell(1/u)$, $u \in [0,\ 0.37796447]$, where the value at $u = 0$ is defined and coincides with $k_{(\infty)}$:

$$k_{u,1}(u) = \frac{16.6\,u^8 - 128.8\,u^6 + 221.2\,u^4 - 128.8\,u^2 + 16.6}{8\left(u^8 - 6\,u^6 + 6\,u^2 - 1\right)},$$

$$k_{u,2}(u) = \frac{0.2\,u^8 + 0.8\,u^6 + 1.2\,u^4 + 0.8\,u^2 + 0.2}{8\left(u^8 - 6\,u^6 + 6\,u^2 - 1\right)}.$$

Note that the resulting parameterization coincides with the original one in w up to the sign, and the interval coincides with the interval of the first component. This is due to the symmetry of the original root localization set of the discrete system (the unit circle) and its parameterization. ♦

CONCLUSIONS

For a polynomial linearly dependent on two parameters, the boundaries of each D-partition region, including the stability region, have been explicitly described by parameterizing the curves and segments on the parameter plane. The stability of the polynomial has been understood in a generalized sense: all its roots lie in a given subset of the complex plane (a root localization region), which may differ from the left half-plane. The constructive D-partition method has been proposed, including an algorithm for finding the boundaries of all stability region components without unnecessary parts. Moreover, if the boundary of a root localization region is described by a piecewise rational curve, then the boundary of the stability region is a finite set of rational curve arcs and segments. In this case, the arcs of rational curves are defined on closed finite intervals of parameters. The results are applied to approximate the stability region and its boundary, as well as to analyze robustness; see part II of the study.

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