CONSTRUCTIVE D-PARTITION FOR TWO PARAMETERS ENTERING A POLYNOMIAL LINEARLY. PART II: Approximation of Stability Regions and Robustness Analysis

A. A. Tremba

Trapeznikov Institute of Control Sciences, Russian Academy of Sciences, Moscow, Russia
Moscow Institute of Physics and Technology, Dolgoprudny, Russia

✉ atremba@ipu.ru

Abstract. For a polynomial linearly dependent on two parameters, several methods are proposed to approximate its stability region with respect to a given root localization region (also called a root clustering set in the literature). The first method is to apply a sufficiently uniform grid to the stability region boundary that ensures its complete coverage with a given accuracy. The second (semi-grid) method yields an internal approximation of the stability region using line segments or curve arcs bounded by the stability region. The third method is to cover the stability region boundary with simple sets (cells) in order to obtain piecewise linear internal and external approximations of the stability region. All methods are based on the constructive D-partition (constructive D-decomposition) method, which describes the stability region boundary as a set of line segments and rational curve arcs. The exact stability radius and its simple estimate are derived in the parameter plane. Implementation of all methods and algorithms is reduced to finding the real roots of polynomials.

Keywords: constructive D-partition, rational curves, approximation of the stability region, sufficiently uniform grid, grid methods, semi-grid methods, support function, stability radius.

INTRODUCTION

We analyze the location of the roots of a polynomial of degree n with two parameters entering it linearly:

$$G(s,\ k_1,\ k_2)=k_1P(s)+k_2Q(s)+R(s). \tag{1}$$

For a given root localization region $\mathbf{D}\subset\mathbb{C}$, it is necessary to determine a set, called the *stability region*, on the parameter plane $(k_1,\ k_2)$, each point of which corresponds to a stable polynomial:

$$D_n=\{(k_1,\ k_2) : \text{all } n \text{ roots of } G(s,\ k_1,\ k_2) \text{ lie in } \mathbf{D}\}. \tag{2}$$

Recall that the stability (D-stability[1]) of roots and a polynomial is defined relative to a given root localization region $\mathbf{D}$ and generalizes the concept of Hurwitz and Schur polynomials corresponding to stable (in the classical sense) continuous- and discrete-time systems. The problem of finding the stability region arises in the controller design of linear control systems in the case when two controller parameters enter the characteristic polynomial of degree n linearly, or in the stability analysis of dynamics systems depending on two parameters (or one parameter, see subsection 2.1).

By selecting a root localization region $\mathbf{D}$, it is possible to ensure a desired stability degree or damping ratio of a closed-loop system, etc. [2–4]. By assumption, $\mathbf{D}$ is a regular open set. In this case, the stability region is also an open set.

This paper is the second part of the study [5], where the constructive D-partition was proposed, i.e., a description of the stability region boundary using the D-partition of the parameter plane. Recall the idea of this method; for details, see [6–10].

The first set is defined by a mapping of the boundary $\partial\mathbf{D}$ of the root localization region onto the parameter plane using the *main equation*

[1] There exists the concept of *D-stable matrices* with a significantly different meaning [1].

$$K_{bnd} = \{(k_1, k_2) : G(s,\ k_1, k_2) = 0_{\mathbb{C}},\ s \in \partial\mathbf{D}\}. \quad (3)$$

The second set is defined by the *degree drop condition* for the coefficient at s^n:

$$K_{deg} = \{(k_1, k_2) : G_n(k_1, k_2) = 0_{\mathbb{C}}\}. \quad (4)$$

The sets (3) and (4) define a D-partition, i.e., a division of the parameter plane into simply connected regions, some forming the stability region. In each D-partition region, the number of stable roots remains the same when continuously varying the parameters within this region, provided that the coefficients of the polynomial depend continuously on the parameters. The equation in formula (3) defines the marginal case in which at least one of the polynomial roots lies on the boundary $\partial\mathbf{D}$.

Let us present several properties of the D-partition for the polynomial (1), together with the results from part I of the study [5].

First, the set K_{bnd} consists of the so-called main curve and a set of (straight) lines, called *singular lines*, defined by the equations

$$a_i k_1 + b_i k_2 + c_i = 0,\ \ i = 1,\ldots,\ M, \quad (5)$$

and K_{deg} is a singular line (possibly an empty set) defined by the equation

$$P_n k_1 + Q_n k_2 + R_n = 0.$$

The stability region boundary belongs to the union of these sets:

$$\partial D_n \subset K_{bnd} \cup K_{deg}.$$

Second, if the boundary $\Gamma = \partial\mathbf{D}$ of a root localization region consists of a finite set of rational curve arcs,

$$\Gamma = \bigcup\nolimits_{\ell} \Gamma_{\ell},\quad \Gamma_{\ell} = \{s_{\ell}(w) \in \mathbb{C} : w \in W_{\ell}\},\quad \ell = 1,\ldots,\ L,$$

then the main equation in (3) is equivalent to L systems of two polynomial equations with some polynomials $P_{\ell,1}(w),\ P_{\ell,2}(w),\ Q_{\ell,1}(w),\ Q_{\ell,2}(w),\ R_{\ell,1}(w)$, and $R_{\ell,2}(w)$, depending on the initial polynomials P, Q, R and the rational functions $s_{\ell}(w)$:

$$\begin{cases} k_1 P_{\ell,1}(w) + k_2 Q_{\ell,1}(w) + R_{\ell,1}(w) = 0 \\ k_1 P_{\ell,2}(w) + k_2 Q_{\ell,2}(w) + R_{\ell,2}(w) = 0 \\ w \in W_{\ell},\ \ell = 1,\ \ldots,\ L. \end{cases} \quad (6)$$

The solution of each system is a rational curve (*main curve*) of the form

$$\begin{aligned} k_{\ell,1}(w) &= \frac{1}{\det T_{\ell}(w)}\big(R_{\ell,2}(w)\ Q_{\ell,1}(w) - R_{\ell,1}(w)\ Q_{\ell,2}(w)\big), \\ k_{\ell,2}(w) &= \frac{1}{\det T_{\ell}(w)}\big(R_{\ell,1}(w)\ P_{\ell,2}(w) - R_{\ell,2}(w)\ P_{\ell,1}(w)\big), \\ & w \in W_{\ell}, \end{aligned} \quad (7)$$

and a set of singular lines of the form (5) corresponding, for each $\ell = 1,\ \ldots,\ L$, to such real roots (critical frequencies) of the equation

$$\det T_{\ell}(w) = P_{\ell,1}(w) Q_{\ell,2}(w) - Q_{\ell,1}(w) P_{\ell,2}(w) = 0,\ \ w \in W_{\ell},$$

for which system (6) has a solution. Hereinafter, the subscript ℓ will be occasionally omitted for brevity.

Third, it has been proven that for a localized D-partition (bounded to a compact set $\boldsymbol{K}$), the above rational curves are defined (or redefined) on closed intervals (segments), and singular lines become segments. Moreover, if the boundary of $\boldsymbol{K}$ consists of a finite set of rational curve arcs, the D-partition will also consist of a finite number of rational curve arcs and segments. The constructive D-partition method proposed allows determining the parameterization intervals of these rational curve arcs and segments as sections of singular lines by finding the real roots of polynomials.

In particular, the boundary of the stability region (2) within the set $\boldsymbol{K}$ consists of a finite set of rational functions of the form (7) and segments. It is convenient to represent the segments in parametric form as

$$k(t) = t\ d + p,\ \ t \in [t_1, t_2],\ \ p, d \in \mathbb{R}^2,\ \ d \neq 0,$$

where the vectors $d = (-b,\ a)^{\mathrm{T}}$ and $p = -\left(\dfrac{ac}{a^2 + b^2},\ \dfrac{bc}{a^2 + b^2}\right)$ are obtained from the equations of the lines (5); hereinafter, the subscripts are omitted for brevity.

The parameter and intersection point of the line initially defined as $k(t) = t\ d + p,\ t \in (-\infty,\ +\infty)$, and the line (5) are given by

$$\begin{aligned} t^* &= -\frac{c + a p_1 + b p_2}{a d_1 + b d_2} = -\frac{c + (a,\ b) p}{(a,\ b) d}, \\ k^* &= t^* d + p = -\frac{c + (a,\ b) p}{(a,\ b) d}\ d + p. \end{aligned} \quad (8)$$

We emphasize one feature of constructing a localized D-partition. The rational curve arcs of the boundaries of the stability region (7) are defined on closed intervals, but there may exist an infinite interval, e.g., $(-\infty, +\infty)$ or $[w_1, +\infty)$. The domain is reduced to a finite closed interval by an appropriate change of the parameter, see subsection 1.2.

The Intersection of Main Curves, Lines, and Localized D-Partition

Recall some results concerning the intersection of rational curves and lines. In particular, the intersection of the singular lines (5) and the main curve (7) of the D-partition is given by the equations

$$\begin{gathered} a_i\left(R_2(w)Q_1(w)-R_1(w)Q_2(w)\right) \\ +b_i\left(R_1(w)P_2(w)-R_2(w)P_1(w)\right) \\ +c_i\left(P_1(w)Q_2(w)-Q_1(w)P_2(w)\right)=0, \\ i=0,\ldots,K. \end{gathered} \tag{9}$$

Here, the subscript 0 corresponds to the singular line K_{deg}. The equations are polynomial with respect to w, and their real roots w_m, $m=1,\ldots$, can be calculated explicitly. These roots, together with the critical frequencies w_i, split the interval W into segments and intervals corresponding to the simple continuous parts of the D-partition boundaries, the arcs of the main curve. In fact, these arcs form the curved part of the D-partition. The intersection points themselves are determined from equation (7) as $k(w_m)=(k_1(w_m), k_2(w_m))$. Together with the limit points $\lim_{w\to w_m} k(w)$ (if any), they divide the main curve into arcs and, moreover, singular lines into segments or infinite intervals (rays). In the case of a localized D-partition, there are no infinite intervals. Similarly, one can find the self-intersection and intersection points of several main curves; see part I of the study [5].

It is convenient to perform a localized D-partition in a rectangle $\boldsymbol{K}=[\underline{k}_1, \overline{k}_1]\times[\underline{k}_2, \overline{k}_2]$, whose boundaries are vertical and horizontal segments. For these segments, equation (9) gets simplified since it suffices to consider each component of the rational curve (7) separately:

$$\begin{gathered} R_2(w)Q_1(w)-R_1(w)Q_2(w) \\ =x\left(P_1(w)Q_2(w)-Q_1(w)P_2(w)\right), \\ x=\underline{k}_1, \overline{k}_1, \end{gathered} \tag{10}$$

$$\begin{gathered} R_1(w)P_2(w)-R_2(w)P_1(w) \\ =y\left(P_1(w)Q_2(w)-Q_1(w)P_2(w)\right), \\ y=\underline{k}_2, \overline{k}_2. \end{gathered} \tag{11}$$

Having solved each of these equations, one should check that the second coordinate is within the desired interval (i.e., the main curve intersects a rectangle's side). For example, if the roots of equation (10) are w_m, then only those are selected for which $k_2(w_m)\in[\underline{k}_2, \overline{k}_2]$, and vice versa.

In what follows, based on the constructive description of the stability region boundary, we propose several approximations for the stability region boundary or the region itself. In addition, the method proposed is applied to perform stability region analysis (subsection 4.3) and robust analysis (subsection 4.4). The resulting parameterization of the stability region boundary can be used for subsequent optimization within the stability region and other problems related to the analysis and design of controllers with performance characteristics determined by the root localization region.

1. POINTWISE APPROXIMATION OF THE D-PARTITION BOUNDARY

Consider a localized D-partition on a bounded set $\boldsymbol{K}$, e.g., on a rectangle $[\underline{k}_1, \overline{k}_1]\times[\underline{k}_2, \overline{k}_2]$. After applying the constructive D-partition algorithm (8), (10), (11), it is possible to determine the intersection of the singular lines (5) and main curves (7) of the D-partition with the boundaries of the set $\boldsymbol{K}$. The resulting stability region D_n will be bounded by a finite number of segments $k_m(t)$ and main curve arcs $k_\ell(w)$:

$$\partial D_n=\cup_m k_m(t)\cup_\ell k_\ell(w).$$

According to Lemmas 1 and 4 from part I of the study [5], each of the boundary parts $k_m(t)$, $k_\ell(w)$ is parameterized by a segment $W_\ell=[w_{\ell,1}, w_{\ell,2}]$ or a closed infinite interval. Moreover, by Theorem 1 [5], it is possible to replace an infinite parameterization interval with a segment (or segments). This replacement, in the context of curve approximation, will be considered separately in subsection 1.2.

The problem is to select a finite set of points (*nodes*) forming a grid $K_{grid}=\{k_r,\ r=1,\ldots\}$ on these curves and segments in order to approximate the stability region boundary with a given fineness $\rho>0$.

The grid fineness relative to a set is defined by the Hausdorff distance between the set of nodes and the given set (in the case under consideration, the stability region boundary). Let us select the nodes lying on the boundary: $k_r \in \partial D_n$. They satisfy the condition

$$\mathrm{dist_H}\left(K_{grid}, \partial D_n\right) = \max_{k \in \partial D_n} \min_r \|k - k_r\| \le \rho. \quad (12)$$

Thus, an appropriate grid is built if and only if circles of radius ρ, centered at k_r, cover the boundary. Such an approximation and the corresponding grid will be called *sufficiently uniform*.

The boundary is easily approximated for each of its arcs separately, by explicitly adding nodes at the junction points of the boundary arcs. For the segments $k_m(t) = t\,d_m + p_m$, $t \in \left[t_{m,1}, t_{m,2}\right]$, we propose an obvious uniform approximation with N intervals, including both endpoints of the segment:

$$k_{m,r} = \left(t_{m,1} + r\delta\right) d_m + p_m, \; r = 0, 1, \ldots, N,$$
$$\delta = \frac{t_{m,2} - t_{m,1}}{N}, \; N = \left\lceil \frac{t_{m,2} - t_{m,1}}{2\rho} \right\rceil. \quad (13)$$

The distance between two nodes does not exceed 2ρ. This grid is optimal among all grids containing endpoints, in the sense of its uniformity and a minimum number of nodes.

The situation is more complicated for boundary arcs $k_\ell(w)$ defined on closed intervals. Recall that on these intervals, the function $k_\ell(w)$ is continuous. Consider one arc, $k(w)$, $w \in W$, omitting the subscript.

Formally, a smooth curve arc can be uniformly approximated, with any given accuracy, by the so-called natural parameterization of a curve arc using the curve length function from the point $k(w_1)$ in the direction of increasing the parameter value. The length function can be defined by the "speed" of a point along the curve, $v(w) = \|k'(w)\| = \sqrt{k_1'(w)^2 + k_2'(w)^2}$, which will be called the parametric speed (or simply the speed). The length function is given by

$$\lambda_{w_1}(w) = \int_{w_1}^{w} v(\tau)\, d\tau, \; w \ge w_1,$$
$$\text{and} \quad (14)$$
$$\lambda_{w_1}(w) = -\int_{w_1}^{w} v(\tau)\, d\tau, \; w < w_1.$$

For a monotonic function $\lambda(w)$, $w \ge w_1$, the inverse function $w(\lambda)$ is built. Then, the segment $\left[0, \lambda(w_2)\right]$ is evenly split by the numbers λ_r into segments of a maximum length of 2ρ, similar to formula (13). The resulting set $k_r = k\left(w(\lambda_r)\right)$ divides the curve $k(w)$, $w \in W$, into arcs of equal length, each no longer than 2ρ. Due to the continuity of the curve and the triangle rule, the distance from each arc point to one of the neighboring nodes, including the endpoints $k(w_1)$ and $k(w_2)$, does not exceed ρ. Unfortunately, the integral (and then the inverse function) cannot usually be calculated in analytic form since the speed vector components $k_1'(w)$, $k_2'(w)$ are defined by rational functions.

1.1. Algorithms for Building a Sufficiently Uniform Grid

We propose two algorithms with an upper estimate of the speed on subintervals and sequential addition of grid nodes. A feature of the algorithms is non-uniform speed estimation for more efficient use of grid nodes. The resulting grid satisfying condition (12) will be sufficiently uniform; in the first algorithm, the nodes on the curve will be located regularly (with respect to w).

Let us calculate the set of points containing the extremum values of the speed within the segment W_ℓ using the necessary optimality condition:

$$\mathrm{Arg\ extr}_{w \in W_\ell} v(w)$$
$$\in \mathrm{Arg\ extr}_{w \in W_\ell} v(w)^2 \in \{w : \left(v(w)^2\right)' = 0\} \quad (15)$$
$$= \{w : k_{\ell,1}'(w) k_{\ell,1}''(w) + k_{\ell,2}'(w) k_{\ell,2}''(w) = 0\}.$$

According to the last expression, due to the rationality of the functions, stationary points are found by calculating the roots of the corresponding polynomial. Denoting by $\{w_s\}$ the set of stationary points, we define a function $v_{\max}(w_a, w_b)$ returning the maximum value of the speed on an arbitrary segment $\left[w_a, w_b\right] \in W_\ell$, $w_a < w_b$:

$$v_{\max}(w_a, w_b) = \max\{v(w_a), v(w_b),$$
$$\max\{v(w_s) : w_s \in (w_a, w_b)\}\}. \quad (16)$$

Formula (16) employs Fermat's theorem, i.e., the fact that on a segment (closed interval), the maximum value of a continuous differentiable function is achieved either at the ends of or within this segment.

For inner maximum points, the necessary optimality condition holds, which is valid for the points from the set $\{w_s\}$. The function $v_{\max}(w_a,\ w_b)$ can be calculated more easily by considering, among the stationary points, only those of (local) maximum defined by the second-order optimality condition $v''(w_s)<0$. If the second derivative is equal to zero, it is necessary to analyze higher-order derivatives or treat such points as candidates.

Thus, according to formula (14), on the segment $[w_a,\ w_b]$, the length of the curve from the endpoint $k_\ell(w_a)$ to the intermediate point $k_\ell(w)$ is estimated as

$$\lambda_{w_a}(w) \le v_{\max}(w_a, w_b)\cdot(w-w_a),\ w\in[w_a,\ w_b]. \quad (17)$$

A similar estimate is valid for the curve length in the opposite direction, from w_b to w. Based on these estimates and the triangle rule, for each half of the curve before and after $w_c=(w_a+w_b)/2$, the curve lengths satisfy the following upper and lower bounds:

$$\|k_\ell(w_a)-k_\ell(w)\| \le \lambda_{w_a}(w) \le \lambda_{w_a}(w_c)$$

$$\le v_{\max}(w_a,\ w_b)\cdot\frac{w_b-w_a}{2},\ w\in[w_a,\ w_c],$$

$$\|k_\ell(w_b)-k_\ell(w)\| \le \lambda_{w_b}(w) \le \lambda_{w_b}(w_c)$$

$$\le v_{\max}(w_a,\ w_b)\cdot\frac{w_b-w_a}{2},\ w\in[w_c,\ w_b].$$

Therefore, one arrives at the sufficient criterion: if $v_{\max}(w_a,\ w_b)\cdot(w_b-w_a)\le 2\rho$, then the arc of the curve $k_\ell(w)$, $w\in[w_a,\ w_b]$, lies in the union of two circles of radius ρ with centers at $k(w_a)$ and $k(w_b)$.

We propose the following iterative algorithm for finding a sufficiently uniform grid K_{set} that covers the arc of the curve $k_\ell(w)$, $w\in W_\ell$, using the function $v_{\max}(w_a,\ w_b)$ on subintervals.

Algorithm 1. A sufficiently uniform grid for a rational curve arc.

Input: a rational curve $k_\ell(w)$, $w\in W_\ell=[w_1,\ w_2]$, whose denominator does not vanish on the interval W_ℓ, and a fineness parameter $\rho>0$.

1. Compute the stationary points $\{w_s\}$ for the speed by formula (15) (or the local maximum points) belonging to W_ℓ, and determine the upper bound function $v_{\max}(w_a,\ w_b)$.

2. Set the initial list of nodes as a list of two nodes $\{k(w_1),\ k(w_2)\}$. Set the initial list of intervals as a list containing one element, i.e., the interval $[w_1,\ w_2]$.

3. If the list of intervals is empty, terminate the algorithm and return the list of nodes. Otherwise, select any interval (e.g., the first or leftmost one), and designate it as $[w_a,\ w_b]$; remove this interval from the list.

4. If $v_{\max}(w_a, w_b)\cdot(w_b-w_a)\le 2\rho$, go to Step 3.

5. Split the interval $[w_a,\ w_b]$ in half at the point $w_c=(w_a+w_b)/2\in(w_a, w_b)$; add $k(w_c)$ to the list of nodes, and add the intervals $[w_a,\ w_c]$ and $[w_c,\ w_b]$ to the list of intervals; get back to Step 3.

Output: a set of parameters from the interval $[w_1,\ w_2]$ and the corresponding set of nodes (a sufficiently uniform grid).

Algorithm 1 is finite, as the parametric speed is bounded above on the entire interval W_ℓ; see the estimate (18) below. By construction, the grid obtained using Algorithm 1 covers the curve with circles of radius ρ and, moreover, divides the curve into arcs of a maximum length of 2ρ. Algorithm 1 can be trivially generalized to any curves with a speed estimate of the form $\bar v(w)\ge v(w)$. In addition, the algorithm is effectively implemented if this speed estimate is a polynomial or other function with simple or precomputed extrema (maxima). Figure 1 shows the parametric speed and a sufficiently uniform grid for the arc of the stability region boundary in Example 1 (see Section 5).

In the particular case of using the maximum speed $\bar v(w)=v_{\max}(w_1,\ w_2)=\text{const}$ as the upper estimate, Algorithm 1 outputs a uniform (binary) grid of the form

$$k_r=k(w_r),\ \ w_r=w_1+r\,\delta_2,\ \ \delta_2=\frac{w_2-w_1}{2^M},$$

$$M=\max\left\{0,\left\lceil \log_2\frac{v_{\max}(w_1,w_2)(w_2-w_1)}{\rho}\right\rceil-1\right\}, \quad (18)$$

$$r=0,\ \ldots,\ 2^M.$$

In terms of the number of nodes, the uniform binary grid is generally worse than (but at most twice as bad as) the uniform grid based on the maximum speed

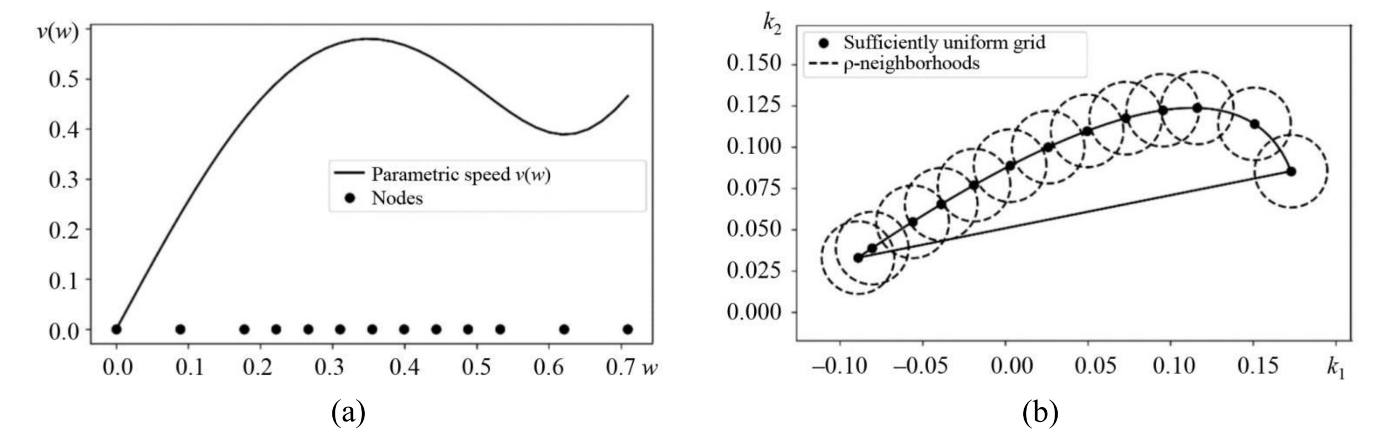

Fig. 1. (a) The parametric speed for the curved part of the stability region boundary in Example 1 and (b) the sufficiently uniform grid obtained by Algorithm 1, together with the set of covering circles.

estimate $v_{\max}(w_1, w_2)$, built similarly to formula (13) as

$$w_r = w_1 + r\delta,\ r = 0, 1, \ldots, N,\ \delta = \frac{w_2 - w_1}{N},$$
$$N = \left\lceil \frac{v_{\max}(w_1, w_2)(w_2 - w_1)}{2\rho} \right\rceil. \tag{19}$$

In Algorithm 1, the distance between the grid nodes $k_r = k(w_r)$ is adapted to the maximum speed in each subinterval, and this algorithm has higher efficiency in practice (significantly fewer nodes compared to the uniform grid (19)). The reason consists in that the speed function is related to rational functions, therefore being significantly nonuniform (especially at the edges of the definitional intervals).

Note that the segment is split by taking its middle point $w_c = (w_a + w_b)/2$, and the intervals yielded by Algorithm 1 will be multiples of the smallest interval $\delta_2 = 2^{-M}(w_2 - w_1)$ for the same M. Indeed, the maximum speed value is reached on at least one of the subintervals.

The function $v_{\max}(w_a, w_b)$ in formula (17) is calculated trivially if the interval contains no extrema:

$$v_{\max}(w_a, w_b) = \max\{v(w_a), v(w_b)\}$$
$$\text{if } \nexists w_s \in (w_a, w_b).$$

Using this fact, Algorithm 1 can be simplified by immediately splitting the interval $[w_1, w_2]$ by stationary points (or maximum points). Then, on each subinterval, the maximum speed will be determined exclusively by the speeds at the endpoints of the segment, and the same applies to the subsequent division into smaller segments.

Algorithm 2. A sufficiently uniform grid for a piecewise rational curve arc (the simplified version).

Input: a rational curve $k_\ell(w)$, $w \in W_\ell = [w_1, w_2]$, whose denominator does not vanish on the interval W_ℓ, and a fineness parameter $\rho > 0$.

1. Compute the stationary points $\{w_s\}$ for the speed by formula (15) (or the local maximum points) belonging to W_ℓ; split the interval $[w_1, w_2]$ by these points.

2. Set the initial list of nodes as $\{k(w_1), k(w_2)\} \cup \{k(w_s)\}$. Set the initial list of intervals obtained by splitting the interval $[w_1, w_2]$ by the set $\{w_s\}$.

3. If the list of intervals is empty, terminate the algorithm and return the list of nodes. Otherwise, select any interval (e.g., the first or leftmost one), and designate it as $[w_a, w_b]$; remove this interval from the list.

4. If $\max\{v(w_a), v(w_b)\} \cdot (w_b - w_a) \le 2\rho$, get back to Step 3.

5. Split the interval $[w_a, w_b]$ in half at the point $w_c = (w_a + w_b)/2 \in (w_a, w_b)$; add $k(w_c)$ to the list of nodes, and add the intervals $[w_a, w_c]$ and $[w_c, w_b]$ to the list of intervals; get back to Step 3.

Output: a set of parameters from the interval $[w_1, w_2]$ and the corresponding set of nodes (a sufficiently uniform grid). Algorithms 1 and 2 have the same output.

Compared to Algorithm 1, Algorithm 2 requires fewer computations (no need to check the belonging of the stationary parameters $\{w_s\}$ to the current interval), but the grid turns out to be irregular.

1.2. Infinite Intervals and Reparameterization

Algorithms 1 and 2 are supplemented by the following considerations. First, for a localized D-partition, the finite (possibly redefined) domain interval of an arc of the rational curve $k_\ell(w)$ (7) within the set $\boldsymbol{K}$ is always closed; see Lemmas 1–3 in part I of the study [5]. To apply approximation on infinite intervals with respect to w, a special change of variables is introduced [11]. Without loss of generality, let a rational curve $k_\ell(w)$ be defined and continuous on the interval $W_\ell = [w_1, \infty)$. We choose an endpoint $w_0 < w_1$, i.e., some finite value $w_0 \neq -\infty$, e.g., $w_0 = w_1 - 1$. Since the D-partition is localized, there exists the limit point $k_\infty = \lim_{w\to\infty} k_\ell(w) \in \boldsymbol{K}$; otherwise, Lemma 3 [5] is valid, and it suffices to consider a finite interval on which the curve lies inside the set $\boldsymbol{K}$. With the change of variable $w = w_0 + 1/u$, $u \in U = (0, 1/(w_1 - w_0)] \subset (0, 1]$, we obtain another parameterization of the same curve arc $k_{\ell,u}(u) = k_\ell(w_0 + 1/u)$, $u \in U$; the node $k_{\ell,u}(0)$ is replaced by the same limit point $k_\infty = \lim_{u\to+0} k_{\ell,u}(u)$, and the speed value at this point is considered to be zero. This change allows obtaining a parameterization of the finite boundary arcs on a finite interval. The case $W_\ell = (-\infty, +\infty)$ leads to a division into two finite intervals after replacing the parameter.

A similar redefinition of the speed with a zero value is applied when redefining the rational function at the endpoint of the interval $[w_1, \ldots)$ if w_1 is a root of its denominator but the limit point $\lim_{w\to w_1} k_\ell(w)$ exists.

Note that for a nonlocalized D-partition, parts of the boundary $k_\ell(w)$ may be unbounded due to an unbounded or open interval W_ℓ, including the cases where the boundary of the interval W_ℓ is zero of the denominator for one component of $k_\ell(w)$. In these cases, the limit of $k_\ell(w)$ does not exist, and an unbounded curve cannot be approximated by a finite number of points with a given accuracy. In such cases, the rational curve (7) has asymptotes with the ratio

$$k_{\ell,1}(w) : k_{\ell,2}(w) = R_{\ell,2}(w)\ Q_{\ell,1}(w) - R_{\ell,1}(w)\ Q_{\ell,2}(w)$$
$$: R_{\ell,1}(w)\ P_{\ell,2}(w) - R_{\ell,2}(w)\ P_{\ell,1}(w)$$
$$\text{as } w \to \infty,$$

and a set of points combined with rays can be considered a set approximating the D-partition. Analysis of the quality and accuracy of such an approximation is a separate problem going beyond the scope of this paper.

2. SEMI-GRID METHODS

In addition to the pointwise approximation method (Section 1), several numerical methods can be proposed for estimating both the boundaries and regions of a D-partition. They involve two approaches as follows.

The idea of the first approach is to parameterize the parameter plane k_1, k_2 using two auxiliary parameters and form a discrete grid based on an auxiliary parameter. Next, it is necessary to take the resulting set of lines (continuous with respect to the second auxiliary parameter, e.g., straight lines or curves) and find the intersection of the D-partition boundary and this grid. First, the intersections will provide a pointwise approximation of the boundary. Second, the stability intervals determined on these lines (the latter's parts falling within the stability region) will provide an internal approximation of the stability region. The corresponding methods, based on the stratification of the parameter space, will be called semi-grid ones. The simplest example is a set of horizontal (or vertical) lines.

Essentially, the one-dimensional parameterization (decomposition, slicing, or stratification) of the parameter plane/space and a grid for this parameterization are used. In the case of three or more parameters, a similar idea is employed to visualize a three-dimensional D-partition; see subsection 2.5.

The second approach is to divide the parameter plane into simple cell sets K_i and select those intersecting the stability region boundary. These sets form a covering of the stability region boundary, and, therefore, the complement to their union contains an internal approximation of the stability region on the parameter plane. In this case, an internal approximation of the stability region is formed on one part of the boundary of the covering set union, and an external approximation is formed on the other part (Fig. 3). We propose more effective checks for the intersection of the D-partition boundary and the sets K_i by selecting the latter as elements of a regular grid.

Both approaches use the constructive D-partition in the form of a set of arcs of stability region boundaries or the one-dimensional D-partition. For the second approach, the constructive D-partition is necessary not to "skip" the stability region components lying entirely in one of the cells.

We will present several semi-grid methods, and then, in Section 3, several grid methods, preceded by a description of the one-dimensional D-partition.

2.1. One-Dimensional D-Partition and Its Connection to Constructive D-Partition

The one-dimensional D-partition refers to a polynomial $G(s,\, t)$ whose coefficients depend on a scalar parameter $t \in \mathbb{R}$. In this case, the real line of the parameter is divided, by certain points t_i, into segments and rays corresponding to the D-partition regions. The points t_i defining the boundaries of the one-dimensional D-partition will be called critical points. As in the case of two parameters, they are defined by the main equation

$$G(s,\, t) = G_n(t)s^n + G_{n-1}(t)s^{n-1} + \ldots + G_1(t)s + G_0(t) = 0_{\mathbb{C}},\; s \in \Gamma, \tag{20}$$

and the degree drop condition $G_n(t) = 0$. If the boundary Γ of the root localization region is described by a piecewise rational curve with respect to the parameter w, and the polynomial G depends on t polynomially, then the main equation can be reduced to a system of two polynomial equations with two unknowns.

The situation is simplified if the polynomial linearly depends on t; then the main equation takes the form

$$G(s,\, t) = t\,P_t(s) + R_t(s) = 0_{\mathbb{C}}. \tag{21}$$

It can be solved explicitly, assuming the absence of common roots[2] of the polynomials $P_t(s)$ and $R_t(s)$ on Γ. The critical points t_i splitting the parameter line into segments (and rays) with a constant number of stable roots satisfy the equation

$$t = -\frac{R_t(s(w))}{P_t(s(w))}. \tag{22}$$

[2] Otherwise, the polynomial is obviously unstable since the root on the boundary does not belong to the open root localization region. This case is analogous to the case of two parameters with no common roots for the polynomials *P*, *Q*, and *R*. If we consider non-open sets $\mathbf{D}$ and the common root on the boundary is supposed to be stable, it can be reduced.

In view of the complex-valued right-hand side, we can eliminate the variable t by solving the equation with respect to w:

$$\operatorname{Im}\frac{R_t(s(w))}{P_t(s(w))} = 0.$$

This equation is equivalent to

$$\operatorname{Im} R_t(s(w))\,\operatorname{Re} P_t(s(w)) = \operatorname{Re} R_t(s(w))\,\operatorname{Im} P_t(s(w)),\;\; s \in \Gamma, \tag{23}$$

except those points where $P_t(s(w)) = 0$. Such points are not the solutions of the original equation (21) since $R_t(s(w)) \neq 0$ at them due to the absence of common roots. Equation (23) is reduced to a polynomial one with respect to w for rational functions $s(w)$. Next, the roots w_i found are substituted into equation (22) to get t_i. They are supplemented with the root of the degree drop equation, $t_0 = R_{t,n} / P_{t,n}$, if it is real and $P_{t,n} \neq 0$. Finally, for the intervals $(t_i,\, t_{i+1})$ obtained by splitting the parameter line division, the number of stable roots is determined, e.g., by the polynomial's roots at the midpoints of the intervals, $G(s, (t_i + t_{i+1})/2)$, and at two points outside the interval $[\min_i t_i,\, \max_i t_i]$.

Similar to the two-dimensional D-partition, for a rational boundary function $s(w)$, one can first reduce equation (21) to a polynomial dependence on w and then proceed to the relations (22) and (23). If the boundary Γ of the root localization region consists of several arcs, equation (21) shall be solved for each arc.

In the general case, for a polynomial linearly dependent on m parameters, the one-dimensional D-partition allows explicitly finding the intersection of the stability region and an arbitrary line (or segment)

$$k(t) = t\,d + p,\; t \in [t_1,\, t_2],\; p,\, d \in \mathbb{R}^m,$$

where the boundaries can be either bounded or unbounded. In the two-dimensional case, the polynomial (1) takes the form (21):

$$G(s,\, t) = G(s,\, k_1(t),\, k_2(t)) = (d_1 P(s) + d_2 Q(s))t + p_1 P(s) + p_2 Q(s) + R(s). \tag{24}$$

The critical points t_i are the roots of a certain polynomial according to formula (23). Among them, only those belonging to the interval $[t_1,\, t_2]$ are selected. This method is used to analyze arbitrary lines and segments in the parameter space of any dimension, particularly as an auxiliary method for analyzing implicitly defined sets, i.e., as an oracle determining the intersection of a line and a stability region [12].

Note that if, for a polynomial depending on two parameters, the constructive D-partition is used to characterize the stability region boundaries in the form of a set of curves and lines, then the intersection of the stability region and lines in the parameter space can be easily obtained from the intersection of the D-partition boundaries and a line. As a result, we get the same stability segments on the line under consideration as with the one-dimensional D-partition.

It is difficult to compare the effectiveness of the two approaches (the intersection with boundary curve arcs and the one-dimensional D-partition) a priori, since their complexity depends on the number of arcs of the stability region boundary and the degrees of the auxiliary polynomials. One should keep in mind that the coordinates of the arcs of the stability region boundary are easily estimated on the plane, see subsection 4.3. With the estimated location of the boundary arcs, the absence of intersection of segments (and lines) and the D-partition regions can be checked effectively, e.g., by comparing rectangles containing the boundary arcs of the D-partition regions. On the other hand, the one-dimensional D-partition for the polynomial (24), which is linear with respect to the parameter, is implemented by solving equation (21) directly, without the need to divide the D-partition boundary into the main curve and singular lines.

2.2. The Semi-Grid Method with Parallel Lines

As mentioned above, it is easiest to apply the method by fixing one parameter. In the case of two parameters, the fixed first (second) parameter on the plane k_1, k_2 corresponds to vertical (horizontal, respectively) lines. If a localization region $\boldsymbol{K} = K_1 \times K_2 = \left[\underline{k}_1, \overline{k}_1\right] \times \left[\underline{k}_2, \overline{k}_2\right]$ is defined, it is easiest to take a uniform grid, based either on fineness or on the number of lines $N+1$; see examples in Figs. 2a and 2b. For instance, vertical lines $\boldsymbol{K}_i$ are defined in explicit and parametric form as

$$k_1 - k_{1,i} = 0,\ k_{1,i} = \underline{k}_1 + i\frac{\overline{k}_1 - \underline{k}_1}{N},\ \ i = 0,\ldots,N,$$
$$\begin{pmatrix} 0 \\ \overline{k}_2 - \underline{k}_2 \end{pmatrix} t + \begin{pmatrix} k_{1,i} \\ \underline{k}_2 \end{pmatrix},\ t \in [0,1],\ \ i = 0,\ldots,N. \tag{25}$$

The intersection of the D-partition boundaries is given by equations (10) and (11) for the main curve and (8) for lines; alternatively, one can apply the one-dimensional D-partition of the polynomial (24) with respect to t using the parameterization (25). On each line, segments are selected for which the polynomial is stable. The set of such segments on all lines $\boldsymbol{K}_i$ corresponding to the stability regions forms an internal approximation of the stability region.

2.3. The Semi-Grid Method with Rays (Angular Grid)

Consider a point k_0 on the parameter plane, e.g., corresponding to a stable polynomial. We select several rays passing through this point and analyze the intersection of these rays and the D-partition boundaries. For construction purposes, it is more convenient to use lines (rather than rays) whose inclination angle relative to the axis is uniformly distributed on a semicircle. For example, for N lines, a grid is described by the equation

$$\left(k_1 - k_{0,1}\right)\sin\frac{i\pi}{N} - \left(k_2 - k_{0,2}\right)\cos\frac{i\pi}{N} = 0,$$
$$i = 0,\ \ldots,\ N-1, \tag{26}$$

or, in the parametric form, $k_0 + t\left(\cos\frac{i\pi}{N},\ \ \sin\frac{i\pi}{N}\right)^{\mathrm{T}}$. In contrast to the case of lines parallel to the coordinate axes, the intersection points with the boundary arcs of the stability region shall be found using formulas (8) and (9). An alternative is to use the one-dimensional D-partition. If the point k_0 passed by the lines corresponds to a stable polynomial, then the minimum positive and negative parameters of the line shall be taken: $\max_{m:t_m<0} t_m \le t \le \min_{m:t_m>0} t_m$, where all t_m correspond to the intersection points with the main curve and singular curves. In this case, we find the intersection of the line and the component of the stability region containing the point k_0. An example of an angular grid is shown in Fig. 2c.

Note that when building a grid by angle, a set of randomly directed lines can be taken as well, and the point k_0 does not necessarily correspond to a stable polynomial.

2.4. The Semi-Grid Method with Concentric Circles (Radial Grid)

Finally, consider a parameterization using a set of concentric circles $\boldsymbol{K}_i = \left\{k : \left\|k - k_0\right\| = r_i\right\}$. The intersection points of the boundaries of the D-partition and the circles are found explicitly: for singular lines in the parametric form $t\,d + p$, from the quadratic equation

$$\left(t\,d_1 + p_1 - k_{0,1}\right)^2 + \left(t\,d_2 + p_2 - k_{0,2}\right)^2 = r_i^2\,; \tag{27}$$

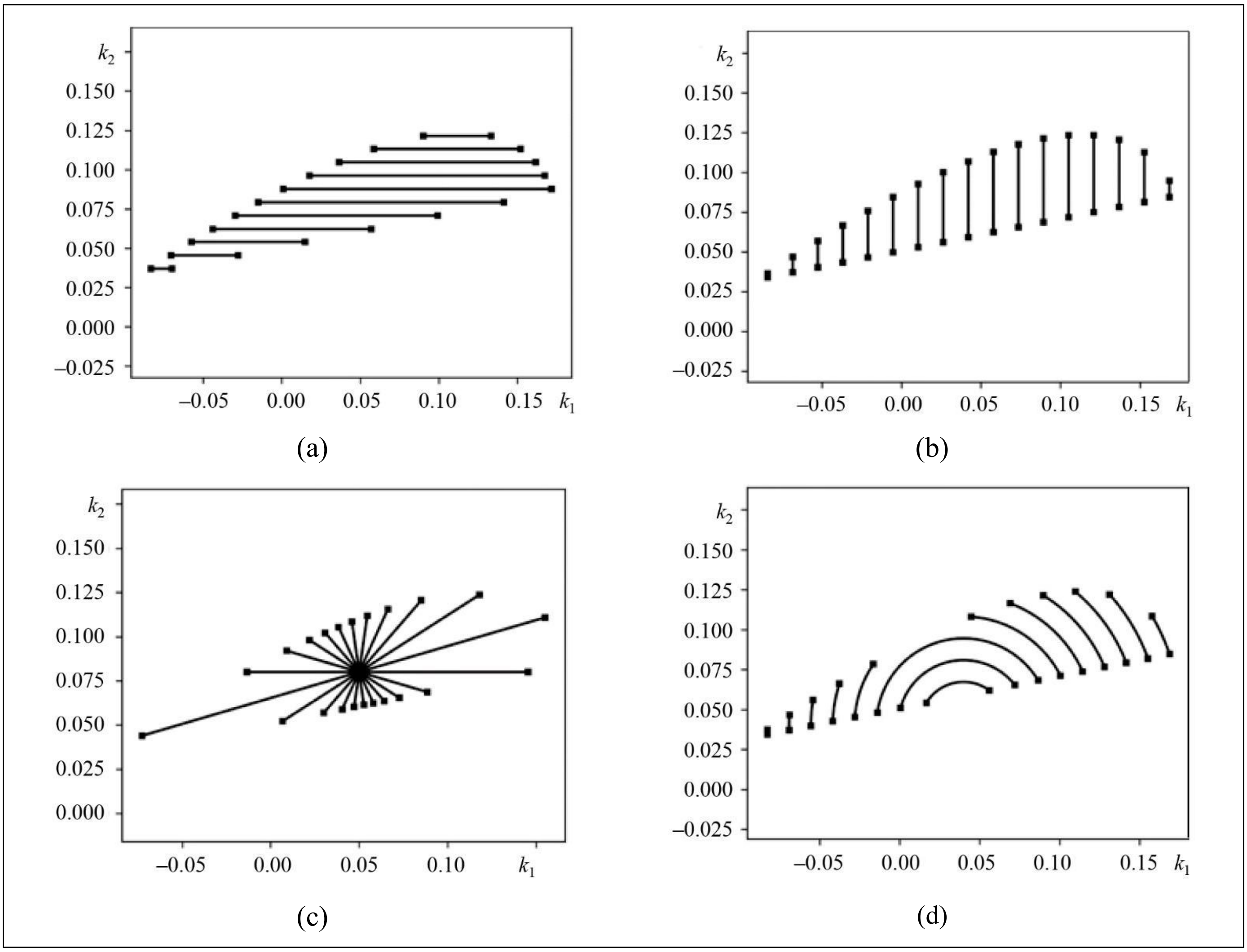

Fig. 2. Internal approximations of the stability region on grids: (a) horizontal lines, (b) vertical lines, (c) concentric circles, and (d) an angular grid.

and for the arcs of the main curve $k_\ell(w),\ w \in W_\ell$, from equations reducible to polynomial ones with respect to w:

$$\left(k_{\ell,1}(w) - k_{0,1}\right)^2 + \left(k_{\ell,2}(w) - k_{0,2}\right)^2 = r_i^2. \quad (28)$$

Fig. 2d presents an example of a radial grid centered outside the set under study.

2.5. The Semi-Grid Method with Parallel (Hyper-)planes

For a polynomial dependent on three parameters (arising, in particular, in the analysis of a closed-loop system with a PID controller), a grid is taken for one parameter. For each fixed value of this parameter, the D-partition is constructed for the remaining two parameters, and the resulting planes are "joined" together. Thus, the construction and visualization of a three-dimensional region is reduced to the construction of a series of two-dimensional stability regions. This classical technique is used to design stabilizing controllers, H_∞-controllers, etc. [3].

In the general case, fixing one parameter reduces the number of free parameters, and it is possible to construct (or approximate) a D-partition for the remaining parameters.

3. GRID METHODS (PARAMETER PLANE PARTITION INTO CELLS)

In semi-grid methods, there is no estimate of how the stability region boundary behaves outside the sets forming the slicing of the parameter space. Only its intersection points with the lines (circles, hyperplanes) that form the slicing boundaries are known. Of course, if the localization region $\boldsymbol{K}$ is defined, the slicing generate a series of bounded sets (segments or circle arcs), but their size is comparable to that of the region $\boldsymbol{K}$.

In grid methods, a finer partition of the parameter space or localization region into sets K_i is used, controlled by *two* grids. Let us emphasize the difference between the method proposed in this paper and the one described in [13]. The common idea is to search for regions for which all polynomials are stable. In this paper, we determine the stability region boundary explicitly using the constructive D-partition and then check its intersection with a *finite* set of the boundaries of the sets K_i. In other words, we explicitly search for the intersection of cell boundaries and the boundaries of the stability region.

Also, we use a cellular grid, particularly formed by a finite set of lines, and explicitly find the cells K_i containing the boundary, based on the intersections of the stability region boundaries and the lines forming the grid. These lines are simultaneously the boundaries of the cells. Clearly, the other cells either entirely belong to some D-partition region, including the stability region, or entirely contain some D-partition region.[3] In this sense, the approach is close to the ideas of adaptive partition of the parameter plane: *quadtree* [14] or the partition by intersecting arcs [11]. The method proposed differs from tracking methods along the boundary of a set using intersecting segments (e.g., orthogonal, simplex, caterpillar, and other methods [2]), as it surely finds all sets covering the boundary.

3.1. An Orthogonal Linear Grid

An obvious combination of one-dimensional grids for each of the parameters k_1, k_2 (subsection 2.2) suggests itself, leading to the analysis of a set of rectangles. This approach is convenient in the sense that it suffices to study the intersection of the D-partition boundary and vertical and horizontal lines (see formulas (28) and (29)), and the intersection points of the grid lines with each other are obvious. And one automatically obtains the rectangular cells K_i containing the stability region boundary; see an example in Fig. 3a.

3.2. A Polar Grid

A similar approach works when combining angular and radial grids. The resulting polar grid consists of concentric circles and radial lines passing through the center of the circles. The radii of the circles are not necessarily uniform, as is the case with the grid at the angles of the lines. If the selected center lies inside the stability region, the polar grid allows finding a simple

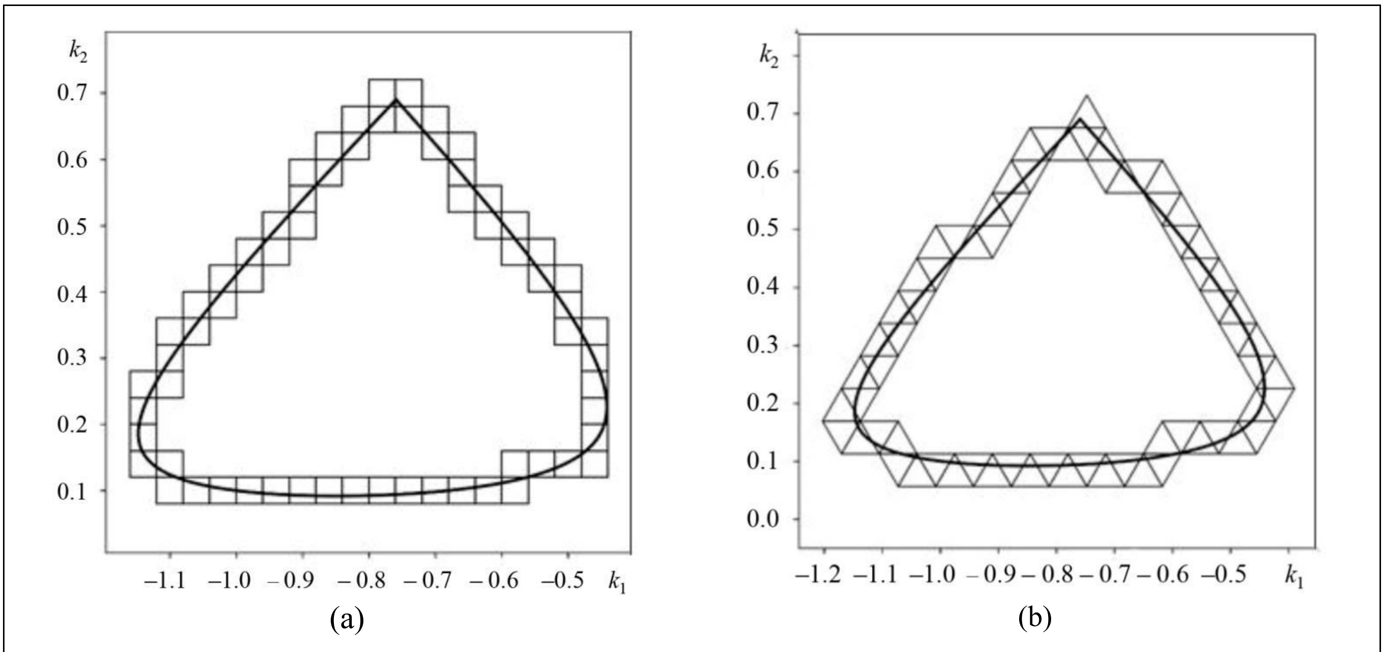

Fig. 3. The boundary of the stability region component in Example 2, approximated by (a) an orthogonal grid and (b) a triangular grid.

[3] In fact, this is the main drawback of the traditional grid approach—it fails to detect explicitly the sets lying inside the cell K_i. This fundamental feature cannot be directly circumvented by reducing the size of cells. However, under the above assumptions, Theorem 1 from part I of the study [5] is valid, and the stable region boundary consists of a finite number of arcs, all explicitly listed. Thus, it is possible to check the localization of each of these arcs (see subsection 4.3) as well as their position relative to the cells K_i.

sectoral internal (and external) approximation of the stability region.

3.3. A Triangular Grid

Here is another type of grids formed by lines—a triangular grid—which requires 1.5 times more computations but yields a "smoother" approximation of the boundary. It is necessary to take a series of horizontal lines and a series of inclined lines; see an example for uniform triangles in Fig. 3b. A triangular grid is convenient because the boundary of the set covering the D-partition boundary has higher "uniformity" and "smoothness." Also, it consists of segments of multiple lengths (with angles of 60°, 120°, and 240°) if the triangles are equilateral.

4. APPLICATION OF CONSTRUCTIVE D-PARTITION

Without loss of generality, the results in this section are also considered relative to the stability region D_n rather than an arbitrary D-partition region D_d.

4.1. Stability Regions on a Curve

Constructive D-partition allows finding stability regions along a curve in the parameter space. Let a curve $\boldsymbol{K}$ on the parameter plane be explicitly defined as

$$F(k_1,\ k_2)=0. \tag{29}$$

Its intersection points with the boundaries of the D-partition regions divide the curve into arcs where the number of stable roots is constant. In particular, to find the "stability arcs" corresponding to stable polynomials, it suffices to consider the intersection of the curve and the stability region boundaries $k_\ell(w)$. These points satisfy the equations

$$F\big(k_{\ell,1}(w),\ k_{\ell,2}(w)\big)=0,\ \ w\in W_\ell, \tag{30}$$

for the intersection with the boundaries representing arcs of the main curve $k_\ell(w)$, and the equations

$$F\left(d_1 t+p_1,\, d_2 t+p_2\right)=0 \tag{31}$$

for the intersection with the boundaries representing singular lines defined in the parametric form $t\,d+p$.

If the curve $\boldsymbol{K}$ is algebraic, i.e., the function $F(k_1, k_2)$ is a polynomial, then the solution of both equations reduces to calculating the roots of a certain polynomial. This technique has been adopted in subsection 2.4 to analyze the intersections of the D-partition boundary and circles using equations (27) and (28) instead of the circle representation (26). Note also that equations (30) and (31) can be treated as boundary equations for the curve $k_\ell(w)$ [15].

Moreover, for the algebraic curves of the form (29), we can obtain the parameterization $\boldsymbol{K}=\{\boldsymbol{k}(v): v\in\mathbb{R}\}$, and from the intersection points with the D-partition boundaries, we can restore the parameters v_i [16]. These points split the curve into arcs with a constant number of roots, and some of the roots correspond to stability arcs. Also, with a known curve parameterization $\boldsymbol{K}=\{\boldsymbol{k}(v): v\in\mathbb{R}\}$, the curve equation can be immediately substituted into the main equation of the D-partition, $G\big(s,\, \boldsymbol{k}_1(v), \boldsymbol{k}_2(v)\big)=0_{\mathbb{C}}$, $s\in\partial\boldsymbol{D}$, and then the one-dimensional D-partition with respect to the parameter v can be performed. If the curve parameterization is rational, this partition can be reduced to the one-dimensional D-partition for a polynomial of the form (20).

The above approach serves to analyze polynomials with coefficients nonlinearly dependent on one parameter, e.g., $G(s,\, t)=f(t)P(s)+g(t)Q(s)+R(s)$. First, the constructive D-partition of the polynomial $G(s,\, f,\, g)=fP(s)+gQ(s)+R(s)$ is found with respect to f and g as free parameters. Then, the curve $\big(f(t),\, g(t)\big)^{\mathrm{T}}$ is parameterized in the form (29), and equations (30) and (31) are solved. A similar technique can be employed to analyze polynomials with several parameters entering the coefficients nonlinearly.

4.2. The Intersection with Sets of Additional Constraints

The stability region description as a set of boundary arcs obtained by the constructive D-partition makes it easy to consider additional constraints on the coefficients k_1, k_2. Let these constraints be defined by sets $\boldsymbol{K}_i$. In this case, first, the parts of the boundary $\partial\boldsymbol{K}_i$ lying inside the stability region are determined, e.g., as indicated in subsection 4.1. Then the stability region boundaries are updated by taking these (new) boundaries into account and excluding the arcs of the stability region boundary outside the sets $\boldsymbol{K}_i$. This procedure is repeated for all regions of the additional constraints $\boldsymbol{K}_i$. In essence, each time the main curves and singular lines are cut off, by analogy with the localized D-partition.

4.3. Localization of D-Partition Regions on the Parameter Plane and the Support Function

Besides limiting the set of parameters of interest a priori, the term "localization" can be used for the opposite purpose, i.e., identifying the position of the stability region or its components on the plane.

Let a set (in particular, a stability region) be described by a set of boundary arcs $k_i(w),\ w \in W_i = [w_{i,a},\ w_{i,b}]$, including both main curve arcs and segments on finite intervals W_i. Then it is not difficult to describe the position of this set. For instance, endpoints can be found using the fact that boundary arcs are defined by rational curves. For example, the minimum and maximum values of the components (i.e., the rectangle containing the boundary arc $k_i(w)$) are determined by the extreme points in the same way as for the speed in subsection 1.1. To this end, we find the roots $w_{i,1,m}$, $m = 1, \ldots,$ $w_{i,2,\ell}$, $\ell = 1, \ldots,$ of the two equations

$$w_{i,1,m} : k_{i,1}^{'}(w) = 0,\ w \in [w_{i,a},\ w_{i,b}],$$

$$w_{i,2,\ell} : k_{i,2}^{'}(w) = 0,\ \ w \in [w_{i,a},\ w_{i,b}].$$

As for the speed estimate (16), there are inequalities, and the extreme values are reached at the endpoints of the curve arcs or at the stationary points $w_{i,1,m}, w_{i,1,\ell}$:

$$\min\left\{k_{i,1}(w_{i,a}),\ k_{i,1}(w_{i,b}),\ \min_m k_{i,1}(w_{i,1,m})\right\} \le k_{i,1}(w)$$
$$\le \max\left\{k_{i,1}(w_{i,a}),\ k_{i,1}(w_{i,b}),\ \max_m k_{i,1}(w_{i,1,m})\right\},$$

$$\min\left\{k_{i,2}(w_{i,a}), k_{i,2}(w_{i,b}),\ \min_\ell k_{i,2}(w_{i,2,\ell})\right\} \le k_{i,2}(w)$$
$$\le \max\left\{k_{i,2}(w_{i,a}), k_{i,2}(w_{i,b}),\ \max_\ell k_{i,2}(w_{i,2,\ell})\right\}.$$

Extreme points can be replaced by only minima (maxima) of the lower (upper, respectively) bounds.

For segments, the intervals are determined by the endpoints: for example, for k_1,

$$\min\left\{k_{i,1}(w_{i,a}),\ k_{i,1}(w_{i,b})\right\}$$
$$\le k_{i,1}(w) \le \max\left\{k_{i,1}(w_{i,a}),\ k_{i,1}(w_{i,b})\right\},$$

and analogously for the second coordinate $k_{i,2}(w)$.

Fig. 4 shows an example of the localization of individual components of the stability region.

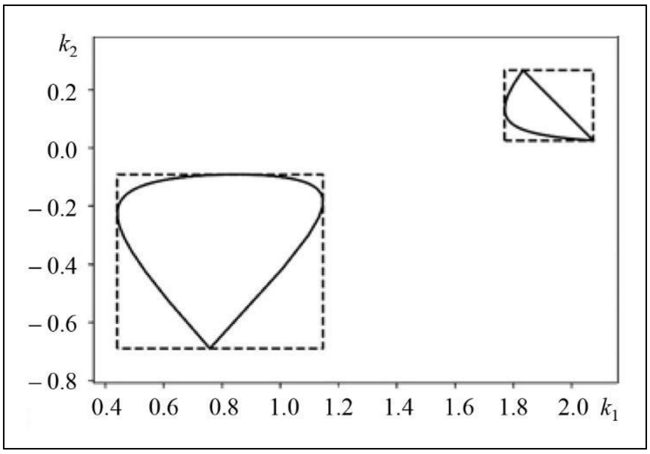

Fig. 4. The localization of two stability region components in Example 2 (see Section 5).

Similarly, it is easy to obtain the support function $\operatorname{supp}_{D_n}(d) = \max_{k \in D_n} d^{\mathrm{T}} k$ and its support element $\operatorname{suppel}(d)$ for the stability region in the direction of the vector $d \in \mathbb{R}^2$. For this purpose, the support function for each arc of the boundary $k_i(w)$ is used:

$$\operatorname{supp}_{D_n}(d) = \max_i \max_{w \in W_i} d_1 k_{i,1}(w) + d_2 k_{i,2}(w),$$

$$\operatorname{suppel}(d) = k_j(w_j),$$

$$w_i = \arg\max_{w \in W_i} d_1 k_{i,1}(w) + d_2 k_{i,2}(w),$$

$$j = \arg\max_i d_1 k_{i,1}(w_i) + d_2 k_{i,2}(w_i).$$

The maximum of a rational function on an interval where its denominator does not vanish is found by calculating the roots of the polynomial in the numerator of its derivative. Moreover, a support function can be built not only for the entire stability region but also for its individual component, by considering only the boundaries of this component.

In the general case, we propose a three-stage approach for the primary analysis of the stability region.

1. Obtain a constructive description of the stability region boundary.

2. Select a sufficiently large localization region $\boldsymbol{K}$ to get finite and closed parameterization intervals W_i, also applying the results from subsection 3.2 on the parameterization change.

3. Localize the stability region by the boundary arcs inside the set $\boldsymbol{K}$.

For the initial localization with a large set $\boldsymbol{K}$, equations (8), (10), and (11) are used for a rectangle, equations (27) and (28) for a circle, etc.

In addition, the stability region can be localized numerically, using a sufficiently uniform grid of fineness ρ. Assume that, according to subsection 1.1, such a grid is obtained by applying Algorithm 1 or 2 to all arcs of the stability region boundary: $K_{grid} = \{k_r, r = 1, \ldots\}$. By sequentially connecting the grid nodes, we get a polygon approximating the stability region with an accuracy no worse than ρ (the grid parameter). For this polygon, we can also explicitly express the support function and the support element through its vertices:

$$\mathrm{supp}_{D_n}(d) = \max_r d^{\mathrm{T}} k_r,$$

$$\mathrm{suppel}_{D_n}(d) = k_r, \; r = \arg\max_r d^{\mathrm{T}} k_r.$$

The above expressions are approximations to the support function and element, and the stability region itself surely lies inside the rectangle

$$\left[-\rho + \min_r k_{r,1}, \; \max_r k_{r,1} + \rho\right] \times \left[-\rho + \min_r k_{r,2}, \; \max_r k_{r,2} + \rho\right],$$

where ρ denotes the fineness of a sufficiently uniform grid.

Analogously, for a set of points, it is possible to calculate the minimum covering circle with center k_0 and radius R. A circle with the same center and radius $R + \rho$ will contain the stability region.

4.4. The Distance to the Stability/Instability Region

For a point k_0 characterizing a stable polynomial $G(s, k_{0,1}, k_{0,2})$, the natural problem is to find the distance to the nearest unstable point, i.e., to determine the circle of maximum radius $\|k - k_0\| \le R$ lying entirely in the stability region. This radius is called the stability radius [3].

Given the constructive D-partition, the stability radius can be calculated exactly or estimated using a sufficiently uniform grid on the boundary. The exact solution is determined by a minimization problem considering all stability region parts $k_i(w)$:

$$R = \min_i \min_{w \in W_i} \left\| k_i(w) - k_0 \right\|.$$

This problem is decomposed into a set of subproblems of minimizing rational functions on the interval $\min_{w \in W_i} \left\| k_i(w) - k_0{}^2 \right\|$. In turn, each of the subproblems is reduced to calculating the roots of a certain polynomial, similar to the parametric speed (15), and checking the endpoints of the segments.

Analogously, by maximizing the distance from the point k_0 to the boundary points, we can find the minimum circle containing the stability region, see Fig. 5.

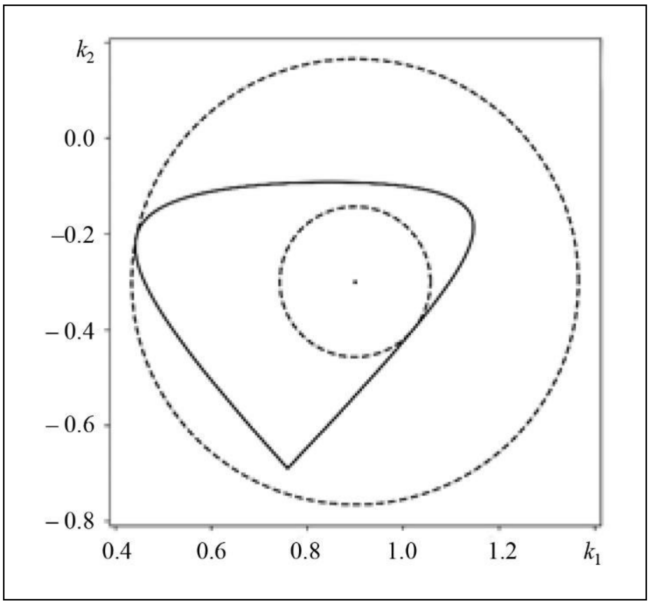

Fig. 5. The circle of maximum radius with a given center contained in the stability region and the circle of minimum radius containing the stability region component, in Example 2 (see Section 5).

The distance to the stability region boundary can also be estimated by the points of a sufficiently uniform grid on the boundary as

$$\min_i \min_r \|k_r - k_0\| - \rho \le R \le \min_i \min_r \|k_r - k_0\| + \rho.$$

Here, k_r are the nodes of the grid covering the stability region boundary. It consists of the union of the grid nodes of each boundary part $k_i(w)$, $w \in W_i$ (without repetitions).

4.5. Application to Robust D-Partition

Robust analysis problems involve polynomials depending on not only "control" but also uncertain parameters, further denoted by q:

$$G(s, k_1, k_2, q) = k_1 P(s) + k_2 Q(s) + R_0(s) + \sum_i q_i R_i(s), \; q \in Q. \tag{32}$$

Here, the uncertainty in the polynomial is characterized by a finite set Q. Thus, for each fixed k, a family of polynomials is considered. The polynomial for particular (chosen) parameter values $q = q_0$ is called the nominal polynomial. In this case, the polynomial (32) takes the form (1) with $R(s)=R_0(s)+\sum_i q_{0,i}R_i(s)$.

The problem is to determine robust stability regions, i.e., such k under which all roots of the polynomial are stable *for all* parameters $q \in Q$. The construction of such regions, albeit being conceptually the same as that of a D-partition, is significantly more complex since the D-partition regions with a constant (for any $q \in Q$) number of stable roots are separated by two sets instead of one-dimensional lines. These sets are defined by the zero exclusion principle, which generalizes equations (3) and (4):

$$K_{bnd} = \{k_1, k_2 : G(s,k_1,k_2,q) = 0_{\mathbb{C}}, s \in \partial\mathbf{D}, q \in Q\}$$

and

$$K_{deg} = \{G_n(k_1, k_2, q) = 0_{\mathbb{C}}, q \in Q\}.$$

Accordingly, the boundaries of these two sets are the boundaries of robust D-partition regions. The boundaries of a robust stability region always lie inside the stability region of the nominal polynomial. This follows from the fact that each polynomial of the family (32) is stable, including the nominal one $G(s, k_1, k_2, q_0)$. Thus, when describing the boundaries of the robust stability region, it suffices to consider only those boundaries of the sets K_{bnd} and K_{deg} that lie inside the stability region of the nominal polynomial.

In addition, one can repeat the constructive D-partition many times for the polynomials $G(s, k_1, k_2, q_r)$, $r = 1, 2, \ldots,$ where each polynomial is defined by randomly selected parameters $q_r \in Q$. The robust stability region lies in the intersection of the stability regions of all selected polynomials.

Let the boundary $\partial\mathbf{D}$ of a root localization region have a parameterization $s(w)$, and let the boundary of the nominal polynomial be parameterized by a set of curves and lines $k_i(w)$, $w \in W_i$. As it turns out, the boundary of robust D-partition regions is characterized similarly: K_{bnd} is the envelope of the family of sets $K_{bnd}(w) = \{k_1, k_2 : G(s(w), k_1, k_2, q) = 0_{\mathbb{C}},\ q \in Q,\ s(w) \in \Gamma\}$ [3, 17].

One might expect that the intervals of the sets $K_{bnd}(w)$ generating the boundary of the robust stability region (with respect to w) are within the intervals W_i of the stability region boundaries without uncertain parameters. Unfortunately, in the general case, the hypothesis of nested intervals is incorrect, since the boundary of the set $K_{bnd}(w)$ with $w \notin W_i$ can affect the final robust stability region. This depends on the values of the main curve function $k(w)$ for the nominal D-partition and the *size* of the set Q. The analysis and design of the constructive robust D-partition are an open research problem even for the case of parameters entering the characteristic polynomial linearly.

5. EXAMPLES

The figures use stability regions or their components from two examples discussed in detail in part I of the study [5].

Example 1 [18 , *p. 77*]. Consider a closed-loop continuous-time system with the characteristic polynomial

$$\begin{aligned} G(s, k_1, k_2) &= k_1 s(s-1)(s-2) \\ &+ k_2(s-1)(s-2) + s(s+1)(s^2+s+1). \end{aligned} \tag{33}$$

It is required to analyze its stability with respect to the root localization region $\boldsymbol{D} = \{s : \operatorname{Re} s < 0.2\}$ with the boundary parameterization $s(w) = -0.2 + jw.$ Since the polynomial (33) has real coefficients, it suffices to take the upper part of the boundary, $W = [0, \infty)$.

There is one main curve, $w \in [0, \infty)$, with the components

$$k_1(w) = \frac{4.6\,w^4 - 7.112\,w^2 + 0.62016}{-w^4 - 6.28\,w^2 - 6.9696},$$

$$k_2(w) = \frac{-w^6 + 8.68\,w^4 - 5.4208\,w^2 - 0.230784}{-w^4 - 6.28\,w^2 - 6.9696}.$$

The critical frequency $w_0 = 0$ is associated with the unique singular line $-0.528k_1 + 2.64k_2 - 0.1344 = 0,$ with the parameterization $p + td$, where $p = (-0.00979021; 0.04895105)^{\mathrm{T}}$ and $d = (-2.64; -0.528)^{\mathrm{T}}$. The stability region is bounded by one arc of the main curve $k(w)$, $w \in [0, 0.70951628]$, and the segment $p + td$, $t \in [-0.06916025, 0.02999640]$ (Fig. 6).

Figure 2 shows a series of internal approximations of the stability region by one-dimensional lines. Next, Fig. 3a presents the parametric speed $v(w)$ of the curve arc specified and the sufficiently uniform parameter values yielded by Algorithm 1 for $\rho = 0.02$; Fig. 3b, the nodes and their neighborhoods of radius ρ covering this curve arc.

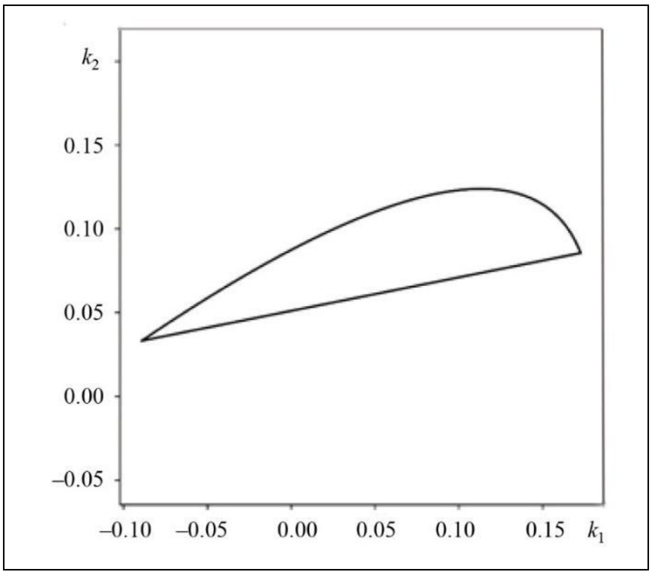

Fig. 6. The stability region in Example 1.

Example 2 [19]. Consider the characteristic polynomial $G_0(z, k_1, k_2) = z^n + k_1 z^{n-1} + (1+\varepsilon) z^{n-2} + k_2$, $n = 5$, $\varepsilon = 0.1$, of a discrete-time system. Its stability is equivalent to the Hurwitz property of the polynomial

$$G(s, k_1, k_2) = (s+1)^5 + (1+\varepsilon)(s-1)^2(s+1)^3 + k_1(s-1)(s+1)^4 + k_2(s-1)^5.$$

The boundary of the D-partition regions consists of the one main curve

$$k_1(w) = \frac{-16.6\, w^8 + 128.8\, w^6 - 221.2\, w^4 + 128.8\, w^2 - 16.6}{8\left(w^8 - 6\, w^6 + 6\, w^2 - 1\right)},$$

$$k_2(w) = \frac{-0.2\, w^8 - 0.8\, w^6 - 1.2\, w^4 - 0.8\, w^2 - 0.2}{8\left(w^8 - 6\, w^6 + 6\, w^2 - 1\right)}$$

and two singular lines. One of them corresponds to $w = 0$ and has the parameterization $t\,d + p$, where $p = (1.05; 1.05)$ and $d = (1, -1)$. The other singular line is defined by the degree drop condition (4) with the parameterization $t\,d + p$, where $p = (-1.05; -1.05)$ and $d = (-1; 1)$.

The stability region consists of four components (Fig. 7):

1) the segment of the first singular line for $t \in [1.025, 1.45]$ and the arc of the main curve for $w \in [0, 0.37796447]$;

2) the arc of the main curve for $w \in [0.42972375, 0.96431209]$;

3) the arc of the main curve for $w \in [1.03700867, 2.32707640]$;

4) the segment of the second singular line for $t \in [1.025, 1.45]$ and the arc of the main curve for $w \in [2.64575131, \infty)$.

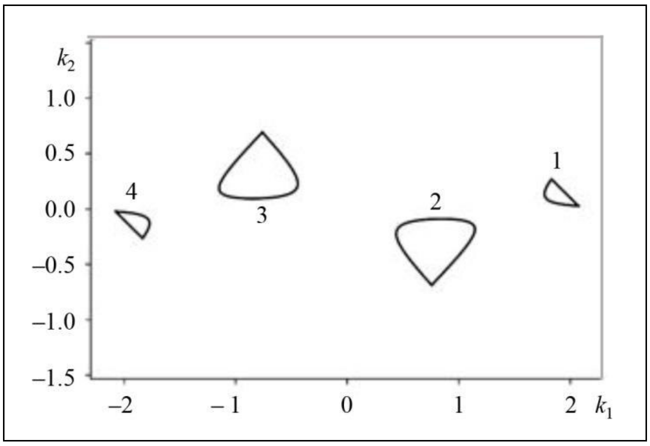

Fig. 7. The stability region components in Example 2.

The last arc of the curve can be written as $k_u(u) = k_\ell(1/u)$, $u \in [0, 0.37796447]$ (see subsection 3.2), where the value at $u = 0$ is defined and coincides with $k_{(\infty)}$:

$$k_{u,1}(u) = \frac{16.6\, u^8 - 128.8\, u^6 + 221.2\, u^4 - 128.8\, u^2 + 16.6}{8\left(u^8 - 6\, u^6 + 6\, u^2 - 1\right)},$$

$$k_{u,2}(u) = \frac{0.2\, u^8 + 0.8\, u^6 + 1.2\, u^4 + 0.8\, u^2 + 0.2}{8\left(u^8 - 6\, u^6 + 6\, u^2 - 1\right)}.$$

Note that the resulting parameterization coincides with the original one in w up to the sign, and the interval coincides with the interval of the first component. This is due to the symmetry of the original root localization region of the discrete system (the unit circle) and its parameterization.

Figure 3 shows the approximations of the second component of the stability region using orthogonal and triangular grids. In addition, Fig. 4 presents the bounding regions $[1.7706, 2.075] \times [0.025, 0.2667]$ for the first component and $[0.4414, 1.1472] \times [-0.6901, -0.09185]$ for the second component of the stability region; the third and fourth components are symmetric to the second and first, respectively. Finally, Fig. 5 demonstrates two circles with centers $k_1 = 0.9$, $k_2 = -0.3$: the smaller lies entirely within the stability region and determines a stability radius of 0.4417 for the above controller; the larger circle with radius 0.9538 contains the entire second component of the stability region.

CONCLUSIONS

Based on the constructive D-partition method (see part I of the study [5]), two algorithms have been proposed to build a sufficiently uniform grid on a stability region boundary with a given fineness. The algorithms involve an estimate of the parametric speed of rational curves. Several semi-grid internal approximations of a stability region have been proposed, together with a regular covering method for the boundary of a stability region with rectangles and triangles. The above methods have been applied in stability analysis problems, including the localization of stability region components on a plane, the construction of their support functions and their approximations, as well as the calculation of the stability radius.

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Author information

Tremba, Andrey Aleksandrovich. Cand. Sci. (Phys.–Math.), Trapeznikov Institute of Control Sciences, Russian Academy of Sciences, Moscow, Russia; Moscow Institute of Physics and Technology, Dolgoprudny, Russia
✉ atremba@ipu.ru
ORCID ID: https://orcid.org/0000-0001-5783-7600

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Translated into English by *Alexander Yu. Mazurov*,
Cand. Sci. (Phys.–Math.),
Trapeznikov Institute of Control Sciences,
Russian Academy of Sciences, Moscow, Russia
✉ alexander.mazurov08@gmail.com