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COURNOT OLIGOPOLY: STRATEGY CHOICE UNDER UNCERTAINTY AND OTHER PROBLEMS[1]

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Abstract. Firms operating in a market economy naturally strive to increase revenues. When large firms affect prices by their actions, this task involves nontrivial mathematics, i.e., game-theoretic oligopoly models. The survey is more concerned with Cournot competition than with Bertrand competition. The existence, uniqueness, and stability of Cournot equilibrium are discussed. The other issues under consideration are as follows: the entry of new firms into the market; the barriers that can be imposed for this; and the impact of such an entry on society's welfare as well as on total surplus and consumer surplus. The problems of collusion between firms are touched upon. Publications comparing the prices of goods, the profits of firms, and society's welfare under Cournot and Bertrand competition are overviewed. Much attention is paid to the problems faced by firms due to the ignorance of some current or future market conditions and the existing uncertainty. The issues of information sharing among firms are considered. One approach to reducing marginal cost is the purchase of licenses; licensing in a Cournot duopoly is also described. Computational methods for Cournot equilibria in the case of multi-product firms are presented. Finally, publications with particular applications of Cournot equilibria are considered.

Keywords: Cournot equilibrium, social efficiency, Bertrand equilibrium, information sharing, uncertainty, licensing, cartel formation, complementarity problem.

INTRODUCTION

If large firms operate in an industry, the prices of goods are determined not only by demand and production costs but also by the strategies of producers. The theory of oligopoly plays an important role when forming the strategies of firms. According to the classification in the book [1], the interactions occurring in an industry with a small number of firms can be quantity competition (firms determine their outputs), price competition (firms determine the prices of their products), or collusion. The interaction where all firms simultaneously choose their outputs, trying to predict the outputs of other firms, is called Cournot competition. The interaction where all firms simultaneously establish the prices for their products, trying to predict the prices of other firms, is called Bertrand competition. Cournot competition and Bertrand competition were also discussed in the book [2] (as Cournot oligopoly and Bertrand oligopoly, respectively). Cournot competition-based approaches may be preferred when outputs must be determined long before production. Information sharing among firms, e.g., costs and market demand, (or its absence) is essential. According to the paper [3], advertising and R&D expenditures are other elements of firms' strategies in addition to outputs and prices; this makes the model algebraically more complex but does not change it completely. Of course, mathematical modeling of non-simultaneous decision-making by firms is of certain interest, but such models will not be addressed below.

The classical Cournot competition model is as follows. Assume that n firms produce a homogeneous good sold at a uniform price. If L_i is the *capacity* of firm i, then its strategy is to produce a *quantity* (*output*) q_i, where $0 \le q_i \le L_i$, $i = 1,..., n$. By assumption, the firm's *costs* C_i depend only on the output q_i

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whereas the *price of the* good P depends only on the *total output* $Q=\sum_{i=1}^{n} q_i$. Then the profit of firm i is given by

$$\pi_i(q_i, Q) = P(Q) q_i - C_i(q_i). \quad (1)$$

The function $P(Q)$ is monotonically decreasing, and each function $C_i(q_i)$ is monotonically increasing (except for the models with constant costs). The function $P(Q)$ is called the *inverse demand function* and the functions $C_i(q_i)$ are called *cost functions*. The objective of firm i is to maximize the profit π_i. In modern terms, this problem is a noncooperative game. (Note that Cournot's and Bertrand's works date back to the 19th century.) It is required to find a Nash equilibrium in the class of pure strategies profiles $q_1, ..., q_n$. When applied to this problem, Nash equilibrium is often called *Cournot equilibrium* or Cournot–Nash equilibrium. A more general concept of equilibrium (with collusion between firms regarding their outputs) was discussed, e.g., in [3]. As it turns out, Cournot equilibrium is a special case of this equilibrium.

The problem formulated above motivated the appearance of numerous interesting and important mathematical works. However, Cournot's goal was to describe the economic realities lying between monopoly and perfect competition, in particular, to answer the following question: Does an increase in the number of firms match public interests? Among other achievements, scientists of the 19th century recognized that it is impossible to answer such questions without applying mathematical methods and solving optimization problems. But what is the level of mathematical sophistication required? According to some authors, search engines provide over 50 000 references for the query "Cournot equilibrium." In some papers, the presentation was limited to the formulation and proof of mathematical theorems; in others, economic conclusions were drawn or specific economic problems were considered. Undoubtedly, it would be wrong to claim that some link in the "theoretical mathematics–applied mathematics–particular applications" chain is more important than others. In addition, with such voluminous literature on the subject, not mentioning some publications in this survey does not mean their insignificance from the author's point of view.

The rapid development of Cournot oligopoly as a mathematical discipline began in the late 1950s–early 1960s. First, the existence and uniqueness of Cournot equilibrium were studied. Second, dynamic models were considered, and the stability of Cournot equilibria was analyzed. These problems are presented in Section 1 of the survey. Note that the aim of this paper is not to overview dynamic models of Cournot competition: the focus is on static models. However, at the initial evolution stage of Cournot oligopoly as a mathematical discipline, the issues of existence, uniqueness, and stability were considered in close connection.

Is the entry of a new firm into the market always desirable in terms of public interests? May such an entry reduce the output of existing firms and increase the price of goods, becoming therefore profitable for the entering firm, but decrease society's welfare? Should the authorities impose entry barriers for new firms, and what should these barriers be? How will these barriers affect total surplus and consumer surplus? The questions listed above are interrelated. Relevant publications are overviewed in Section 2. Many authors compared Cournot and Bertrand competition in terms of equilibrium prices, outputs, society's welfare, etc. These studies are presented in Section 3. To a large extent, the survey concerns probability theory methods when applied to the interaction of large firms. Section 4 is devoted to the works on strategy choice under uncertainty and information sharing among firms. Also, this section describes works on the sale of licenses. Cartel formation is discussed in Section 5. Finally, Section 6 considers the practical calculation of Cournot equilibria and their use in particular applications.

1. THE EXISTENCE, UNIQUENESS, AND STABILITY OF COURNOT EQUILIBRIA

Leaving aside the existence and uniqueness of Cournot equilibria for a while, we begin with the paper [4], which analyzed the stability of such equilibria. That paper considered a discrete-time model as follows. At each time step, by determining its output, each firm maximizes its profit under a linear inverse demand function $P(Q)$ known to all firms. Each firm assumes that the outputs of other firms will not change. The stability of the resulting system of difference equations was investigated. As it turned out, the solution is stable in the case of 2 firms only; if the number of firms exceeds 3, the solution becomes unstable.

Note that by that time, there was considerable literature on the stability of Walrasian equilibria; it was well known that the (in)stability of the solution depends on the adjustment process used in the model (e.g., see [5, *p. 643*]). The judgmental and undoubted practical applicability of the conclusions of [4], and the fact that the adjustment process in that work is chosen arbitrarily, stimulated further studies of the stability of Cournot equilibria. For a wider class of adjustment processes, it was shown in [6] that Cournot equilibrium can be stable for any number of firms.

The author [7] considered continuous-time models with different cost functions of firms and a nonlinear inverse demand function. Lyapunov functions were used to prove the stability of Cournot equilibria. The paper [8] combined the approaches from [7] and [5] to formulate general conditions for adjustment processes under which Cournot equilibria are stable. The researcher [9] relaxed the assumption that all firms produce a homogeneous good, but each firm produces only one good. (Such models are called differentiated goods models, in contrast to models where each firm produces multiple goods.) The stability of Cournot equilibrium was proved under "a sufficiently weak link between goods." The stability results from [7] were extended in [10] using the concept of conjectural variations. (For more details on conjectural variations, we refer to, e.g., [11].) Further results on the stability of Cournot equilibria were overviewed in [12–14].

In the case of firms producing a homogeneous good, the existence of Cournot equilibria was considered in [15] as follows. Assume that the functions $P(Q)$ and $C_i(q_i)$ are continuous (but not necessarily differentiable) and the function $\pi_i(q_i, Q)$ is concave in the argument q_i. Under these conditions, the existence of Cournot equilibrium was established. In addition, previous works containing special cases of the corresponding theorem were cited. The uniqueness theorem of Cournot equilibrium was proved in [16]. Also, we should mention the work [17] on concave noncooperative games: the results presented therein can be used to prove the existence and uniqueness of Cournot equilibria, but Cournot oligopoly was not considered as an example in that paper. The authors [15] and [17] showed the existence of equilibria using Kakutani's fixed point theorem. A simpler proof of the existence and uniqueness of Cournot equilibria for differentiable functions $P(Q)$ and $C_i(q_i)$ was provided in [18]. According to the counterexamples in the book [19, *pp. 4, 5*], there may exist no Cournot equilibrium at all or there may exist several (nonunique) Cournot equilibria. The existence, uniqueness, and stability of Cournot equilibria can be considered with respect to firms producing multiple goods as well (e.g., see [20]). These issues were also covered in the book [19]. The existence of Cournot equilibria for the case of biconcave inverse demand functions was established in [21].

In a number of works, the economic processes under consideration were studied along the path of increasing the complexity of the mathematical tools involved. In the paper [22], the inverse demand function was understood as a multivalued mapping. In [23], the goal was to create a unified conceptual approach to Walrasian and Cournot equilibria as follows. Initially, the notion of economy was introduced in a way common for Walrasian equilibria with production. Cournot production was then defined as some probability measure, and Cournot equilibrium was understood as an equilibrium in the class of mixed (not pure) strategies. Other mathematical tools are also adopted to study Cournot oligopoly; for example, in [24], the theory of attractors was used to compare Walrasian and Cournot equilibria within discrete-time models.

The publication [25] considered firms producing a homogeneous good with different cost functions. The existence of Cournot equilibria was investigated. The focus was on the following question: how far can the previous conditions on the inverse demand function and cost functions (see [15, 18] and others) be relaxed without violating the existence theorem of Cournot equilibria? The results of that work partially overlap with those obtained independently in [26]. The paper [27] belongs to the same line of research. Subsequent publications on the subject were overviewed in [28] for the inverse demand function $P(Q) = a - bQ^{\beta}$, where β can be either positive or negative; the existence, uniqueness, and stability of Cournot equilibria were studied. The author [29] studied the effect of risk aversion on the strategies of firms.

Many researchers developed the classical Cournot competition model with application to certain economic problems. In [30], oil production was studied, and the profits of firms were maximized over a long time interval with discounting. Within the model described in [31], firms choose which labor (as a factor of production) to use, and their outputs are determined by their choice. According to [32], a mixed oligopoly is an oligopoly with one state-owned firm and several private firms. The state-owned firm seeks to maximize society's welfare, whereas private firms maximize profit (all firms produce a homogeneous good). The conditions were formulated under which Cournot equilibrium exists and is unique in such a mixed oligopoly. The effect of taxes on equilibrium outputs in a Cournot oligopoly was discussed in [33, 34]. In [35], the existence of Cournot equilibrium was proved under fixed cap prices. In [36], random yield models with Cournot competition were considered: the output of each firm has the form $q_i = \xi_i \bar{q}_i$, $i = 1, ..., n$, where $\bar{q}_i$ is the target quantity and ξ_i is a random variable. Total output, consumer surplus, and the entry of new firms into the market were investigated.

Correlated equilibrium (by definition, some probability measure) is a generalization of Nash equilibrium in pure strategies. The following result was established in [37, 38] under appropriate conditions: if there exists a unique Cournot equilibrium, it will also be a unique correlated equilibrium.

In several publications, reflexive game theory methods [39] were used to study Cournot competition. The conditions presented in [40] ensure convergence to a Cournot equilibrium when each firm gives the best response (in terms of maximizing its profit) to the outputs of other firms. In the case of one Leader and several Follower firms, the convergence to a Cournot equilibrium was discussed in [41]. Some examples of nonunique Cournot equilibria were given in [42]. Efficiency issues for static and dynamic games under different organizational modes of firms were studied in [43].

2. MARKET ENTRY AND EFFICIENCY ISSUES

Let the inverse demand function have the form $P(Q)=a-bQ$, where $a>0$ and $b>0$; let the cost functions have the form $C_i(q_i)=c_iq_i$, where $c_i\ge 0$. Assume that the capacity of each firm is unbounded. The values c_i, key for the subsequent analysis, are called *marginal cost* or unit production cost. Then, in accordance with formula (1),

$$\pi_i(q_1,\ldots,q_n)=\left(a-b\sum_{j=1}^{n}q_j\right)q_i-c_iq_i$$

and

$$\frac{\partial\pi_i}{\partial q_i}(q_1,\ldots,q_n)=a-2bq_i-b\sum_{j\ne i}q_j-c_i\,.$$

A strategy profile $q_1^*,\ldots,q_n^*$ is a Nash equilibrium if

$$\pi_i(q_1^*,\ldots,q_{i-1}^*,q_i^*,q_{i+1}^*,\ldots,q_n^*)$$
$$=\max_{q_i}\pi_i(q_1^*,\ldots,q_{i-1}^*,q_i,q_{i+1}^*,\ldots,q_n^*)$$

for all $i=1,\ldots,n$. The necessary condition of the maximum is the zero value of all partial derivatives:

$$\frac{\partial\pi_i}{\partial q_i}(q_1^*,\ldots,q_{i-1}^*,q_i^*,q_{i+1}^*,\ldots,q_n^*)=0$$

or

$$bq_i^*=a-c_i-b\sum_{j=1}^{n}q_j^*\,,\ i=1,\ldots,n\,.$$

Summing the last n equations gives

$$Q^*=\frac{n}{b(n+1)}(a-\bar c),\qquad(2)$$

where $\bar c=\frac{1}{n}\sum_{i=1}^{n}c_i$ is the average marginal cost. By assumption, $a>\bar c$. Then

$$P(Q^*)=a-\frac{n}{n+1}(a-\bar c).\qquad(3)$$

The values q_i^* and $\pi_i(q_1^*,\ldots,q_n^*)$ can be easily calculated:

$$q_i^*=\frac{1}{b(n+1)}\big(a-\bar c+(n+1)(\bar c-c_i)\big),\qquad(4)$$

$$\pi_i(q_1^*,\ldots,q_n^*)=\frac{1}{b(n+1)^2}\big(a-\bar c+(n+1)(\bar c-c_i)\big)^2.\qquad(5)$$

The point q_i^* is the maximum point of the profit function $\pi_i(q_1^*,\ldots,q_{i-1}^*,q_i,q_{i+1}^*,\ldots,q_n^*)$ since π_i is concave in the argument q_i. In the case $c_i>\bar c$, the problem of negative (zero) outputs q_i^* may arise. This problem will be considered in detail in Section 6. Here, we suppose that $c_i<\bar c+(a-\bar c)/(n+1)$ for all *i*. Note that the share of the output of firm *i* in total output is easy to calculate as well:

$$\frac{q_i^*}{Q^*}=\frac{1}{n}\frac{a-c_i}{a-\bar c}+\frac{\bar c-c_i}{a-\bar c}.$$

Due to formulas (4) and (5), if n is large enough, the difference ($\bar c-c_i$) can play a greater role than the difference ($a-\bar c$). For a firm, reducing the marginal cost c_i is among the most important tasks.

According to Cournot's conclusions, total output grows and the price of goods decreases when increasing the number of firms. In the linear model case, these conclusions are true; see formulas (2) and (3). For the nonlinear model, Cournot's conclusions remain valid under the conditions formulated in [44].

It follows from formula (3) that

$$P(Q^*)-\bar c\to 0\ \text{as}\ n\to\infty.$$

In other words, oligopoly turns into perfect competition with an unlimited increase in the number of firms. Oligopoly with firms producing a homogeneous good is called quasi-competition if total output grows and the price of goods decreases when increasing the number of firms. Under the identical cost functions of all firms, quasi-competitiveness alone is not sufficient for an oligopoly to become perfect competition with an unlimited increase in the number of firms; for details, see [45]. The case of different cost functions of firms was studied in [46]; as was shown therein, oligopoly is quasi-competitive under the conditions ensuring the uniqueness of Cournot equilibrium in the paper [16].

The dependence of the price of goods and the share of output produced by one firm on the number of firms was further examined in [47]. The author [48] also studied the transition from Cournot to perfect competition when the number of firms increases, including the stability of Cournot equilibrium; as it turned out, the system is stable if the capacities of firms decrease with increasing the number of firms. In the case of collusion between firms, the same issues were considered in [49] using conjectural variations, by analogy with [10]. The author [50] provided a rigorous mathematical proof of the so-called "folk theorem": if firms are small relative to the market, then there exists a Cournot equilibrium approximately representing perfect competition. By assumption, firms can enter and leave the market freely. Note that the average cost function of firms was adopted in the analysis. The research [50] was continued in [51], where the asymptotic properties of Cournot equilibria were analyzed. For the equilibrium with free entry of firms into the market, the dependence of the number of firms and total output on the demand and cost functions was investigated in [52].

The works [53–55] addressed society's welfare under Cournot competition. As it turned out within the mathematical models considered, the free entry of firms into the market can worsen rather than improve society's welfare. Consumer surplus and irrecoverable losses were also studied in [56]. With a certain simplification of the mathematical model, explicit elementary functions-based expressions were derived in [57] for some efficiency measures. In [58], firms producing several goods were considered to compare Cournot competition and collusion in terms of consumer surplus, total surplus, and the profit losses of firms. Note that uncertainty, which can make collusion inefficient, was neglected in that paper. The publication [59] was devoted to a Cournot oligopoly with free entry of new firms into the market and the following issues: how do the number of firms, the output of an individual firm, and total output depend on entry cost and on the market size? In [60], the restrictions of authorities for new firms entering the market were analyzed; one goal of the authorities is to improve society's welfare.

The paper [61] continued the studies initiated in [54, 55] that the entry of new firms into the market may be redundant. The R&D investment of firms to reduce costs was studied; the literature on Cournot equilibria was linked to the previous literature on R&D investment. In [62], models with firms choosing between two production technologies were considered. (The type of the cost function depends on this choice.) As was shown, in some cases, Cournot equilibria may not exist.

3. COURNOT AND BERTRAND OLIGOPOLIES

When studying Bertrand oligopoly, a usual assumption is that firms produce differentiated goods. In this case, firm *i* sets a price p_i for its good, $i=1,..., n$. The demand q_i for this good depends on all prices $p_1,..., p_n$. Then the profit of firm *I* is given by

$$\pi_i\left(p_1,..., p_n\right)=p_i q_i\left(p_1,..., p_n\right)-C_i\left(q_i\left(p_1,..., p_n\right)\right).$$

Firm *i* seeks to maximize the profit π_i. It is required to find a Nash equilibrium in the class of pure strategy profiles $p_1,..., p_n$.

The reason to consider firms with differentiated goods in Bertrand oligopoly is that, in the case of producing a homogeneous good, the unique Nash equilibrium may be the prices equal to marginal cost, i.e., zero profit for each firm. (Of course, to obtain this result, some assumptions must be accepted regarding the functions q_i and C_i; for example, see [2, 63, 64].) The conclusions concerning Bertrand oligopoly, particularly the comparison of Cournot and Bertrand oligopolies, essentially depend on whether the goods are substitutes or complements. (For the definition of substitutes and complements, we refer, e.g., to the monograph [1].)

Firms producing differentiated goods can also be considered within Cournot oligopoly. Then the profit of firm *i* is given by (instead of (1))

$$\pi_i\left(q_1,...,q_n\right)=P_i\left(q_1,...,q_n\right)q_i-C_i\left(q_i\right).$$

The equilibrium prices of goods under Cournot competition were compared with their counterparts under Bertrand competition, e.g., in the book [65, *pp. 68–78*]. In the duopoly problem considered therein, the equilibrium prices under Cournot competition are not lower than those under Bertrand competition. This result was confirmed in [66]; a linear duopoly model was investigated in detail, including the comparison of equilibrium prices and outputs, the profits of firms, consumer surplus, and total surplus. In the paper [67], similar results were established for nonlinear duopoly models. However, as was shown by the author [68], the conclusions of [66] may break when considering an arbitrary number of firms (not two).

In [69], Cournot and Bertrand equilibria were compared for firms producing a homogeneous good. Cournot and Bertrand oligopolies were compared by various criteria in [70–75] and other publications. In the linear case, Cournot competition and Bertrand competition were compared in detail in the paper [76]. Innovations to reduce cost were compared for Cournot and Bertrand oligopolies in [77]. The author [78] considered a duopoly with firms producing differentiated goods; the marginal cost of production was reduced using R&D expenditures both in a given firm and (with some factor) in the other firm; society's welfare under the two types of competition (Cournot and Bertrand) was investigated as a function of this reduction factor. Also, Cournot competition was compared with Bertrand competition in [79], but without using the concept of Nash equilibrium.

The paper [80] overviewed models in which output is the strategy of some firms and price is the strategy of the other firms (the so-called Cournot–Bertrand oligopoly). In addition, such an oligopoly was discussed in [81]. The paper [82] considered the environmental impact of production; the amount of pollution was compared for three models (Cournot duopoly, Bertrand duopoly, and Cournot–Bertrand duopoly).

4. STRATEGY CHOICE UNDER UNCERTAINTY. INFORMATION SHARING BETWEEN FIRMS. SELLING LICENSES

Consider a duopoly with $P(Q) = a - Q = a - (q_1 + q_2)$ and zero cost for both firms. Let *a* be a random variable taking values 60 and 120 with an equal probability of 0.5. The first firm is informed, i.e., knows the current (true) value *a*, and determines its output by formula (4) based on this value. The second (uninformed) firm proceeds from the fixed value $a = 90$ and also determines its output by formula (4).

For the current value $a = 120$, we obtain

$$q_1 = 40,\ q_2 = 30,\ Q = 70.$$

Then the formula $P = a - Q$ yields $P = 50$. The resulting profits are $\pi_1 = Pq_1 = 2\,000$ and $\pi_2 = Pq_2 = 1\,500$. But if the first firm informs the second one about the current value a, then $q_2 = 40$, $Q = 80$, and $P = 40$; in this case, $\pi_1 = \pi_2 = 1\,600$. In other words, the profit of the first firm decreases whereas the profit of the second firm increases.

For the current value $a = 60$, we obtain

$$q_1 = 20,\ q_2 = 30,\ Q = 50.$$

Then the formula $P = a - Q$ yields $P = 10$. The resulting profits are $\pi_1 = Pq_1 = 200$ and $\pi_2 = Pq_2 = 300$. But if the first firm informs the second one about the current value *a*, then $q_2 = 20$, $Q = 40$, and $P = 20$; in this case, $\pi_1 = \pi_2 = 400$.

Thus, the average profit of the first firm is $0.5(2\,000 + 200) = 1100$ if it does not share information with the second firm and is $0.5(1\,600 + 400) = 1\,000$ otherwise. That is, the first firm benefits nothing from sharing information with the second firm. Similarly, for the second firm, the average profit is $0.5(1\,500 + 300) = 900$ if it receives no information from the first firm and is $0.5(1\,600 + 400) = 1\,000$ otherwise. Therefore, information sharing is beneficial for the second firm.

This example can be somewhat generalized. Suppose that the first and second firms have information $(a + \xi_1)$ and $(a + \xi_2)$ about the value *a*, respectively, where ξ_1 and ξ_2 are random variables with zero mean. Is it beneficial for the firms to share this information? Does the answer depend on the correlation of the random variables ξ_1 and ξ_2? The information may be more complex than just a single numerical parameter. Is it then beneficial to share some part of this information with another firm? Which part should it be?

For an arbitrary number of firms, such a problem with a random variable *a* was considered in [83]. Assume that the inverse demand function and cost functions of all firms are linear. The main question addressed in the paper is as follows: it is beneficial for the firms to research the market jointly? As it turned out, the benefit of reduced market research costs may be smaller than the losses due to the information available to competitors. In addition, the impact of market research on consumer surplus was studied. This impact is found to be positive, i.e., market research increases consumer satisfaction. A similar model for duopoly was considered in [84], and the authors arrived at the following conclusion: information sharing between firms increases the correlation of their outputs, reducing their expected profits. The difference between the works [85] and [84] is the use of Gaussian random variables. As a result, in some cases, the exact values of means can be calculated. Besides, information sharing between firms in [85] was associated with subsequent collusion to reduce output.

For a similar problem, the possibility of partial information sharing between firms was considered in [86]. The author [87] compared Cournot and Bertrand competition. Also, information sharing under Cournot and Bertrand competition was examined in [88]. In that paper, a duopoly with firms producing differentiated goods was considered in the following statement: the inverse demand functions are linear; the free terms in the price equations form a random vector with the 2D Gaussian distribution. In [89], similar problems were considered for the equation $P = a - bQ$ with b as a random variable. The example at the beginning of this section shows that the expected profit of a more informed firm is higher. In [90], an example of the opposite nature was provided for a duopoly with firms producing differentiated goods and uncertainty in some cross-effects.

Similar problems, but when exchanging information about cost rather than market demand, were considered in [91]. According to the conclusions, within the linear model, such information sharing increases the expected profits of firms but decreases the expected surplus of consumers. The author [92] studied problems where firms may (or may not) share both cost and market demand information with other firms. Theorems on the existence of Cournot equilibria and on convergence to perfect competition as the number of firms tends to infinity were obtained. The paper [93] challenged the conclusion of [91] that information sharing between firms reduces expected consumer surplus. In [94], the profits of firms depend on their strategies (outputs and prices) and, moreover, on the unknown state of the environment (each firm receives some signal about this state). The equilibria obtained with and without information sharing about these signals were compared.

The potentially disorienting character of information shared with other firms was discussed in [95]. The paper [96] presented many of the previous results on information sharing between firms within a unified model. In [97], similar issues were touched upon with respect to the Cournot–Bertrand duopoly (the mixed case with outputs and prices as the strategies of different firms). Several researchers considered the problem where each firm maximizes revenue or some weighted average of profit and revenue. In this case, the interests of owners and managers running the firm are separated. In [98], the reasonability of information sharing between firms in such a model was analyzed, including the effect of information dissemination on society's welfare.

In the problem described in [99], firms receive noisy signals about market demand and cost and determine Cournot equilibrium outputs based on these signals. Using the strong law of large numbers, it was proved that total output converges almost surely to the total output corresponding to perfect competition as the number of firms tends to infinity. Note that the linear model with the same marginal cost for all firms was adopted therein. The paper [100] developed the results and approaches from [92] and [99]; a continuum of firms was considered, and some convergence results were obtained for Bayesian–Nash equilibria.

A duopoly with firms producing differentiated goods was studied in [101]. In this model, the inverse demand function is subjected to random disturbances: the prices of goods under given outputs become random variables. The strategy of each firm takes the following form. First, the firm selects the type of strategy (the output or price of the good); then it decides on the output or price, respectively. All decisions are made simultaneously by both firms. Some results on the existence of Bayesian–Nash equilibria and on the expected profits of firms were given. A similar problem was considered in [102], where the firm's decision is related to the degree of substitutability of goods.

In [103], the following model was presented. The uncertainty regarding demand and cost is described by some probability space (Ω, F, μ). The awareness of firm i is given by a σ-subalgebra F_i included in the σ-algebra F. It was discussed under which conditions the better-informed firm (i.e., the firm with a larger σ-subalgebra F_i) would obtain a higher expected profit. In [104], the existence of Bayesian–Nash equilibria was investigated within the same problem statement.

The paper [105] considered a linear model with all completely known parameters. Each firm believes that all other firms will determine their outputs with probability p based on Cournot equilibrium and with probability $(1 - p)$ in another way, $0 < p < 1$. The pessimistic firm supposes "the worst-case" way whereas the optimist firm "the best-case" way. The best responses of firms as well as their possible outputs, prices, and profits were examined.

In [106], the inverse demand function has the form $P(Q) = \max(a - Q, 0)$, where a is a random variable taking two values ("large" and "small" with equal probabilities of 0.5). The ranges ensuring the existence of a unique Cournot equilibrium and exactly two Cournot equilibria were found. The expected profits and expected total surplus in such a model were compared with those in the model without uncertainty (i.e., in the model with the same "large" and "small" values). The paper [107] confirmed the conclusion of [106] that under uncertainty regarding market demand, the requirement of nonnegative prices may lead to a non-unique Cournot equilibrium. However, for models with free sales (firms can sell less goods than they produce), such non-uniqueness does not arise. In [108], uncertainty was related to the capacity of firms. In [109], Pareto equilibria in Cournot oligopoly were studied under uncertainty regarding demand.

In the publication [110], under a limited demand d for a good, the profit to be maximized is given by

$$\pi_i(q_i, Q) = P(Q)\min\left(q_i, \frac{q_i d}{Q}\right) - C_i(q_i),$$

where $C_i(q_i) = c_i q_i$ (c.f. formula (1)). Cournot equilibria were determined in the duopoly with this profit function. The demand d_t was then assumed to be a continuous-time random process satisfying some stochastic differential equation, $t \in [0, T]$. For random processes q_{1t} and q_{2t}, a control problem was studied to maximize functionals expressing the total profits of firms over the time interval from 0 to *T* with discounting.

The paper [111] was devoted to a duopoly where each firm produces two goods, *X* and *Y*. For technological reasons, the output of each good is a certain share of the firm's total output: $q_{iX} = \gamma q_i$ and $q_{iY} = (1-\gamma) q_i$, where $0 < \gamma < 1$, $i = 1, 2$. The inverse demand function is specific for each good:

$$P_X = a_X - b_X \left(q_{1X} + q_{2X}\right) \text{ and}$$

$$P_Y = a_Y - b_Y \left(q_{1Y} + q_{2Y}\right).$$

The random vector $\left(a_X, a_Y\right)$ was supposed to obey the bivariate Gaussian distribution. Cournot competition was considered: firms determine the values q_1 and q_2. It was investigated how the availability of information about the true inverse demand function would affect the firm's expected profit.

In [112], a nonlinear duopoly model was considered as follows. One firm (e.g., firm 1) receives revenue from selling a license to another firm (firm 2). By doing so, firm 2 reduces cost. In this setup, formula (1) is replaced by

$$\pi_1\left(q_1, q_2\right) = P(Q) q_1 - C_1\left(q_1\right) + \left(r q_2 + f\right),$$

$$\pi_2\left(q_1, q_2\right) = P(Q) q_2 - C_1\left(q_2\right) - \left(r q_2 + f\right),$$

where f is a fixed fee and r denotes the royalty. It was studied under what conditions firm 2 would buy a license, both firms would remain active, or firm 1 would become a monopolist. The cases of drastic and non-drastic technologies were examined separately.

In [112], the linear model analysis from [113] (with calculations similar to those yielding (2)–(5)) was transferred to the nonlinear case. For the linear case, the author [113] presented the relative advantages and disadvantages of fixed fees and royalties. Also, his considerations were transferred to the nonlinear quadratic case in [114]; the differences from the linear case were discussed.

The researchers [115] studied a similar problem within the linear duopoly model with non-drastic technologies and foreign (firm 1) and domestic (firm 2) agents. Consumer surplus and society's welfare were studied for different cases ($f > 0$ and $r = 0$; $f = 0$ and $r > 0$; $f > 0$ and $r > 0$). The paper [116] investigated the possible impact of the limited capacity of the firm selling the license. In [117], the sales of license was considered in the case where firms 1 and 2 produce differentiated goods; Cournot competition and Bertrand competition were compared. Also, Cournot competition and Bertrand competition in the sales of licenses with R&D cost were compared in [118]; partially, the conclusions are opposite to those of [66].

5. CARTEL FORMATION

If a cartel is formed, the optimization problem will change. Cartelized firms seek to maximize total profit instead of their individual profits. The distribution of total profit among the firms may or may not be included in the analysis. Thus, within the mathematical model under consideration, cartel formation does not differ from the merger of firms, although it is not the same in reality.

Consider the model with the linear inverse demand function $P(Q) = a - bQ$ and the same cost function $C_i\left(q_i\right) = c q_i$ for all firms. By $\pi(m)$ we denote the profit of one firm given m firms in the market. Assume that n firms initially operate in the market, and k firms form a cartel. Total profit is distributed equally among all cartelized firms. Then the member firms benefit from forming the cartel if

$$\pi(n-k+1) - k\pi(n)$$
$$= \frac{(a-c)^2}{b}\left(\frac{1}{(n-k+2)^2} - \frac{k}{(n+1)^2}\right) > 0.$$

(Here, formula (5) is used.)

The paper [119] established a result called paradoxical in many subsequent works: the difference above is positive only for k close enough to n, approximately for $k \geq 0.8n$.

One drawback is that this model does not distinguish the cartel from other firms and neglects the "big" size of the newly created firm. To eliminate this drawback, the authors [120, 121] considered the same problem with the cost function $C_i\left(q_i\right) = \frac{q_i^2}{2k_i}$, where k_i is the capital of firm i. The impact of cartel formation on society's welfare was also studied in [121].

The problem of cartel formation under Bertrand competition was considered in [122]. The results obtained therein radically differ from those of [119]. Cartel formation increases the profits of all firms, while the profits of non-cartelized firms are greater than the profits of cartelized firms. In addition, an increase in the number of cartelized firms raises the profits of the cartelized firm.

Within the model proposed in [123], n firms produce a homogeneous good; among them, k firms are leaders and the rest are followers. Followers compete in the Cournot sense given the total output of leaders. Leaders realize this fact (i.e., they know the reaction functions of followers to their outputs) and also have Cournot competition among themselves by determining their outputs. Initially, all firms are assumed to be identical. The author [124] studied society's welfare for such a model. Similar problems with quadratic cost functions were considered in [125].

The following model was used to analyze cartel stability in [126]. In an economy sector, there are *n* identical firms producing a homogeneous good. Then k firms form a cartel and set the price of the good to maximize their total profit. Let $\pi_c(k)$ and $\pi_f(k)$ denote the profit of the cartelized and non-cartelized firms, respectively. A cartel is called *internally stable* if $\pi_c(k)/k \geq \pi_f(k-1)$, i.e., a cartelized firm will not increase its profit by leaving the cartel. A cartel is called *externally stable* if $\pi_f(k) \geq \pi_c(k+1)/(k+1)$, i.e., a non-cartelized firm will not increase its profit by joining the cartel. A cartel is called stable if it is both internally stable and externally stable. The paper [127] examined the process of forming a stable cartel by combining the ideas from [126, 128]. Note that neither Cournot nor Bertrand competition was considered in [126]. The concept of cartel stability in relation to Cournot competition, Bertrand competition, and leader-follower games was studied in many publications; for example, see [129–133].

Choosing a coalition structure that best matches the interests of players is a central problem in cooperative game theory. Such approaches are also applied to oligopoly games. In [134], a very general notion of equilibrium was considered; it includes the endogenous determination of the best coalition structure under a given set of admissible coalition structures. Particular cases of this equilibrium are Nash equilibrium for a noncooperative game and the core for a cooperative game with non-transferable utility. In both cases mentioned, a given set of admissible coalition structures is a singleton. In the former case, only coalitions consisting of one player are allowed. In the latter case, only coalitions consisting of all players are allowed. The paper [135] separately considered coalitions with positive and negative externalities, i.e., players not included in the coalition benefit (lose, respectively) from coalition formation. Stability was analyzed for different coalition formation rules (e.g., the consent of all coalition members is required for new entrants to join the coalition; the coalition can break up or merge with other coalitions, etc.). Research works focused on *partition function form games* (instead of cooperative games in the characteristic function form) were surveyed in [136].

6. PRACTICAL CALCULATION OF COURNOT EQUILIBRIA AND THEIR USE IN APPLICATIONS

Consider n firms producing *m* goods; q_i, the output of firm *i*, is an *m*-dimensional vector; L_i, the capacity of firm *i*, is also an m-dimensional vector; $P(Q)$ is the *m*-dimensional vector of prices; $P(Q)q_i$ in formula (1) is understood as an inner product. If $q^* = \left(q_1^*,..., q_n^*\right)$ is an equilibrium, then q_i^* maximizes $\pi_i\left(q_1^*,..., q_{i-1}^*, q_i, q_{i+1}^*,..., q_n^*\right)$ as a function of the argument q_i for any *i*. A common approach is to find Cournot equilibrium by solving the complementarity problem.

Let π_i be a concave and continuously differentiable function of the argument q_i. Then for any $k = 1,..., m$, the partial derivative $\frac{\partial \pi_i}{\partial q_{ik}}\left(q_i^*\right)$ is nonpositive if $q_{ik}^* = 0$; equal to zero if $0 < q_{ik}^* < L_{ik}$; and nonnegative if $q_{ik}^* = L_{ik}$. In other words, there exist values u_{ik} and v_{ik} such that

$$\frac{\partial \pi_i}{\partial q_{ik}}\left(q_i^*\right) + u_{ik} - v_{ik} = 0, \tag{6}$$

where $u_{ik} \geq 0$ if $q_{ik}^* = 0$ and $u_{ik} = 0$ if $q_{ik}^* > 0$; $v_{ik} = 0$ if $q_{ik}^* < L_{ik}$ and $v_{ik} \geq 0$ if $q_{ik}^* = L_{ik}$. By the definition of u_{ik} and v_{ik}, it follows that

$$u_{ik} q_{ik}^* = 0 \text{ and } v_{ik}\left(L_{ik} - q_{ik}^*\right) = 0. \tag{7}$$

With the m-dimensional vectors $u_i = \left(u_{i1},..., u_{im}\right)'$ and $v_i = \left(v_{i1},..., v_{im}\right)'$ (prime means transpose), the relation (6) can be written as

$$\operatorname{grad} \pi_i\left(q_i^*\right) + u_i - v_i = 0,$$

and the relations (7) as the zero scalar products

$$u_i q_i^* = 0 \text{ and } v_i\left(L_i - q_i^*\right) = 0.$$

In addition, $u_i \geq 0$, $v_i \geq 0$, $q_i^* \geq 0$, and $L_i - q_i^* \geq 0$. (For vectors, the nonstrict inequalities with "≥ 0" hold componentwise.)

Consider the $2m$-dimensional vectors

$$w_i = \begin{pmatrix} q_i \\ v_i \end{pmatrix}, \; i = 1,..., n,$$

and the function

$$f_i\left(w_i\right)=\begin{pmatrix} v_i-\text{grad }\pi_i\left(q_i\right) \\ L_i-q_i \end{pmatrix}$$

that translates $2m$-dimensional vectors into $2m$-dimensional vectors. For $q_i=q_i^*$, the scalar product satisfies the equality

$$w_i f_i\left(w_i\right)=0\,, \quad i=1,\ldots,n\,. \qquad (8)$$

We introduce the notations

$$q=\begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix},\ v=\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix},\ L=\begin{pmatrix} L_1 \\ \vdots \\ L_n \end{pmatrix},$$

$$\text{grad }\pi(q)=\begin{pmatrix} \text{grad }\pi_1\left(q_1\right) \\ \vdots \\ \text{grad }\pi_n\left(q_n\right) \end{pmatrix},\ w=\begin{pmatrix} q \\ v \end{pmatrix},$$

$$\text{and } \ f(w)=\begin{pmatrix} v-\text{grad }\pi(q) \\ L-q \end{pmatrix}.$$

Thus, Cournot equilibria can be determined by solving the complementarity problem: it is required to find vectors $w\geq 0$ such that $f(w)\geq 0$ and

$$w\,f\left(w\right)=0\,.$$

The existence of solutions for complementarity problems was explored in [137, 138]. Due to (8), if q^* is a Cournot equilibrium, then there exists a vector v^* with nonnegative components such that the vector

$$w^*=\begin{pmatrix} q^* \\ v^* \end{pmatrix}$$

is the solution of the complementarity problem.

Some literature on the application of this approach for finding Cournot equilibria was cited, e.g., in [139]. Based on the same ideas, Cournot equilibria can also be determined in the case of nondifferentiable demand functions [140]. The authors [141, 142] proposed to calculate Cournot equilibria as solutions of some mathematical programming problems. In [143], another algorithm for finding Cournot equilibria was presented using the ideas from [137, 138, 142]. Based on complementarity problems, the paper [35] considered the existence and uniqueness of Cournot equilibria under price constraints as well as algorithms for finding Cournot equilibria.

Also, we note the following aspect: if firms produce a homogeneous good, then in some cases, Cournot equilibria can be determined by directly analyzing the multivalued mapping that associates with each strategy profile of other firms the best responses of a given firm [144]. Cournot equilibria can be found using the so-called tâtonnement process, which also involves best responses [145, *pp. 84–97*]. For firms with differentiated goods, this process was discussed in [146].

In [147], solutions of complementarity problems were adopted to study Cournot competition in electricity markets with uncertainty. Cournot equilibria were also used to analyze electricity markets in [139, 140, 148].

The researchers [149] compared Cournot competition and Bertrand competition in the software industry. One firm sells a platform (e.g., an operating system), and the other two firms supply application software. The comparison results for this problem quite differ from the ones obtained commonly.

In [150], competition between air carriers and railroad companies in high-speed transportation was treated as a Cournot duopoly. A model with linear inverse demand functions and zero cost was applied. By assumption, air carriers can have price discrimination (sell tickets to different groups of passengers at different prices) whereas railroad companies cannot. As it turned out, price discrimination increases the profit of air carriers. Consumer surplus and society's welfare were also studied.

CONCLUSIONS

This survey has presented the main sections of Cournot oligopoly as a mathematical discipline:

– the existence, uniqueness, and stability of equilibria;

– market entry for new firms and efficiency issues;

– a comparison of Cournot oligopoly, Bertrand oligopoly, and Cournot–Bertrand oligopoly;

– consideration of uncertainty by probabilistic methods (model parameters, deviations of real outputs from the planned ones, etc.);

– information sharing among firms and the sale of licenses;

– cartel formation;

– numerical methods for determining Cournot equilibria, primarily for multidimensional problems.

(Consideration of uncertainty by fuzzy set theory methods goes beyond the scope of this paper.)

The application of this mathematical theory in various branches of the economy has been considered.

The oligopoly theory has been evolving toward including more and more real processes in the mathematical model. In this way, recommendations for control and management can be developed. Mathematical modeling of information sharing among firms and cartel formation is important to determine which information exchange or the level of interaction between firms best matches the interests of firms. Also, such mathematical models are crucial from the society's welfare point of view to prevent a significant increase in prices for goods.

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