THE VICTORY FUNCTION AND ITS APPLICATION IN CONFLICT MODELING

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Abstract. The victory function (VF) in combat and special operations is a separate class of contest success functions (CSFs) describing the probability of a participant's success in a competition or conflict. This paper briefly characterizes aggregate functions, which include production functions, utility functions, CSFs, power (might) indices of countries, etc. Substantive (postulates) and formal (axioms, properties) requirements for the VF are given. Probabilistic (based on A.N. Kolmogorov's law of target destruction and the Weibull distribution) and substantive (based on the framework of military science and practice) justifications of the VF are presented. Economic models of conflict and appropriation are overviewed. Using the security model and the VF, three problems of distributing a disputed resource (territory and population) between countries are formulated. A promising line of further research is to develop conflict theory at several levels, namely, in the theater of military operations, the intercountry, and geopolitical levels.

Keywords: mathematical model, security, combat and special operations, victory function, postulates of conflict technology, models of conflict and appropriation, military cybernetics.

INTRODUCTION

Aggregate functions are used in many fields. They represent mathematical relationships between the expected result (opportunities) and several factors (efforts, resources, technologies, etc.). An aggregate function maps a vector of quantitative indicators (resource types) to a real number. This number characterizes the potential or latent capability of a system rather than the result observed.

In computer science and statistics, aggregation conventionally refers to operations like summation, averaging, or computing extremal values (minimum, maximum, median, etc.). In systems analysis, aggregation—combining several elements into a single whole—is closely related to the emergence property (the presence of some properties in a system that are not inherent in its individual components) [1] and uncertainty (the category of uncertainty may be sufficient to consider the factors of complexity and emergent behavior [2]).

Among the most extensively studied aggregate functions we note production functions [3], utility functions [4–6], contest success functions (CSFs, see the review [7]), the indices (models) of geopolitical power and might of countries (see the paper [8] and the comprehensive review in the monograph [9]), etc. Despite criticism of aggregate production functions (the so-called "Cambridge debates" [10, 11]), they remain the foundation of applied statistical studies and economic growth theories.

In the 1950s, the scope of traditional economics was expanded: in addition to production and trade, researchers began considering the problems of appropriation (seizure of foreign products or protection of domestic ones) [12]. In conflict theory, CSFs, which describe the probability of a participant's success in a competition (conflict), are analogous to production functions in production theory and utility functions in consumption theory. Conflict was usually considered to be the result of incomplete and asymmetric information or even the result of irrationality. It was shown that military operations change the strategic positions of opponents in the long term. In this case, the party (to conflict) considering short- and long-term effects may choose war, and this choice will correspond to rational and far-sighted behavior based on complete information. If one or more parties consider the future to be sufficiently important, the outbreak of war should be expected [7].

The objective of this paper is to study the victory function (VF) in combat and special operations as a distinct subclass of CSFs and to formulate problems related to its application in conflict and appropriation models.

1. REQUIREMENTS FOR CONTEST SUCCESS FUNCTIONS

Let us categorize the requirements for the aggregate functions of contest success (victory in combat and special operations) as *substantive* (implied by the postulates of military science and systems analysis) and *formal* (axiomatic requirements).

1.1. Substantive Requirements

Figure 1 shows a selection of works that have significantly influenced the development of military science and its practical application.

J. Bernoulli's treatise on the art of conjectures [13] was the first systematic book of probability theory. It presented combinatorics, binomial distribution, and the first version of the law of large numbers.

The "classical stage" in the development of probability theory was completed by P. Laplace's work on the analytical theory of probability [14]. The author examined discrete and continuous random variables[1], introduced the concepts of probability density and characteristic function, gave the formula for total probability, and proved the convergence of the binomial distribution to the normal distribution (the Moivre–Laplace theorem) as the number of trials tends to infinity. As often been the case in later periods (e.g., the 20th century, with the successes of cybernetics and disappointment in it [15]), the number of works on probability theory continued to grow in the 19th century, and attempts were undertaken to extend its methods far beyond reasonable limits: to ethics, psychology, law, theology, etc., compromising the science.

C. Clausewitz undoubtedly belongs to the classics of European and world science of the modern era. His fundamental *Von Krieg* [16] had a profound influence on the development of military science, being still relevant nowadays. This study does not aim to provide a comprehensive exposition of Clausewitz's intellectual legacy—a task better suited to dedicated military historians and practitioners. Instead, following the established scholarly tradition [17, 18], we will comparatively analyze the perspectives on the role of uncertainty in war and combat as held by C. Clausewitz and L. Tolstoy [19].

Adhering to the approach described in [20, 2], we define the *uncertainty* of military (combat) operations as the possibility of certain events accompanying these operations that affect their implementation and result, but may/may not occur. Due to the uncertainty of military operations, it is impossible to predict a priori the characteristics of its result, as well as the time and effort (resources) to achieve the result.

The *measurable uncertainty* of military operations is defined as the possibility of events described by certain regularities that may/may not occur. Such events can be analyzed using quantitative methods (e.g., probabilistic and statistical) based on previous measurements or fundamental laws (together with the assumption of invariable conditions and regularities).

The *true uncertainty* of military operations is the possibility of unique (or rarely recurring) events that cannot be explained by known regularities.

The fundamental distinction between true uncertainty and measurable uncertainty is that the former's events occur due to unknown factors (a frequent and important, albeit special, case is the active choice of an individual), whereas the latter's events, although unpredictable, are described by known regularities [2].

As famously observed by Clausewitz [16], war is a realm of uncertainty; three-quarters of war action's foundation lies in the fog of the unknown. His way of eliminating uncertainty is a subtle, flexible, and penetrating mind. The unreliability of information and assumptions leads to the fact that those fighting in reality face a completely different situation than they expected. Many reports received during war contradict each other; there are even more false reports, and the vast majority of them are unreliable. Another recipe to eliminate uncertainty at the strategic level is to provide a decisive numerical superiority (one and a half to two times) over the enemy in terms of forces and means and take the moral factor into account. Uncertainty decreases when passing from the tactical to strategic level. The third approach to deal with uncertainty is to have reserves. Reserves are a means to counter

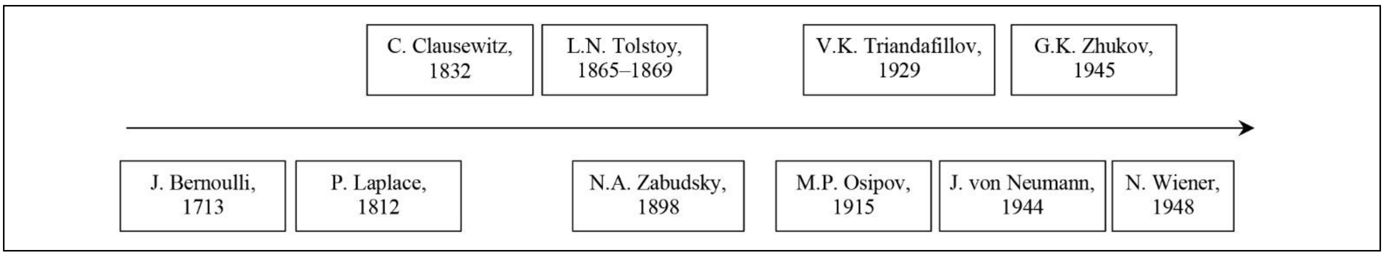

Fig. 1. Key influential works of military science and practice.

[1] The concepts of a "random variable" and "distribution function" appeared in the 20th century in the works of the Russian probability school.

the enemy's unforeseen actions and, moreover, to correct the unpredictable outcome of a battle under an unfavorable turn of events.

The moral factor decisively influences on the outcome of a battle. At the tactical level, morale gradually recovers. The moral influence of victory grows exponentially depending on the scale of military operations. This may also indicate a lesser degree of uncertainty at the strategic level. Another condition determining the moral weight of victory is the numerical ratio of the forces fighting each other. Defeating the enemy with small forces is evidence of overall superiority over it. However, in reality, the actual ratio of forces at the tactical level is usually unknown (with rare exceptions when the battlefield is clearly defined), and at the strategic level, it becomes known many years later. Therefore, such information about the moral ratio of the parties has no effect on current events.

In conclusion, Clausewitz argued that the bravery and spirit of armed forces have always increased their physical strength, and this will continue to be the case. However, in some periods in history, a sharp superiority in the organization and armament of forces gave a significant moral advantage; in other periods, superiority was provided by the mobility of forces[1]; later on, newly introduced tactical systems had an impact, then the art of war was dominated by the desire to use the terrain skillfully, etc. Nowadays, armies have become so similar in terms of weapons, equipment, and training that there is no noticeable difference between the best and worst of them in this respect. The degree of the development of scientific forces still significantly differs between armies, but this difference only makes some armies the initiators and inventors of various improvements and the others their quick imitators. As the above factors tend to equilibrium, Clausewitz concluded, the numerical preponderance of forces regains its status as the most decisive element in strategic calculations.

Tolstoy considered wars from a broader perspective (philosophical, civilizational, political, cultural, ethical, etc.) and highlighted the true uncertainty of military activity: "Napoleon, who seems to us to have been the leader of all this movement (as the figurehead on the prow of a ship may seem to a savage to guide the vessel), Napoleon during all that time of his activity was like a child who, holding on to the straps inside a carriage, imagines that he is driving it." [19]. Countless free forces (for nowhere is a person freer than in a battle, where it is a matter of life and death) influence the direction of the battle, and this direction can never be known in advance and never coincides with the direction of any force. Historians write after the fact: *chance* created the situation, and *genius* took advantage of it. But the words "chance" and "genius" mean nothing that really exists and therefore cannot be defined. The officers—the heroes of Tolstoy's novel—wander across the battlefield, blinded by gun smoke, without the slightest idea of what is happening. The author explained the true uncertainty of war by the fact that the human mind cannot comprehend the causes of events in their entirety, but the desire to find them is inherent in the human soul.

Today, Tolstoy's ideas are particularly relevant and significant for military experts and researchers, particularly in connection with the adoption of artificial intelligence (AI) systems. Undoubtedly, AI will be used to solve simple and typical tasks (machine vision, sensor data processing in near real time, etc.). According to T. Lipsky (the United States Military Academy, West Point) , given its interest in adopting AI, the army risks forgetting the importance of keeping the boundary between man and machine, not only for ethical reasons but also because, as Tolstoy showed[2], a commander cannot obey algorithms and templates in a battle. Command remains an "artistic and psychological" task [18]. AI is ill-suited to rapidly changing situations, where actions depend largely on context and require human judgment. This is how any war can be characterized. Therefore, it is important to include healthy skepticism about AI in army command instructions. Military personnel must learn to make plans and manage their implementation without external assistance before incorporating AI into the process, and they must undergo regular recertification in this area throughout their military careers [18]. As shown in the paper [21], machine learning does not lead to general AI, and the myth of AI weakens belief in human potential. O.P. Kuznetsov noted that understanding is interpretation in terms of a person's worldview; our brain constructs a worldview, and it is structured through the categorization of human experience; meanings (senses) are formed earlier than conceptual structures; meanings are based on biological and social

[1] Superiority in mobility was observed in the 1930s–1950s (the transition of armies to continuous motorization of forces, the creation of motorized rifle and mechanized units and formations): motorized groups (divisions) were created for deep operations and deep encirclement of enemy's infantry units and formations.

[2] In *War and Peace*, the role of M.I. Kutuzov was described as follows: "By long years of military experience he knew, and with the wisdom of age understood, that it is impossible for one man to direct hundreds of thousands of others struggling with death, and he knew that the result of a battle is decided not by the orders of a commander in chief, nor the place where the troops are stationed, nor by the number of cannon or of slaughtered men, but by that intangible force called the spirit of the army, and he watched this force and guided it in as far as that was in his power." [19].

goals; cognitive processes involve not only the brain but also the body, and understanding is connected with actions in the environment, and knowledge of the latter is contained in the worldview.

Starting from the second half of the 19th century, in Russia and European countries, the achievements of natural science and probability theory were actively adopted in the theory and practice of artillery and firing control [22]. During World War I, the first works on modeling military and combat operations were published (the Osipov–Lanchester models [23, 24]).

Based on military statistics on the largest battles of regular armies, M.P. Osipov developed the so-called quadratic model of a battle, laid the foundations of the theory of combat potential, and quantitatively assessed the role of the moral factor and the art of command. (According to his approach, victory depends not on the duration of a battle but mainly on the losses suffered by the parties; therefore, it would be more accurate to assume that a battle lasts until the losses of one party reach a certain percentage. This percentage can be considered to be 20% on average [23].)

V. K. Triandafilov's book on the nature of operations of modern armies is noteworthy in two respects. First, the author developed a new branch of military art, i.e., operational art (in addition to tactics and strategy); second, the book gives every reason to consider Triandafilov the founder of systems analysis. In the preface to the first edition of the book, the author outlined the methodology of his research as follows. First, the material basis of military operations is examined—the armament of forces, their number, organization, and other important data about the situation affecting the nature of military operations. Then, based on all the data above, the issues of modern tactics, individual operations, and a series of consecutive operations are examined. All postulates are expressed in terms of particular numbers as well as tactical and operational norms. This is the only way to show the difference between the present and the past and the direction in which military science is evolving. Moreover, all the numerical data are *indicative*; norms may change for each particular situation. The task of a commander (practical leader) is to determine the nature and extent of such changes [25, *pp. 13, 14*]. Triandafilov's contribution to military science and systems analysis (as a branch of cybernetics) was analyzed in the monograph [9].

In the 1930s–1940s, there was a rare case in history when both game theory (a branch of operations research) and military practice solved the same problem in parallel: increasing efficiency by eliminating patterns (using the so-called "mixed strategies"). In 1944, the seminal work *Game Theory and Economic Behavior* was published [5]. The cornerstone of modern noncooperative game theory is the concept of Nash equilibrium, which exists for all finite games [26]. Game-theoretic problem statements were reviewed in [9].

In the legacy of outstanding Soviet commander G.K. Zhukov, we mention only two results (see the publication [27]). First, Zhukov identified the same factors affecting the success of any battle (combat and military operation), i.e., those valid for all levels: tactical, operational, and strategic (operational-strategic). Second, starting from 1943, he utilized the ideas of "mixed strategies" when planning strategic offensive operations (breaking through the enemy's prepared defense).

Operations research, as an applied mathematical discipline, appeared during World War II. Its purpose was defined as providing commanders with quantitative grounds for decision-making. Subsequently, this discipline became a branch of cybernetics [28], which is now understood as the science of organizing and managing systems [15]. Today, the network-centric approach is extremely fashionable and productive. It includes the principles of organizing and analyzing any networks in general and, in particular, those "assembled" for a combat mission, at the right time and in the right place [15].

Based on the above considerations, we draw the following conclusions (substantive requirements for the VF).

- *Zhukov's postulate:* the VF shall have the same form for describing military operations at all levels (tactical, operational, and operational-strategic), including special operations (guerrilla, anti-guerrilla, sabotage, reconnaissance and diversion, counterterrorism, etc.);
- *Clausewitz –Tolstoy's postulate:* the VF shall reflect both true and measurable uncertainty;
- *Osipov–Triandafilov's postulate:* the VF shall consider the number of armed forces of each party, their morale and technological characteristics, situational specifics, and the existing and prospective weapon systems.

1.2. Formal Requirements

Historically, the first VF stems from Osipov's battle model. In the absence of operational losses and reserves, a typical battle is described by the system of differential equations

$$\frac{dx(t)}{dt} = -a_y y(t), \quad \frac{dy(t)}{dt} = -a_x x(t),$$

where $x(t)$ and $y(t)$ denote the number of armed forces of the first and second parties, respectively, at a time instant t; a_x and a_y are their striking efficiencies. From the equal forces condition $y_0 = x_0\sqrt{a_x / a_y}$, we obtain Osipov's indicator VF (the probability of the first party's victory):

$$p_x(x, y) = \begin{cases} 1, & x\sqrt{a_x} > y\sqrt{a_y}, \\ 0.5, & x\sqrt{a_x} = y\sqrt{a_y}, \\ 0, & x\sqrt{a_x} < y\sqrt{a_y}. \end{cases} \quad (1)$$

One model of a guerrilla war [29] has the form

$$\frac{dx(t)}{dt} = -a_y y(t), \; \frac{dy(t)}{dt} = -a_x x(t)\frac{y(t)}{y_0},$$

where $x(t)$ is the number of regular armed forces, and $y(t)$ is the number of guerrillas. From the equal forces condition $y_0\sqrt{2a_y} = x_0\sqrt{a_x}$, we obtain S. Deitchman's VF

$$p_x(x, y) = \begin{cases} 1, & x\sqrt{a_x} > y\sqrt{2a_y}, \\ 0.5, & x\sqrt{a_x} = y\sqrt{2a_y}, \\ 0, & x\sqrt{a_x} < y\sqrt{2a_y}. \end{cases}$$

(As a matter of fact, it follows from the guerrilla war model [29].)

In other words, *ceteris paribus*, parity is achieved if the number of regular armed forces exceeds, by $\sqrt{2} \approx 1.4$ times, that of guerrillas (cf. the expression (1)).

The following class of VFs (those of success in a competition or auction) has been fairly well investigated by now:

$$p_x(x, y) = \frac{f_x(x)}{f_x(x) + f_y(y)}, \quad (2)$$

where $f_x(\cdot)$ and $f_y(\cdot)$ are nonnegative strictly increasing functions. Here are some of the most common functional forms of model (2) [7, 9].

G. Tullock's model

$$p_x(x, y) = \frac{x^\mu}{x^\mu + y^\mu} = \frac{(x/y)^\mu}{1 + (x/y)^\mu}, \quad (3)$$

where $0 < \mu \le 1$ is the determination parameter of the parties, belongs to the class of models based on the ratio of the forces (power) of the parties involved.

The model proposed by D. McFadden and J. Hirshleifer,

$$p_x(x, y) = \frac{e^{\mu x}}{e^{\mu x} + e^{\mu y}} = \frac{1}{1 + e^{\mu(x-y)}}, \quad (4)$$

belongs to the class of models based on the difference in the forces of the parties. The probit model $p_x(x, y) = \Phi(x - y)$, where Φ indicates the Laplace function, is another representative of this class.

The probabilistic justification of conflict functions proceeds from an analysis of the influence of neglected factors (random errors) on the result. In the general case, regression functions are of the form $Y_x = h(x, \varepsilon_x)$ and $Y_y = h(y, \varepsilon_y)$, where the error functions ε_x and ε_y have zero mean. Then the probability of the first party's victory in a conflict is given by

$$p_x(x, y) = P(Y_x > Y_y) = P\left(h(x, \varepsilon_x) > h(y, \varepsilon_y)\right).$$

Conflict functions have been axiomatized, in particular, by R. Luce [30] and S. Skaperdas [31]. The axiomatic system is based on the *Independence of Irrelevant Alternatives*: in the context of conflict, this property requires that the outcome of a conflict between any two parties depends only on the amount of armaments possessed by them, and not on the amount of armaments possessed by third parties. The next important requirement for conflict functions is *zero-degree homogeneity*, i.e., $p_x(tx, ty) = p_x(x, y)$ for all $t > 0$. Models (3) and (4) have *symmetry or anonymity*: if the efforts of the parties interchange places, the probabilities of their victory will also do so.

Skaperdas et al. noted that despite the rich literature on the modeling of conflicts, contests, and auctions, only a small number of publications have addressed the verification of conflict functions based on real data [32].

2. PROBABILISTIC AND SUBSTANTIVE JUSTIFICATION OF THE VICTORY FUNCTION

2.1. Probabilistic Justification of the Victory Function

In 1945, A.N. Kolmogorov proposed a firing efficiency criterion based on the law of target destruction, i.e., the probability of destroying a single or group target (the expected number of targets destroyed) depending on the number of shots fired at this target. The probability of destroying a target (event A) with x hits is given by [33]

$$P(A | x) = 1 - e^{-\alpha x}, \quad (5)$$

where $\alpha > 0$ is a parameter. The expression (5) describes the exponential distribution, a special case of the Weibull distribution. It has a wide range of applications (the reliability of technical systems, queuing systems, etc.) and is closely related to the concept of a Poisson process. For such a process, the intervals between successive events are independent random variables with the exponential distribution, and α is the mean number of events per unit time.

Let the random variables X and Y be the numbers of hits required to destroy the enemy's targets by the first and second parties, respectively. Using the Weibull distribution, we find the probabilities of successful task completion (destroying the enemy's targets) by the parties:

$$F_x(x) = 1 - e^{-(\alpha_x x)^m}, \ \alpha_x = \beta_x r_x,$$
$$F_y(y) = 1 - e^{-(\alpha_y y)^m}, \ \alpha_y = \beta_y r_y,$$

where $m > 0$ is the scale parameter of combat operations; r_x and r_y are the numbers of combat units at the disposal of the first and second parties, respectively; finally, $\beta_x > 0$ and $\beta_y > 0$ are the combat efficiencies of the units of the parties.

The densities of the random variables X and Y are

$$f_x(x) = \alpha_x m(\alpha_x x)^{m-1} e^{-(\alpha_x x)^m},$$
$$f_y(y) = \alpha_y m(\alpha_y y)^{m-1} e^{-(\alpha_y y)^m}.$$

The first party will defeat the opponent in combat and special operations with the probability

$$P_x = P(x < y) = 1 - P(x > y)$$
$$= 1 - \int_0^\infty f_x(x)\left[\int_0^x f_y(y)dy\right]dx. \qquad (6)$$

(The fewer hits are required for a victory, the more efficient the combat units will be.)

With intermediate calculations omitted, formula (6) yields

$$P_x = \frac{(\beta_x r_x)^m}{(\beta_x r_x)^m + (\beta_y r_y)^m}.$$

A similar approach was used in the paper [34], but with a different interpretation of the model parameters and without the specifics of combat operations.

By denoting $x = r_x$, $y = r_y$, and $\beta = \beta_x/\beta_y$, we obtain the following VF in combat and special operations:

$$p_x(x, y) = \frac{(\beta x)^m}{(\beta x)^m + (y)^m}, \ \beta = \varphi\rho, \qquad (7)$$

where β is the combat superiority parameter of the first party over the second; φ is the moral superiority parameter; finally, ρ is the technological superiority parameter (superiority in coordination of actions, reconnaissance, firepower, and maneuverability; for details, see the paper [35]).

Let $q = \beta x/y$ be the ratio of the forces of the parties. Then

$$p_x = \frac{q^m}{q^m + 1} = 1 - s^{-m}, \ s^m = q^m + 1, \ s > 1, \qquad (8)$$

which is the Pareto distribution. A random variable with the distribution function (8) has the density

$$f(s) = ms^{-m-1}, \ s > 1,$$

and the mean

$$M[S] = \int_1^\infty ms^{-m} ds = \left.\frac{ms^{1-m}}{1-m}\right|_1^\infty.$$

If $m \le 1$, the mean of the distribution (8) is infinite. Consequently, the VF (8) with $m \le 1$ reflects true uncertainty whereas the VF (8) with $m > 1$ measurable uncertainty.

From formula (8), we find the required ratio of forces to achieve victory with a given probability:

$$q = \sqrt[m]{\frac{p_x}{1 - p_x}}. \qquad (9)$$

In the paper [36], the scale parameter was statistically evaluated, and the hypothesis on the conformity of model (7) to statistical data was verified using Pearson's χ^2 test. Based on a sufficiently large volume of statistical data for the 19th–20th centuries, it was shown that the parameter m has the following values:

– for special operations, $m \approx 0.5$;

– for battles (the tactical level), $m \approx 1$;

– for combat operations (the operational level), $m \approx 2$;

– for military operations (the strategic level), $m \approx 3$.

In applied sciences, the most important question is: at what probability may an event be considered almost true?

There are four levels of combat readiness for formations, units, and subunits[3]: combat-ready (at least 75% of organizational structures are combat-ready); boundedly combat-ready (50–75%); partially combat-ready (30–50%); and non-combat-ready (less than 30% of organizational structures are combat-ready).

Concerning preparation for combat and special operations, the following confidence degrees of the first party's victory can be assigned:

– an almost confident victory (the probability of victory ranges from 0.85 to 0.95);

– a sufficient confidence degree of victory (the corresponding probability ranges from 0.80 to 0.85);

– an acceptable confidence degree of victory (the corresponding probability ranges from 0.7 to 0.8).

The confidence degrees are assigned considering the current situation. Assigning high confidence degrees of victory is not always advisable, as this re-

[3] Combat readiness. URL: https://encyclopedia.mil.ru/encyclopedia/dictionary/details.htm?id=3465@morfDictionary (Accessed August 10, 2023.)

quires a sufficiently great concentration of armed forces and, consequently, increases the risk of group defeat by the enemy forces.

According to the results of computations by formulas (8) and (9), as the scale of combat operations increases, uncertainty shifts, more and more, from true to measurable. For instance, at the strategic level, an almost confident victory is achieved with a twofold superiority over the enemy. The greatest uncertainty is characteristic of special operations, where even an acceptable confidence degree of victory over the enemy (guerrillas, sabotage and reconnaissance groups, and terrorist groups) requires nine-fold superiority.

2.2. Substantive Justification of the Victory Function

Based on the experience of the Great Patriotic War, the combat order of a Soviet rifle division was built in two echelons, in a strip 8–12 km wide and 8–10 km deep over the front. The defense strips of the army and the front were relatively narrow but covered a large area. In other words, at the operational and strategic levels of defense, a relatively small share of the front (army) forces could be deployed in a timely manner to hold the strip and launch a counterattack against the invading enemy; see [9, *subsection 3.2.3*]. This corresponds to a higher value of the scale parameter m of model (7).

After World War II, several generations of weapons changed, and the motorization of armed forces was almost completed; moreover, unmanned systems, high-precision weapons, and automated control systems for forces and weapons appeared. The capabilities of divisions and brigades to defeat the enemy in the tactical and operational depth grew significantly.

The impact of modern weapon systems (and, consequently, new tactical tricks and action methods for armed forces) on the significance of the scale parameter of the VF at the operational and strategic levels was assessed in [9]. Due to the greater effective range of enemy reconnaissance and the defeat of its combat units, the values of the scale parameter at the operational and strategic levels decreased to $m \approx 1.5–2$ for the operational level and $m \approx 2–3$ for the strategic level. Estimating the value of the parameter m for various theaters and conditions of warfare is a topical scientific problem.

3. MODELS OF CONFLICT AND APPROPRIATION: A REVIEW

In conventional economics, appropriation is understood as a nonviolent process guaranteed by perfect property rights and their unimpeded enforcement. Conflict economics is based on a model of competition between two or more players (agents), each choosing between producing resources (consumer goods) and producing weapons (tools intended to appropriate the resources produced by other players, individually or jointly). Recall the well-known *free rider problem*: individual consumers of public goods contribute nothing to their provision, hoping that others will do so for them. In view of this problem, it has been established that group structures are less stable the more participants they have; see the reviews in [7, 37].

The following environment was considered in [7]. There are two identical and risk-neutral agents (countries) competing for R units of a resource that can be *consumed directly*. Due to the imperfection of governance and enforcement institutions, the dispute may be resolved by a conflict (threat of conflict). The objective functions of the parties are

$$V_i(G_1, G_2) = p_i(G_1, G_2)R - G_i,$$

$$p_i(G_1, G_2) = \frac{G_i^{\mu}}{G_1^{\mu} + G_2^{\mu}}, \quad i = 1, 2,$$

where $0 < \mu \le 1$ is the determination parameter; G_i is the amount of resources spent by the ith party on weapons production.

The optimal values of the objective functions are

$$V_i(G^*) = V^* = \frac{1 - \mu / 2}{2} R, \quad i = 1, 2.$$

Since $\mu \le 1$, the players benefit by allocating funds for armaments. Next, the authors of [7] considered the costs of conflict and two-stage games: at the first stage, the parties simultaneously and independently allocate resources for weapons production; at the second stage, the parties start negotiations on the distribution of the disputed resource. If they reach an agreement, the resource will be distributed between the parties. Otherwise, the negotiations end in conflict, and the winner will take all of the disputed resource.

A. Alesina and E. Spolaore studied a more general problem, i.e., the relationship between a conflict and the distribution of country sizes in a model where both peaceful negotiations and military conflicts are possible [8]. As is known, when the size of a country and its population grow, the costs of public goods (defense, security, education, etc.) for its citizens decrease; however, the costs of coordinating the interests and preferences of different ethnic and social groups increase accordingly. In the case of high heterogeneity of these groups and attempts to impose certain actions on all groups, the costs may manifest themselves in the form of interethnic conflicts and civil wars. The authors demonstrated the existence of optimal sizes for countries, which can be established through negotiations or conflicts.

4. APPLICATION OF THE VICTORY FUNCTION IN CONFLICT MODELING

Consider two countries, $i = 1, 2$. (Without loss of generality, these can be two blocs of countries governed by two centers.) For the sake of clarity, assume that there is a disputed territory ($i = 3$), e.g., previously under the joint control of the first and second countries (Fig. 2).

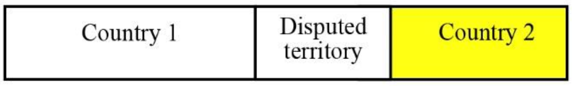

Fig. 2. Two countries and a disputed resource.

Let us introduce the following notation: s_i is the area of country/disputed territory i; z_i is the population of country i; $\mu_{ij} \geq 1$ is the parameter of ethnic diversity between the populations of countries i and j; δ_i is the parameter of attraction of country i; finally, A_i is the socio-technological development index of country i. Suppose that $A_1 = A_2 = A_3 = A$ and $z_1 > z_2 > z_3$.

Following the established tradition (see the paper [8]), we will assign each country an analog of a production function, i.e., a security function of the form [38]

$$u_i = w_i q_i, \quad w_i = A\left(\frac{z_i}{z_{\max}}\right)^{\omega}\left(\frac{s_i}{s_{\max}}\right)^{1-\omega},$$

$$q_i = \left(\frac{\xi_i}{z_i}\right)^{\frac{\xi_i + \mu_i (z_i - \xi_i)}{\delta_i z_i}},$$

where $z_{\max}$ is the population of the country with the largest population (currently India); $s_{\max}$ is the area of the country with the largest territory (Russia); $\omega \approx 0.5$ is the population elasticity parameter; ξ_i is the population of the main ethnic group in country i; μ_i is the parameter of ethnic diversity in country i (between the main ethnic group and the others); δ_i is the parameter of attraction of the main ethnic group in country i; w_i is the sovereignty function of country i; finally, q_i is the preservation function of country i.

If $\mu_i = 1$, there are no ethnic differences in the country (in terms of the participation of ethnic groups in socially significant activities). As the value of μ_i increases, these differences grow. If $\delta_i > 1$, an ethnic group is capable of effectively integrating other nationalities into society. Small values of the parameter ($\delta_i < 0.5$–0.6) are characteristic of peoples without established statehood [38].

The entry of new countries (regions, territories) into a country (union, bloc) increases the value of the sovereignty function (and, consequently, reduces the costs of individuals for the production of public goods), but at the same time reduces the value of the preservation function (an increase in costs associated with interethnic conflicts).[4] Figure 3 shows an example of the dynamics of the functions of sovereignty, preservation, and security.

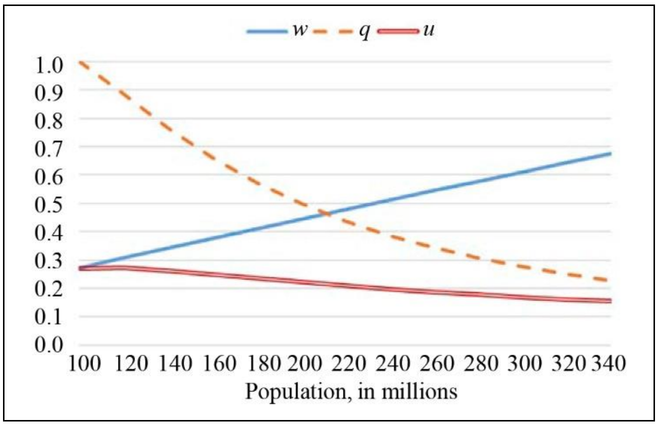

Fig. 3. The functions of sovereignty (*w*), preservation (*q*), and security (*u*).

At ***the first stage*** (*negotiations*), we calculate the security functions of the new unions of countries (territories). The security function of country 1 and the disputed territory is

$$U_{13} = A\left(\frac{z_1 + z_3}{z_{\max}}\right)^{\omega}\left(\frac{s_1 + s_3}{s_{\max}}\right)^{1-\omega}\left(\frac{\xi_1}{z_1 + z_3}\right)^{\frac{\mu_1 z_1 + \mu_{13} z_3}{\delta_1 (z_1 + z_3)}}.$$

The security function of country 2 and the disputed territory is

$$U_{23} = A\left(\frac{z_2 + z_3}{z_{\max}}\right)^{\omega}\left(\frac{s_2 + s_3}{s_{\max}}\right)^{1-\omega}\left(\frac{\xi_2}{z_2 + z_3}\right)^{\frac{\mu_2 z_2 + \mu_{23} z_3}{\delta_2 (z_2 + z_3)}}.$$

The security function of countries 1 and 2 and the disputed territory is

$$U_{123} = A\left(\frac{z_1 + z_2 + z_3}{z_{\max}}\right)^{\omega}\left(\frac{s_1 + s_2 + s_3}{s_{\max}}\right)^{1-\omega} \times\left(\frac{\xi_1}{z_1 + z_2 + z_3}\right)^{\frac{\mu_1 z_1 + \mu_{12} z_2 + \mu_{13} z_3}{\delta_1 (z_1 + z_2 + z_3)}}.$$

[4] The ethnic composition of a country (hence, the value of its security function) can change as a result of uncontrolled migration.

In the absence of conflict, the following equilibria (payoffs from negotiations) are possible depending on the values of the security functions of individual countries and their unions:

– The disputed territory declares sovereignty.
– Country 1 and the disputed territory unite.
– Country 2 and the disputed territory unite.
– Countries 1, 2, and the disputed territory unite.

In the event of disagreement by one or more agents (countries, governments), ***the second stage*** (conflict, combat operations) may begin. The subject of the conflict may be the struggle between countries 1 and 2 for the disputed territory. We define the payoff functions of these countries as follows:

$$H_i = V_i \pi_i - C_i x_i\,,\ \pi_1 = \frac{(\beta x_1)^m}{(\beta x_1)^m + (x_2)^m},$$
$$\pi_2 = 1 - \pi_1,\ V_i = \frac{(z_i + z_3)\delta_i}{z_i \mu_{i3}},\ i = 1, 2,$$

where V_i is the value of the disputed territory for country i; π_i is the probability of country i's victory; x_1 (x_2) is the resource allocated by the first (second, respectively) country for combat operations; β is the parameter of the combat superiority of the first country's armed forces over those of the second country; finally, C_i is the cost of acquiring and maintaining the resources of country i. The parameter μ_{i3} of the ethnic diversity between country i and the disputed territory reflects the former's costs of conducting combat operations in the disputed territory, and the attraction parameter δ_i reflects the capability to reduce these costs. Thus, the object's value is proportional to the degree of increase in the country's population and the attraction parameter and inversely proportional to the diversity parameter.

Let $C_1 = C_2 = C$. To find the Nash equilibrium, we apply the first-order necessary optimality conditions. Omitting the intermediate calculations of the partial derivatives, we obtain

$$x_1^* = \frac{m(V_1V_2)^m}{C[(V_1)^m + (V_2)^m]^2}\frac{V_1}{\beta},\ x_2^* = \frac{m(V_1V_2)^m}{C[(V_1)^m + (V_2)^m]^2}V_2\,.$$

In this conflict, the probability of the first country's victory over the opponent is

$$\pi_1 = \frac{(V_1)^m}{(V_1)^m + (V_2)^m} = \frac{1}{1 + \left(\dfrac{\delta_2(1 + z_3/z_2)\mu_{13}}{\delta_1(1 + z_3/z_1)\mu_{23}}\right)^m}\,.$$

Having determined the expected outcomes of the conflict, the parties proceed to ***the third stage***—estimating the costs of integrating the disputed territory. We define the objective functions of individuals (citizens) in the first and second countries as follows:

$$G_i = Y_i - T_i - (\mu_{i3} - 1)S_i\frac{\pi_i z_3}{z_i},\ i = 1, 2,$$

where Y_i is the citizen's income in country i; T_i is the citizen's taxes; finally, S_i is the citizen's costs due to counterterrorism and special operations.

The last expression shows that the citizen's costs depend significantly on the parameter of the ethnic diversity between the country and the population of the disputed territory. These costs also depend on the ratio of the population of the disputed territory to the population of the country.

Based on the security indicator, conflict costs, and the impact of integrating a new population into the country on the citizens' welfare, the government can make well-grounded decisions and gain public support.

CONCLUSIONS

In this study, we have formulated a comprehensive set of substantive and formal requirements for the victory function (VF) in combat and special operations as a variation of contest success functions (CSFs).

Using A.N. Kolmogorov's law of target destruction and the Weibull distribution, a particular type of VFs has been justified in probabilistic terms. Also, the function and its parameters have been substantively justified based on the postulates of military science and operational practice. With the above justification, the VF in combat and special operations should be treated as a separate class of CSFs.

In recent decades, the scope of political economy and economics has expanded. In addition to the conventional problems of production and distribution of goods, the issues of appropriation as a result of conflict between agents (countries) and the establishment of optimal boundaries between countries have begun to be investigated. Using a security model and the VF, three tasks (stages) have been set for the distribution of disputed resources (territory and population) between countries. At the first stage, to justify its negotiating position, each country calculates the security function under the condition that the disputed territory will become part of it. If the issue is not resolved peacefully, the governments of the countries threaten conflict (combat operations), allocating the appropriate resources to the armed forces. The second stage is to determine the expected outcome of the conflict. At the third stage, the costs of integrating the disputed territory into one of the countries are estimated. As a result, the government can gain the support of the public during negotiations or in the course of the conflict.

A promising line of further research is to develop conflict theory at several levels, namely, in the theater of military operations, the intercountry, and geopolitical levels.

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