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CONSTRUCTING A MAP OF LOCALLY OPTIMAL PATHS FOR A CONTROLLED MOVING OBJECT IN A THREAT ENVIRONMENT

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Abstract. In some path planning problems for controlled objects, the main criterion is to reduce the integral risk of detection when moving in a threat environment with a given map of potential threats. In this paper, we construct all locally optimal paths in a 2D threat environment. The environment is represented by a fixed number of detectors whose positions are known to an evasive object. This object and the detectors are material points. The original problem is formalized as an optimal control problem and reduced to a boundary value problem based on Pontryagin's maximum principle. The boundary value problem is solved numerically by the shooting method. The case of point-to-point transition of the evasive object with and without the path length constraint is studied, and the results of numerical simulation are provided. A parametric analysis of the problem is carried out.

Keywords: threat environment, evasive object, maximum principle, path optimization, numerical simulation.

INTRODUCTION

Due to the widespread use of mobile autonomous vehicles in recent years and the continuing development of this field, problems related to moving one or several vehicles to a given point are highly topical. Such missions can be complicated by the presence of different sensors and transducers, stationary or mobile detectors, forming a threat environment for a controlled object [1–4]. In the literature, this class of problems is known as path planning in a threat environment (sometimes, called a conflict environment as well). Methods for solving such problems are well studied in the case where the signal level at a sensor does not depend on the moving object's velocity. For example, we mention the method of potential fields in the obstacle evasion problem [5]. The paper [6] provided a general overview of safe path planning methods; in particular, Dijkstra's, A*, genetic, and ant colony algorithms were described. Also, robot path planning based on the ant colony algorithm was considered in [7].

This paper considers a threat environment [8] represented by stationary detectors and one controlled object (CO) that moves in a water medium with a possibly variable velocity between two given points and evades detection. By assumption, the location of the detectors forming a threat map [9] is known to the CO. The CO's path and motion parameters are chosen by minimizing the negative impact of the threat environment on it (the probability of its detection by the environment).

In the works [8, 10], analytical solutions were obtained for optimal velocity modes in the 2D problems with one detector. In [8], the problems with several conflicting objects were also addressed, and their optimal velocity modes were numerically simulated based on Dijkstra's algorithm.

This study proceeds from the assumption that the signal-to-noise ratio at the inputs of the receiving systems of all detectors is small during the object's motion along the entire path. Therefore, the CO minimizes an integral criterion, i.e., the risk of detection over the hydroacoustic field [11, 12].

In [11], the problem with one evasive object and two stationary detectors was solved for such a criterion in polar coordinates. The paper [13] considered a formal statement with stationary sensors and detectors with given detection ranges (circles); Dijkstra's algorithm was verified for the case of one sensor and one detector, and the problem was solved analytically. In [14], the path planning problem with the path length constraint was analytically studied for a CO evading one detector. The publication [15] was devoted to minimizing the risk of aircraft detection in a 3D statement; in the case of one radar, an analytical solution was obtained using variational principles. In [16], a discrete method for optimizing the path of an evasive mobile object was proposed based on dynamic programming.

In this paper, we consider the 2D problem statement with an arbitrary number of detectors and develop a methodology for constructing a map of locally optimal paths of a CO based on indirect optimization methods. Note that the sufficient (second-order) conditions of optimality are not verified for the resulting extremals.

In addition, we present a computational scheme of the shooting method as well as the results of numerical simulation for the problem without and with the path length constraint. Numerical simulation is carried out for one, two, and three detectors.

1. PROBLEM WITHOUT PATH LENGTH CONSTRAINT

1.1. Finding Paths in a Threat Environment: Problem Statement

There are $N \ge 1$ stationary detectors (points S_i) in a 2D space. It is required to optimally move a CO between two given points of this space in a fixed time $T < \infty$. The detectors and the CO are represented by material points. In the simplified motion model under consideration, the control variables are the magnitude and direction of the CO's velocity vector v. The problem is to minimize the performance criterion

$$\int_0^T \left(\sum_{i=1}^N q_i \frac{v^2(t)}{r_i^2(t)} \right) dt \to \min,$$

where $v(t)$ is the magnitude of the CO's velocity vector and $r_i(t)$ is the distance to the point S_i at a time instant t; the value $q_i > 0$ is the weight factors of the impact of the point S_i. The location of all points S_i and the corresponding values q_i are given and known to the CO.

1.2. Mathematical Formalization

This problem is formalized as an optimal control problem. Let us introduce the Cartesian frame of coordinates in the 2D space as follows: the origin coincides with the start; the axis Oy passes through the start and finish, being directed from the former point to the latter one. The unit segment is chosen so that the finish has an ordinate of 1 (Fig. 1). In this frame, the start and finish conditions are as follows:

$$\begin{cases} x(0) = 0 \\ y(0) = 0; \end{cases} \qquad \begin{cases} x(T) = 0 \\ y(T) = 1. \end{cases} \tag{1}$$

The CO's motion in this frame is described by the system of differential equations

$$\begin{cases} \dot{x} = v\cos\varphi \\ \dot{y} = v\sin\varphi, \end{cases} \tag{2}$$

where φ denotes the direction of the velocity vector, the angle counted from the positive direction of the axis Ox.

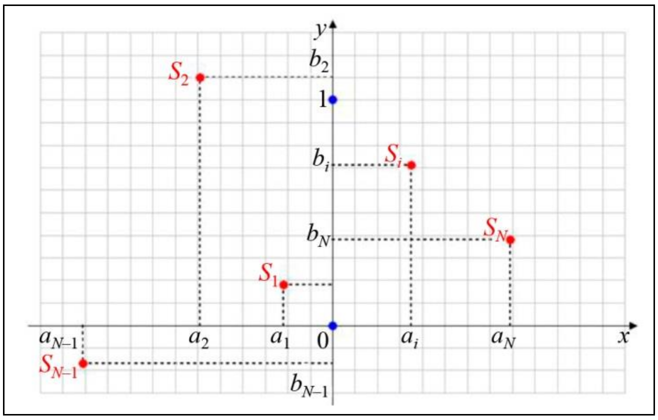

Fig. 1. A threat environment represented by stationary detectors.

The control variables v and φ are assumed to be bounded and piecewise continuous:

$$0 \le v(t) \le v_{\max} \le \infty,$$
$$0 \le \varphi(t) \le 2\pi \quad \forall t \in [0, T], \tag{3}$$

where T specifies a given finish time and $v_{\max}$ is the CO's maximum velocity (a parameter of the problem).

For bounded piecewise continuous control variables v and φ, the phase variables x and y will be continuous piecewise smooth functions satisfying the differential coupling equations (2) on the continuity segments of their derivatives.

The CO starts moving at the time instant $t = 0$.

Let the points S_i have coordinates (a_i, b_i). Then the CO's performance criterion takes the form

$$\int_0^T \left(\sum_{i=1}^N q_i \frac{v^2}{(x - a_i)^2 + (y - b_i)^2} \right) dt \to \min. \tag{4}$$

The globally optimal solution in terms of the risk of CO's detection is the set of unknown phase variables $x(\cdot)$, $y(\cdot)$ and control variables $v(\cdot)$, $\varphi(\cdot)$ that satisfy the system of differential equations (2), the control constraints (3), and the start and finish conditions (1) and minimize the performance criterion (4).

We are interested in all locally optimal paths, i.e., those advantageous for the CO that deliver a strong local minimum in problem (1)–(4).

Note that the criterion (4) can increase arbitrarily when the CO approaches a certain detector: the denominator of the corresponding term of the integrand contains the squared distance between the CO and this detector. The criterion grows infinitely as this distance tends to zero. Therefore, the CO is prohibited from passing through any detector.

1.3. The System of Necessary Optimality Conditions

Let us apply Pontryagin's maximum principle [17, 18] to the optimal control problem under consideration. For this purpose, we write the main elements of the maximum principle:

– the Lagrange function

$$\mathcal{L} = \int_0^T L dt + l,$$

– the Lagrangian

$$L = p_x(\dot{x} - v\cos\varphi) + p_y(\dot{y} - v\sin\varphi) + \lambda_0 \sum_{i=1}^{N} q_i \frac{v^2}{(x-a_i)^2 + (y-b_i)^2}, \tag{5}$$

– the terminant

$$l = \lambda_{x_0} x(0) + \lambda_{y_0} y(0) + \lambda_{x_T} x(T) + \lambda_{y_T} (y(T) - 1),$$

and

– the Pontryagin function

$$H = p_x v\cos\varphi + p_y v\sin\varphi - \lambda_0 \sum_{i=1}^{N} q_i \frac{v^2}{(x-a_i)^2 + (y-b_i)^2}.$$

Assume the existence of a strongly optimal process in problem (1)–(4) such that, for some $\varepsilon > 0$, the CO's path is outside small ε-neighborhoods of the points S_i and the following smoothness conditions hold: the right-hand sides of the systems of differential equations (2) and their partial derivatives with respect to the phase variables x and y are continuous in the neighborhood of the optimal path, and the integrand in (4) is continuously differentiable on $[0, T]$ almost everywhere.

For the optimal control problem (1)–(4) with these conditions, according to Pontryagin's maximum principle, there exist Lagrange multipliers, i.e., constants λ_0, λ_{x_0}, λ_{y_0}, λ_{x_T}, and λ_{y_T} and functions p_x and p_y, at least one of which being nonzero, such that the following system of necessary optimality conditions is satisfied on the CO's optimal path:

– the Euler–Lagrange equations (stationarity with respect to the phase variables)

$$\begin{cases} \dot{p}_x = -\dfrac{\partial H}{\partial x} \\ \quad = -\lambda_0 \displaystyle\sum_{i=1}^{N} q_i \frac{v^2}{\left((x-a_i)^2 + (y-b_i)^2\right)^2} \cdot 2(x-a_i) \\ \dot{p}_y = -\dfrac{\partial H}{\partial y} \\ \quad = -\lambda_0 \displaystyle\sum_{i=1}^{N} q_i \frac{v^2}{\left((x-a_i)^2 + (y-b_i)^2\right)^2} \cdot 2(y-b_i), \end{cases} \tag{6}$$

– the Pontryagin condition (control optimality)

$$(v, \varphi) = \arg\operatorname{abs}\max_{\substack{0 \leqslant v(t) \leqslant v_{\max} \\ 0 \leqslant \varphi(t) < 2\pi}} \left(p_x v\cos\varphi + p_y v\sin\varphi - \lambda_0 \sum_{i=1}^{N} q_i \frac{v^2}{(x-a_i)^2 + (y-b_i)^2} \right),$$

– the transversality conditions with respect to the phase variables

$$\begin{aligned} p_x(0) &= \frac{\partial l}{\partial x(0)} = \lambda_{x_0}, \quad p_x(T) = -\frac{\partial l}{\partial x(T)} = -\lambda_{x_T}, \\ p_y(0) &= \frac{\partial l}{\partial y(0)} = \lambda_{y_0}, \quad p_y(T) = -\frac{\partial l}{\partial y(T)} = -\lambda_{y_T}. \end{aligned} \tag{7}$$

Note that stationarity conditions with respect to time are omitted: the start and finish time instants in problem (1)–(4) are known constants.

Also, the complementary slackness conditions are omitted because problem (1)–(4) includes no "less than or equal to" conditions. The condition of CO's non-passage near the points S_i is checked directly when solving the problem numerically. The nonnegativity condition has the form $\lambda_0 \geq 0$.

The Lagrange function is homogeneous with respect to the Lagrange multipliers. (The Lagrange multipliers can be chosen within a positive cofactor.) The abnormal case $\lambda_0 = 0$ leads to the CO's straight (linear) path from the start to the finish with maximum velocity, which delivers a minimum only in the absence of detectors on the segment connecting these points and $T = \frac{1}{v_{\max}}$; otherwise, the controlled process corresponding to $\lambda_0 = 0$ will be inadmissible. The condition $\lambda_0 = \frac{1}{2}$ is chosen to normalize the problem.

1.4. Boundary Value Problem

Based on Pontryagin's maximum principle, the optimal control problem is reduced to a boundary value problem with the boundary conditions (1) and the system of differential equations

$$\begin{cases} \dot{x} = \hat{v}\cos\hat{\varphi} \\ \dot{y} = \hat{v}\sin\hat{\varphi} \\ \dot{p}_x = -\sum_{i=1}^{N} q_i \dfrac{\hat{v}^2}{\left((x-a_i)^2+(y-b_i)^2\right)^2}\cdot(x-a_i) \\ \dot{p}_y = -\sum_{i=1}^{N} q_i \dfrac{\hat{v}^2}{\left((x-a_i)^2+(y-b_i)^2\right)^2}\cdot(y-b_i), \end{cases} \tag{8}$$

where

$$\hat{v} = \min\left(\frac{\| p \|}{\sum_{i=1}^{N} \dfrac{q_i}{(x-a_i)^2+(y-b_i)^2}}, v_{\max}\right), \tag{9}$$

and the control variable $\hat{\varphi}$ is determined from the relations $\cos\hat{\varphi} = \frac{p_x}{\| p \|}$, $\sin\hat{\varphi} = \frac{p_y}{\| p \|}$ with $\| p \| = \sqrt{{p_x}^2 + {p_y}^2} \neq 0$. For $\| p(\tau) \| = 0$, at a single point $\tau \in [0, T]$ we obtain the control variable $\hat{v}=0$, $\hat{\varphi} \in [0, 2\pi]$. If $\| p \| = 0$, in the resulting solution the CO will stand still for a time Δ. This solution does not deliver a local minimum in the original problem; see Theorem 3 below.

1.5. A Method for Solving the Boundary Value Problem

The resulting boundary value problem (1), (7)–(9) has the fourth order and is nonlinear. In this paper, it will be solved numerically by the shooting method [19, *Ch. 2*; 20, Sect. *2*] using the following computational scheme.

The constants p_0 and ψ_0 for the time instant $t = 0$, missed for solving the Cauchy problems, are chosen as the shooting parameters:

$$p_0 = \|\mathrm{p}(0)\| = \left\| \begin{pmatrix} p_x(0) \\ p_y(0) \end{pmatrix} \right\| = \sqrt{(p_x(0))^2 + (p_y(0))^2}\,;$$

ψ_0 is found from the equation

$$\psi_0 : \begin{pmatrix} p_x(0) \\ p_y(0) \end{pmatrix} = p_0 \begin{pmatrix} \cos\psi_0 \\ \sin\psi_0 \end{pmatrix}.$$

Setting the shooting parameter vector $\boldsymbol{\alpha} = (\mathrm{p}_0, \psi_0)^{\tau}$ somehow and solving the Cauchy problem on the time interval $[0, T]$, we obtain the functions $x(\cdot)[\boldsymbol{\alpha}]$, $y(\cdot)[\boldsymbol{\alpha}]$, $p_x(\cdot)[\boldsymbol{\alpha}]$, and $p_y(\cdot)[\boldsymbol{\alpha}]$ corresponding to the chosen vector $\boldsymbol{\alpha}$ and, in particular, the values of the phase and conjugate variables depending on $\boldsymbol{\alpha}$ at the time instant T. The corresponding expressions from the boundary conditions at the time instant T form the residual vector function

$$\mathbf{X}[\boldsymbol{\alpha}] = \begin{pmatrix} x(T) \\ y(T) - 1 \end{pmatrix}.$$

A series of the Cauchy problems in Newton's method were solved numerically by the explicit Runge–Kutta method of the eighth order based on the 8(7)th order Dorman–Prince 8(7) formulas with automatic step selection [21, *Chs. II.4* and *II.6*; 22].

To solve the boundary value problem, it was necessary to select the values of the shooting parameters $\boldsymbol{\alpha}$ so that $\mathbf{X}[\boldsymbol{\alpha}] = 0$.

Thus, the boundary value problem was reduced to a system of two nonlinear algebraic equations with two unknowns. The solution $\boldsymbol{\alpha}$ of the system of algebraic equations $\mathbf{X}[\boldsymbol{\alpha}] = 0$ was calculated by Newton's method in the Isaev–Sonin modification [23].

The computer program [24] was implemented in the C language to solve the boundary value problem (1), (7)–(9). The program was tested on numerical examples and provides the following functionality: solution of the systems of nonlinear and linear equations, integration of the Cauchy problem, differentiation, and visualization of the results.

The problem under consideration is multi-extremal, and different initial approximations of the shooting parameters $\boldsymbol{\alpha}$ may yield different solutions of the boundary value problem. To classify the CO's paths, the shooting method was repeatedly executed from the points of a bounded rectangular grid in the space of shooting parameters (p_0, ψ_0).

1.6. Numerical Simulation

The methodology described above is applicable to calculations for any number N of detectors. Numerical simulation was carried out for $N \le 10$. This paper presents the simulation results for one, two, and three stationary detectors with $q_i = 1$, $i \in [1, N]$, $v_{\max} = 2$, and $T = 1$.

Figures 2–5 below show the CO's paths in a threat environment for different numbers of detectors. The CO's motion starts from the point (0; 0) and finishes at the point (0; 1); they are indicated by black squares. The locations of the detectors are indicated by red circles. In all figures below, the path with a lower value of the performance criterion corresponds to a bolder curve; in this section, the best paths are highlighted in purple color.

Figure 2 shows the Pontryagin extremals in the case of one detector. The calculations yielded three types of paths: those bypassing the detector from the left and from the right as well as the one with a complete revolution around the detector. Along the blue path, the CO first moves from the start to point A on the curve l_1, then makes a revolution around the detector on the curve l_2, returning to point A, and ultimately moves from point A to the finish point on the curve l_3.

Despite that the paths with a complete revolution around the detector satisfy the necessary optimality conditions and are obtained through calculations, we will discard them. Indeed, eliminating complete revolutions from such paths will decrease the value of the criterion; moreover, the value of the criterion on these paths is much higher compared to the ones without complete revolutions around detectors.

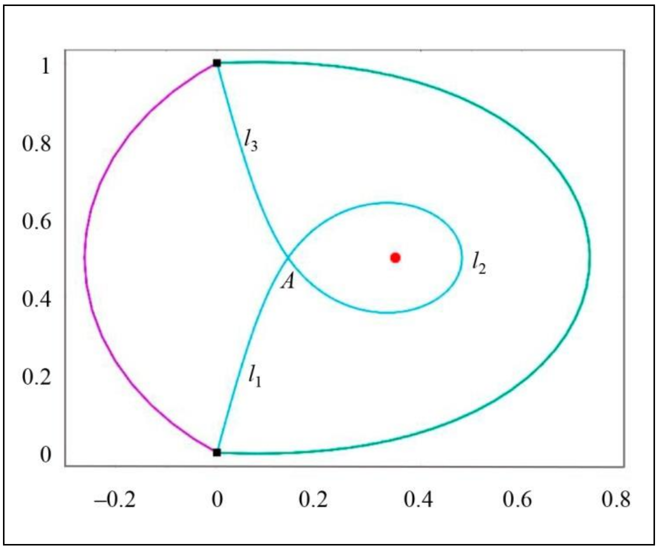

Fig. 2. CO's paths in the case of one detector located at (0.5; 0.5).

Next, Fig. 3 demonstrates the paths in the case of two detectors: (a) all the paths obtained and (b) those remaining after discarding the paths with complete revolutions around detectors.

Figure 4 shows the paths in the case of three detectors: (a) all the paths obtained and (b) those remaining after discarding the paths with complete revolutions around detectors.

For another configuration of the three detectors, the paths without complete revolutions around detectors are presented in Fig. 5.

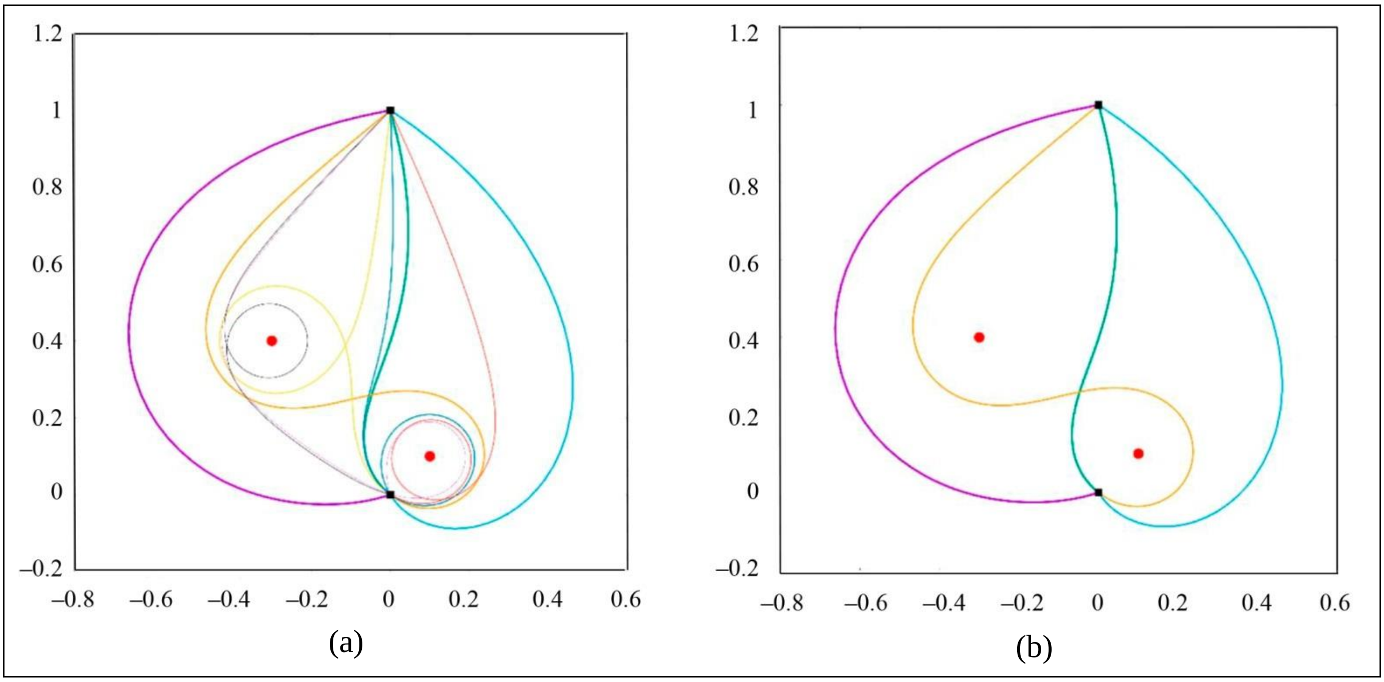

Fig. 3. CO's paths in the case of two detectors located at (0.1; 0.1) and (-0.3; 0.4): (a) all paths and (b) the paths without complete revolutions around detectors.

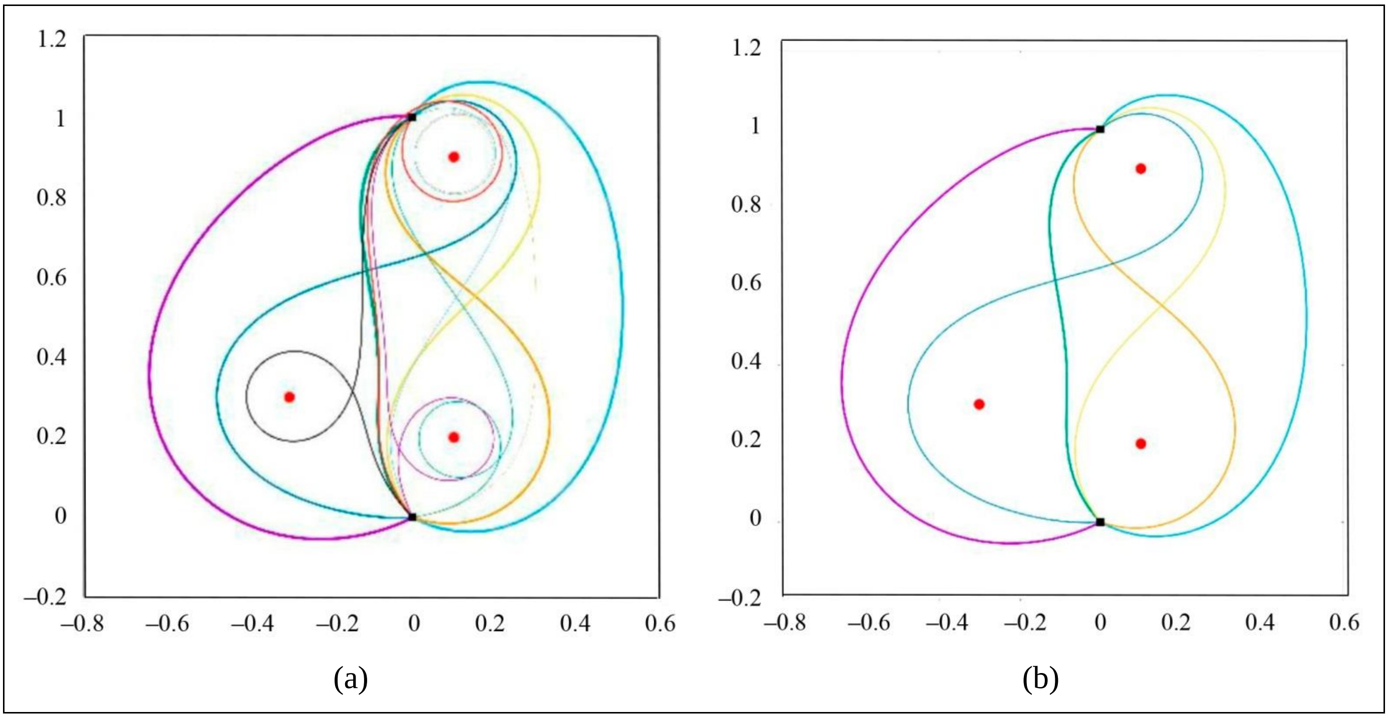

Fig. 4. CO's paths in the case of three detectors located at (0.1; 0.2), (-0.3; 0.3), and (0.1; 0.9): (a) all paths and (b) the paths without complete revolutions around detectors.

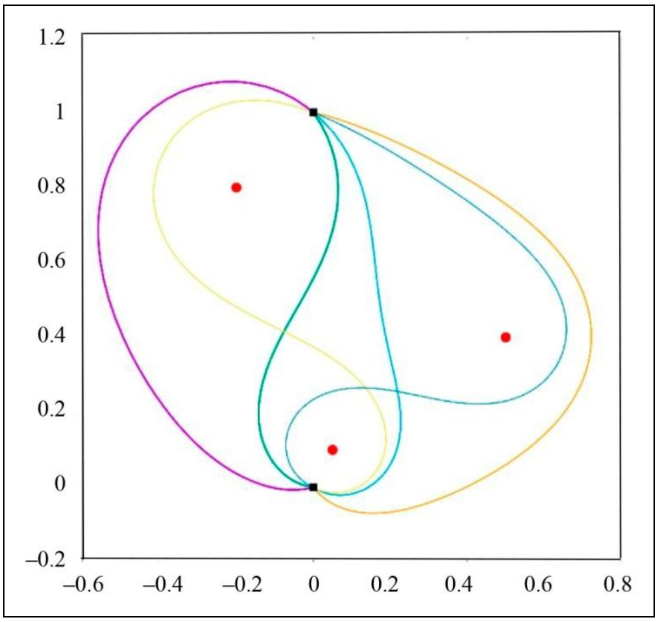

Fig. 5. CO's paths in the case of three detectors located at (0.05; 0.1), (0.5; 0.4), and (-0.2; 0.8), without revolutions around detectors.

1.7. Analysis of Numerical Simulation Results

If the velocity value $v_{\max}$ was not reached on the path, the CO's path $y(x)$ remained unchanged in many numerical simulations when varying the parameter T of the problem. In other words, the corresponding path was still a Pontryagin extremal, but the CO moved along it with a different velocity. Let us formulate this observation as follows.

Theorem 1. *If the constraint* $v \le v_{\max}$ *is not reached on an extremal* $\{\hat{x}(\cdot), \hat{y}(\cdot), \hat{v}(\cdot), \hat{\varphi}(\cdot)\}$ *of problem* (1)–(4), *the path* $\hat{y}(\hat{x})$ *will correspond to some extremal in the modified problem differing from the original one* (1)–(4) *only by an increased constant* T *.*

P r o o f. Let $T = T_1$ be the CO's travel time in the original problem (further called problem 1). Consider problem 2, differing from problem 1 only in the CO's travel time: $T_2 > T_1$. Then the same boundary value problem (1), (7)–(9) with another value of the constant T corresponds to problem 2. Now, we make the v-time substitution [25], i.e., expand the time æ-fold, where $æ = \frac{T_2}{T_1} > 1$. Direct verification shows that an extremal of problem 1 will also be an extremal of problem 2. In this case, the CO's path $\hat{y}(\hat{x})$ is preserved, but the velocity will be reduced æ-fold compared to problem 1. ♦

Remark 1. The converse is false because the values $v_{\max}$ and T determine the CO's maximum possible path length ($v_{\max}T$). Suppose that for a time $T = T_1$, there exists an extremal $\{\hat{x}(\cdot), \hat{y}(\cdot), \hat{v}(\cdot), \hat{\varphi}(\cdot)\}$ of problem (1)–(4) with a path $\hat{y}(\hat{x})$ of length ℓ. For a time $T_2 < \dfrac{\ell}{v_{\max}} \le T_1$, this path will no longer be admissible: either the CO has to take a shorter path closer to some detector to satisfy the constraint, or the solution with bypassing detectors, obtained for a greater time, ceases to exist. ♦

Corresponding path changes can be observed by comparing the left- and right-hand paths in Figs. 6c and 6d.

Remark 2. As the time T increases, new paths may emerge that are extremals violating the natural path length constraint for a smaller value of T. ♦

This can be seen by comparing Fig. 6b and Figs. 6a, 6c, and 6d: under the strict time constraint (Fig. 6b), there are no paths bypassing detectors on the left and right, only a path passing between them.

Proposition. *For the same type of paths* $\hat{y}(\hat{x})$ *within the conditions of Theorem* 1, *the value of the performance criterion* (4) *will decrease* æ*-fold since the criterion depends linearly on time and quadratically on velocity*. ♦

P r o o f. We apply the change of variable $\tau = t / æ$ in the integral, transforming the segment $[0, T_2]$ to $[0, T_1]$. In this case, $v(\tau) = æ v(t)$ and

$$\int_0^{T_2} \sum_{i=1}^{N} \frac{v^2(t)}{r_i^2(t)} dt = \int_0^{T_2} v^2(t) \sum_{i=1}^{N} \frac{1}{r_i^2(t)} dt$$

$$= \int_0^{T_1} \frac{v^2(\tau)}{æ^2} \sum_{i=1}^{N} \frac{1}{r_i^2(\tau)} æ d\tau = \frac{1}{æ} \int_0^{T_1} \sum_{i=1}^{N} \frac{v^2(\tau)}{r_i^2(\tau)} d\tau. ♦$$

Remark 3. The shortest possible path is when the CO moves along the segment (0; 0)–(0; 1) of length 1. Therefore, for $T = \dfrac{1}{v_{\max}}$, the solution exists only in the absence of detectors on the segment connecting the start and finish and represents the CO's movement along this segment with the maximum velocity $v_{\max}$; for $T < \dfrac{1}{v_{\max}}$, there are no solutions in problem (1)–(4) since the shortest path has length 1. ♦

Remark 4. The paths $y(x)$ of all problem extremals lie in some pipe of paths containing the segment (0; 0)–(0; 1). Regardless of the location of detectors, as the time T tends to $\dfrac{1}{v_{\max}}$ on the right, the diameter of this tube vanishes. ♦

The corresponding contraction of the CO's path to the segment connecting the start and finish can be seen in Fig. 6 (an example of the path passing between detectors).

Theorem 2. *Closed curves with a complete revolution around a certain detector do not deliver the global minimum in problem* (1)–(4).

P r o o f. Indeed, let the CO make a complete revolution around a certain detector (Fig. 2). In this case, the entire path can be divided into three segments: motion along the curve l_1 to some point A, where the CO's path will self-intersect; then a complete revolution around the detector (the curve l_2); and finally, motion along the curve l_3 from point A to the finish.

Note that if we remove the segment l_2, the CO's path from the start to point A along the curve l_1 and then immediately from point A to the finish along the curve l_3 will be admissible in problem (1)–(4): the problem statement includes no constraints on the curvature of the CO's path and its rate of turning. The criterion (4) is additive and its value strictly increases when moving along the path; hence, removing the segment l_2 will decrease the value of the criterion. Besides, the total travel time will decrease by ΔT; by the proposition, it is possible to reduce the CO's velocity $\dfrac{T}{T - \Delta T}$-fold on the remaining part of the path, which will additionally reduce the value of the criterion (4).

Thus, the value of the criterion will be smaller when moving along the path $l_1 l_3$. ♦

Remark 5. According to the proof of Theorem 2, the motion along the path $l_1 l_3$ is better than that along the path $l_1 l_2 l_3$ in terms of minimizing the risk of detection. Despite this fact, usually, the CO's path is nonsmooth on such paths at the junction point A of the segments l_1 and l_3, which corresponds to a control jump and a discontinuity of the conjugate variables. As a result, the controlled process with the CO's motion along the path $l_1 l_3$ is not an extremal in problem (1)–(4).

Theorem 3. *The paths of controlled processes* ξ *with* $\| p \| = 0$ *on some interval* Δ *in the boundary value problem* (1), (7)–(9) *do not deliver a local minimum in the optimal control problem* (1)–(4).

P r o o f. From the condition $\| p \| = 0$ it follows that $\hat{v}=0$, i.e., the CO stands still at the same point for a time Δ. For an arbitrarily small $\varepsilon \in (0,1)$, we reduce the time of staying at this point $\dfrac{1}{\varepsilon}$-fold, from Δ to $\varepsilon\Delta$, on the path $y(x)$ corresponding to ξ. In this case, the travel time along the rest of the path will increase from $(T - \Delta)$ to $(T - \varepsilon\Delta)$. Let us decrease the CO's velocity k-fold, where $k = \dfrac{T - \varepsilon\Delta}{T - \Delta}$. On such a controlled process corresponding to the constructed path, which is close to ξ in the solution space, the criterion (4) will decrease k-fold, $k > 1$. ♦

2. PROBLEM WITH PATH LENGTH CONSTRAINT

2.1. Modifications in the Problem Statement. Boundary Value Problem

If the CO's path length is limited, an additional phase variable $z(t)$ (the path length reached at a time instant t) is introduced to consider this constraint.

Then the boundary conditions (1) are supplemented by the conditions

$$z(0) = 0, z(T) \le \ell, \qquad (10)$$

where ℓ is a limit of the CO's path length.

In this case, the system of differential relations (2) also includes the relation

$$\dot{z} = v. \qquad (11)$$

The maximum principle-based analysis of this problem statement leads to the following modifications. In the basic elements (5), the term $p_z(\dot{z} - v)$ is added to the Lagrangian L, the terms $\lambda_{z_0} z(0)$ and $\lambda_{z_T}(z(T) - \ell)$ to the terminant l, and the term $p_z v$ to the Pontryagin function H.

For the conjugate system, besides equations (6), we obtain $\dot{p}_z = 0 \Rightarrow p_z \equiv \text{const}$.

The transversality conditions (7) are supplemented by the relations $p_z(0) = \lambda_{z_0}$ and $p_z(T) = -\lambda_{z_T}$.

In addition, we have the complementary slackness condition

$$\lambda_{z_T}(z(T) - \ell) = 0 \qquad (12)$$

and the extra nonnegativity condition

$$\lambda_{z_T} \ge 0. \qquad (13)$$

Two different cases are possible depending on the zero cofactor in condition (12).

In the first case, the constraint on the path length is valid: $z(T) = \ell$. Then, due to formulas (10) and (11), the optimal control problem reduces to the boundary value problem

$$\begin{cases} \dot{x} = \hat{v}\dfrac{p_x}{\|p\|}, \qquad \dot{y} = \hat{v}\dfrac{p_y}{\|p\|}, \quad \dot{z} = \hat{v} \\ \dot{p}_x = -\sum\limits_{i=1}^{N} q_i \dfrac{\hat{v}^2}{\left((x-a_i)^2 + (y-b_i)^2\right)^2} \cdot (x - a_i) \\ \dot{p}_y = -\sum\limits_{i=1}^{N} q_i \dfrac{\hat{v}^2}{\left((x-a_i)^2 + (y-b_i)^2\right)^2} \cdot (y - b_i), \end{cases}$$

where

$$\hat{v} = \min\left(\max\left(\frac{\|p\| + p_z}{\sum\limits_{i=1}^{N} \dfrac{q_i}{(x-a_i)^2 + (y-b_i)^2}}, 0 \right), v_{\max} \right)$$

and

$$\begin{cases} x(0) = 0, & x(T) = 0 \\ y(0) = 0, & y(T) = 1 \\ z(0) = 0, & z(T) = l. \end{cases}$$

The computational scheme of the shooting method in this case is as follows. The shooting parameters p_0, ψ_0, and λ_{z_T} are used to specify the initial values for the Cauchy problem:

$$\begin{pmatrix} p_x(0) \\ p_y(0) \\ p_z(0) \end{pmatrix} = \begin{pmatrix} p_0 \cos\psi_0 \\ p_0 \sin\psi_0 \\ -\lambda_{z_T} \end{pmatrix}.$$

Then the Cauchy problem is solved, and the residual vector function is calculated:

$$\begin{pmatrix} x(T) \\ y(T) - 1 \\ z(T) - l \end{pmatrix}.$$

Finally, the system of nonlinear equations is solved by the method described in subsection 1.5.

In the second case, where $z(T) < \ell$ (the constraint is invalid), we obtain the boundary value problem (1), (7)–(9), which corresponds to the computational scheme described in subsection 1.6; in addition, however, it is necessary to integrate the variable z in the equation $\dot{z} = \hat{v}$, $z(0) = 0$.

If $z(T) > \ell$ on the resulting solution of the boundary value problem, then the corresponding path is discarded due to its inadmissibility. (This path violates the system of constraints.)

2.2. Numerical Simulation Results

To illustrate the effect of the path length constraint on the solutions, we consider a scenario with two detectors located at (0.2; 0.3) and (-0.3; 0.4) and the constants $q_i = 1$, $v_{\max} = 0.8$, and $T = 2$.

Figure 6a shows three Pontryagin extremals in the problem without the path length constraint. The situation in Fig. 6b corresponds to a rigid path length constraint, which fails on the paths bypassing both detectors on the left or right; consequently, they are removed from the family of admissible solutions. In Fig. 6c, the constraint is weaker and the paths bypassing both detectors on the same side appear again, but the path length constraint forces the CO to move closer to them compared to Fig. 6a. With further relaxation of the path length constraint, we observe paths with a complete revolution around one of the detectors (Fig. 6d); they are disadvantageous in terms of the performance criterion and nonoptimal by Theorem 2.

The best solution for the cases in Figs. 6b–6d is still the bolder green path between the detectors (Fig. 6a), where the path length constraint is invalid. In contrast, the paths passing between the detectors in Figs. 6b–6d are not extremals: the non-negativity condition (13) fails on them.

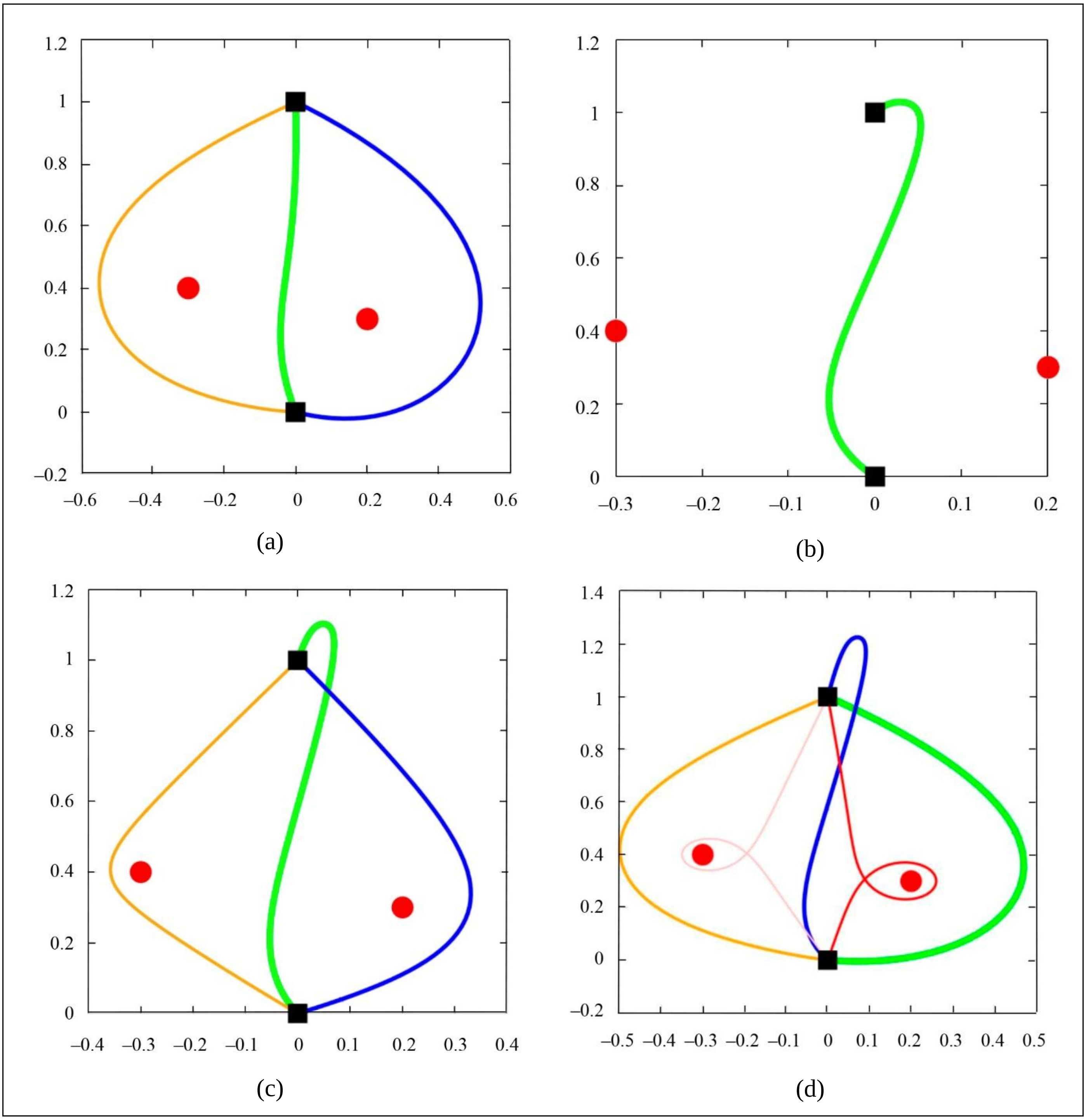

Fig. 6. The effect of the path length constraint on the solution: (a) no constraint, (b) $\ell = 1.1$, (c) $\ell = 1.25$, and (d) $\ell = 1.5$.

The effect of varying the parameters q_i on the CO's paths was studied in [26]; according to the simulation results in Fig. 2, it is advantageous for the CO to pass further away from the detector with a higher weight q_i. This outcome agrees with common sense.

Note also that two identical problems with different values of $v_{\max}$ may have different sets of CO's paths if the limit $v_{\max}$ is reached at least in one of the two cases. For example, see Fig. 5 in [26].

CONCLUSIONS

For a controlled object (CO), the path planning problem with an integral performance criterion (risk) depending on the CO's velocity and the location of several detectors, as well as with an integral path length constraint, has been formalized as an optimal control problem. This problem has been solved by indirect optimization methods based on Pontryagin's maximum principle. The globally optimal path is chosen from the resulting set of extremals, and the optimal velocity law on this path is obtained. Numerical simulation has been carried out to assess the sensitivity of the problem parameters (the coordinates of the detectors and the CO's travel time and path length).

This path planning methodology for a CO in a threat environment can be further applied to the inverse problem of optimizing the arrangement of detectors to counteract the stealthy movements of a CO [27].

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REFERENCES

1. Rafai, A., Adzhar, N., and Jaini, N., A Review on Path Planning and Obstacle Avoidance Algorithms for Autonomous Mobile Robots, *Journal of Robotics*, 2011, pp. 1–14. DOI: 10.1155/2022/2538220.
2. Zhang, Z., Wu, J., Dai, J., and He, C., Rapid Penetration Path Planning Method for Stealth UAV in Complex Environment with BB Threats, *International Journal of Aerospace Engineering*, 2020, pp. 1–15. DOI: 10.1155/2020/8896357.
3. Hu, J., Fan, L., Yifei, L., et al., Reinforcement Learning-Based Low-Altitude Path Planning for UAS Swarm in Diverse Threat Environments, *Drones*, 2023, vol. 7, p. 567. DOI: 10.3390/drones7090567.
4. Chen Q., Zhao Q., and Zou, Z., Threat-Oriented Collaborative Path Planning of Unmanned Reconnaissance Mission for the Target Group, *Aerospace*, 2022, vol. 9, no. 577, pp. 1–23. DOI: 10.3390/aerospace9100577.
5. Filimonov, A.B. and Filimonov, N.B., Constructive Aspects of the Method of Potential Fields in Mobile Robotics, *Optoelectronics, Instrumentation and Data Processing*, 2021, vol. 57, no. 4, pp. 371–377. DOI: 10.15372/AUT20210406.
6. Karur, K., Sharma, N., Dharmatti, Ch., and Siegel, J.E., Survey of Path Planning Algorithms for Mobile Robots, *Vehicles*, 2021, vol. 3, no. 3, pp. 448–468. DOI: 10.3390/vehicles3030027.
7. Ajeil, F.H., Ibraheem, I.K., Azar, A.T., and Humaidi, A.J., Grid-Based Mobile Robot Path Planning Using Aging-Based Ant Colony Optimization Algorithm in Static and Dynamic Environments, *Sensors*, 2020, vol. 20, no. 7, art. no. 1880. DOI: 10.3390/s20071880.
8. Galyaev, A.A., Maslov, E.P., and Rubinovich, E.Ya., On a Motion Control Problem for an Object in a Conflict Environment, *Journal of Computer and Systems Sciences International*, 2009, vol. 48, no. 3, pp. 458–464.
9. Dogan, A. and Zengin, U., Unmanned Aerial Vehicle Dynamic-Target Pursuit by Using Probabilistic Threat Exposure Map, *Journal of Guidance, Control and Dynamics*, 2006, vol. 29, no. 4, pp. 723–732. DOI: 10.2514/1.18386.
10. Galyaev, A.A. and Maslov, E.P., Optimization of a Mobile Object Evasion Laws from Detection, *Journal of Computer and Systems Sciences International*, 2010, vol. 49, no. 4, pp. 560–569.
11. Galyaev, A.A. and Maslov, E.P., Evasion from Detection by a System of Heterogeneous Observers in Threat Environment, *Journal of Computer and Systems Sciences International*, 2014, vol. 53, no. 4, pp. 487–496.
12. Sysoev, L.P., The Criterion of Probability Detecting on the Trajectory in the Problem of Movement Control in Threat Environment, *Control Sciences*, 2010, no. 6, pp. 65–72. (In Russian.)
13. Galyaev, A.A., Problem of Evading Detection by System of Heterogeneous Observers: One Sensor – Group of Detectors, *Control Sciences*, 2016, no. 3, pp. 72–77. (In Russian.)
14. Galyaev, A.A., Lysenko, P.V., and Yakhno, V.P., Moving Object Evasion from Single Detector at Given Speed, *Control Sciences*, 2020, no. 1, pp. 83–91. DOI: 10.25728/pu.2020.1.8. (In Russian.)
15. Zabarankin, M., Uryasev, S., and Murphey, R., Aircraft Routing under the Risk of Detection, *Naval Research Logistics*, 2006, vol. 53, no. 8, pp. 728–747. DOI: 10.1002/nav.20165.
16. Dobrovidov, A.V., Kulida, E.L., and Rud'ko, I.M., Control of Object Movement in Threat Environment, *Control Sciences*, 2011, no. 3, pp. 64–75. (In Russian.)
17. Alekseev, V.M., Tikhomirov, V.M., and Fomin, S.V., *Optimal Control*, Springer US, 1987.
18. Pontryagin, L.S., Boltyanskii, V.G., Gamkrelidze, R.V., and Mishchenko, E.F., *The Mathematical Theory of Optimal Processes*, New York–London: John Wiley & Sons, 1962.
19. Grigor'ev, I.S., *Metodicheskoe posobie po chislennym metodam resheniya kraevykh zadach printsipa maksimuma v zadachakh optimal'nogo upravleniya* (Manual on Numerical Methods for Solving Boundary Value Problems of the Maximum Principle in Optimal Control Problems), Moscow: Center for Applied Research, Faculty of Mechanics and Mathematics, Moscow State University, 2005. (In Russian.)
20. Aleksandrov, V.V., Bakhvalov, N.S., Grigor'ev, K.G., et al., *Praktikum po chislennym metodam v zadachakh optimal'nogo upravleniya* (Tutorial on Numerical Methods in Optimal Control Problems), Moscow: Moscow State University, 1988. (In Russian.)
21. Hairer, E., Nørsett, S.P., and Wanner, G., *Solving Ordinary Differential Equations I: Nonstiff Problems*, Springer Series in Computational Mathematics, vol. 8, 2nd ed., Springer, 1993.

22. Bakhvalov, N.S., Zhidkov, N.P., and Kobel'kov, G.M., *Chislennye metody* (Numerical Methods), Moscow: BINOM. Laboratoriya Znanii, 2008. (In Russian.)
23. Isaev, V.K. and Sonin, V.V., On a Modification of Newton's Methods for the Numerical Solution of Boundary Problems, *USSR Computational Mathematics and Mathematical Physics*, 1963, vol. 3, no. 6, pp. 1525–1528.
24. Samokhin, A.S. and Samokhina, M.A., Solution of the Cauchy Problem by the Multidimensional Runge-Kutta Method Based on the 8(7)th Order Dorman–Prince Formulas with Automatic Step Selection, *Certificate of Registration of Computer Program RU 2020611811*, Registered February 11, 2020. (In Russian.)
25. Milyutin, A.A., Dmitruk, A.V., and Osmolovskii, N.P., *Printsip maksimuma v optimal'nom upravlenii* (The Maximum Principle in Optimal Control), Moscow: Moscow State University, Faculty of Mechanics and Mathematics, 2004. (In Russian.)
26. Galyaev, A.A., Samokhin, A.S, and Samokhina, M.A., On Problem of Optimal Observers' Placement on Plane, *Journal of Physics: Conference Series*, 2021, vol. 1864, no. 1, art. no. 012075. DOI: 10.1088/1742-6596/1864/1/012075.
27. Galyaev, A., Samokhin, A., and Samokhina, M., Application of the Gradient Projection Method to the Problem of Sensors Arrangement for Counteraction to the Evasive Object, *Proc. 28th St. Petersburg International Conference on Integrated Navigation Systems (ICINS)*, St. Petersburg, 2021, p. 9470857.

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