BUILDING A DEFENDER'S 3D PROGRAM PATH IN AN ADT GAME WITH INCOMPLETE A PRIORI TARGET INFORMATION

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Abstract. This paper is devoted to an Attacker–Defender–Target (ADT) game in the 3D space. The Target makes flat circumferential movements with a constant velocity. The Attacker moves uniformly and rectilinearly from an arbitrary point in the upper hemisphere. The distinctive feature of the problem statement is that the Target has an onboard mobile Defender. The Defender is intended to intercept the Attacker's possible paths dangerous to the Target (in the pointwise meeting sense). This task is complicated since the Target and Defender do not see the Attacker during the movements. They know only the initial bearing; the current bearing and the initial and current distances to the Attacker remain uncertain. For this reason, the Target and Defender are assumed to move along a program path.

Keywords: ADT game, program movements, path curvature constraints.

INTRODUCTION

The interest in the mathematical formalization of interaction processes of mobile objects, such as unmanned aerial vehicles (UAVs) or autonomous underwater vehicles (AUVs), has been growing recently. This trend is due to several objective reasons known to everybody.

One formalization is the so-called *Attacker–Defender–Target* (ADT) games, also known as *Missile–Target–Defender* (MTD) games in the literature. In addition to the conventional participants of pursuit–evasion games (Pursuer and its evading Target), they involve the third player (Defender). Acting in coordination, the Defender and Target form a coalition and play against the Attacker. The Defender's task is to intercept the Attacker on its motion path or divert the Attacker from its intended pursuit path in cases where the Defender acts as a false target (Decoy).

In these tasks, a crucial issue concerns a priori and current information available to players during pursuit and evasion.

A flat differential game of one Pursuer against a coalition of a true Target and a Decoy was solved in [1]. In this game, the Pursuer with a circular detection zone of radius R minimizes the time until crossing the detection circle by the Decoy; initially located outside the detection zone, the true Target maximizes the minimum distance to the pursuer.

In [2], the interception problem was solved for the Pursuer equipped with a detection zone; all Pursuer's paths under which the Target enters the detection zone were considered dangerous for it. In this paper, we build and optimize a Decoy's path under incomplete a priori information about the Pursuer's state-space coordinates. The publication [2] continues the research initiated in [3–5].

The basic idea proposed in [2], also used below, is to build an appropriate Decoy's path to intercept the Pursuer on those paths endangering the true Target moving along a chosen evasion path. In this case [2], the path dangerous for the true Target is the only pursuer's path implementing its task (meeting the true Target on a chosen rectilinear evasion path).

Recent results in this field of research were presented in [6–22]. In the papers cited, all statements concern problems with accurate information about the current positions of all players. The problem considered below is related to incomplete information. The Target knows only two parameters about the Attacker: its velocity and initial bearing. Important information about the distance to the Attacker is not available.

This formulation is the simplest mathematical formalization of the subsequent actions in the following real-life situations.

- A heavy bomber is evading attack with an air-to-air missile. By assumption, the aircraft is equipped with a (passive) electro-optical sensor for information transfer.
- A submarine is evading a torpedo attack.
- A submarine is evading a mobile search system.

Note that this research continues R. Boyell's studies [4, 5]. However, it is not a game-theoretic analysis of the Target defense scenario; the main attention is paid to the kinematics of combat under incomplete a priori information. Once again, we emphasize the following features: the Pursuer does not maneuver, maintaining its course, but this course and the distance to the Pursuer are unknown to the coalition defending the Target.

1. PROBLEM STATEMENT

Consider the game of three players in a 3D space, namely, Attacker (*A*), Defender (*D*), and Target (*T*). It is known that the Target moves along a circle of a fixed radius R with a constant linear velocity v_T.

At the initial time instant, the Target receives information that the attacking object A has been launched at the Target from a selected straight line (the initial bearing line). In addition, the Attacker's starting point is uncertain to the Target, but the Attacker is known to move from the upper hemisphere along a pre-selected straight line with a constant velocity $v_A > v_T$. In response, at the same time instant, the Target releases the Defender with a programmed motion of a constant velocity v_D, where $v_T < v_D < v_A$.

It is required to build a Defender's path intercepting the Attacker on all its possible paths dangerous for the Target (in the pointwise meeting sense).

2. THE GEOMETRICAL DESCRIPTION OF THE PROBLEM

Let τ denote the parametric time of the problem. Assume that the Defender plans to intercept the Attacker at some preemptive time instant $t = t(\tau)$. For a fixed value of τ, Fig. 1 illustrates the geometrical description of the problem.

The point O indicates the center of the Target's motion circle. Points *A* and *T* correspond to the positions of the Attacker and Target, respectively, at the

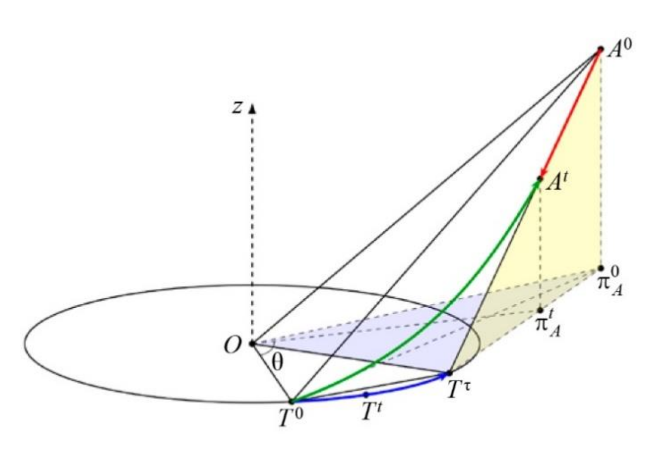

Fig. 1. The geometrical description of the problem.

time instants specified by the superscripts. For example, the points T^0 and A^0 are the initial positions of the Target and Attacker (at the initial time instant $\tau = 0$). In turn, the points π_A^0 and π_A^t are the projections of the points A^0 and A^t on the plane OT^0T^τ, respectively.

The arc $\cup T^0 T^t T^\tau$ (set off in blue in Fig. 1) is the Target's path during the time τ. At the time instant $\tau = 0$, the Target receives the following information: the attacking object is moving straight with a constant velocity from some unknown initial point A^0 in the upper hemisphere located on a known ray coming from the point T^0. The slope of the straight line A^0T^0 is given by two angles: $\angle OT^0\pi_A^0 = \gamma$ and $\angle A^0T^0\pi_A^0 = \lambda$.

Then, to intercept the Target at the point T^τ at the time τ, the Attacker must move along the segment A^0T^τ from the initial point A^0. In this case, the Attacker will travel the path of length $A^0T^\tau = v_A\tau$ since the start.

To counteract the Attacker (to prevent it from intercepting the Target at the point T^τ), the Target releases the Defender at the initial time instant $\tau = 0$. The task of this player is to intercept the Attacker at the preemptive point A^t at some time instant $t(\tau) \le \tau$.

To describe this episode analytically, we introduce a cylindrical frame centered at the point O, with the Oz axis directed upwards perpendicular to the plane OT^0T^τ, and the angle θ counted from the straight line OT^0 in the direction of the Target's motion.

3. SOLUTION OF THE PROBLEM

Let us find the coordinates of the point A^t in the cylindrical frame. For the triangle $\Delta\, OT^0T^\tau$, the following relations are valid:

$$\angle T^0OT^\tau = \eta = \frac{v_T}{R}\tau,\ T^0T^\tau = L = 2R\sin\frac{\eta}{2}. \qquad (1)$$

Then

$$\angle T^\tau T^0\pi_A^0 = \zeta = \frac{\pi-\eta}{2} - \gamma.$$

Denoting $T^0T^\tau = \nu$, for the triangle $\Delta\, OT^0T^\tau$ we have

$$\nu^2\cos^2\lambda + L^2 - 2L\nu\cos\lambda\cos\zeta = \left(T^\tau\pi_A^0\right)^2$$

by the cosine theorem. On the other hand,
$\left(T^\tau\pi_A^0\right)^2 + \left(A^0\pi_A^0\right)^2 = v_A^2\tau^2$, where $A^0\pi_A^0 = \nu\sin\lambda$.
Therefore, ν satisfies the quadratic equation

$$\nu^2 - 2L\nu\cos\lambda\cos\zeta + L^2 - v_A^2\tau^2 = 0.$$

The solution is given by

$$\nu = L\cos\lambda\cos\zeta \pm \sqrt{L^2\left(\cos^2\lambda\cos^2\zeta - 1\right) + v_A^2\tau^2}.$$

Proposition. *For* $v_A > v_T$ *and* $\forall\tau \ge 0$,

$$\nu_1 = L\cos\lambda\cos\zeta - \sqrt{L^2\left(\cos^2\lambda\cos^2\zeta - 1\right) + v_A^2\tau^2} \le 0 .$$

P r o o f.

If the first term is below 0, the inequality will hold automatically. Hence, let the first term be nonnegative.

Transferring the root to the right-hand side of the inequality and squaring the resulting expression, we obtain the inequality

$$L^2\cos^2\lambda\cos^2\zeta \le L^2\left(\cos^2\lambda\cos^2\zeta - 1\right) + v_A^2\tau^2.$$

Obviously, this condition is equivalent to

$$L^2 \le v_A^2\tau^2.$$

In view of the relations (1), the last inequality is equivalent to

$$2R\sin\left(\frac{v_T}{2R}\tau\right) \le v_A\tau.$$

Considering the functions in the left- and right-hand sides, the inequality will be true under the corresponding inequality for the derivatives of these sides at $\tau = 0$:

$$v_T \le v_A,$$

which is the case by the hypotheses of the proposition. ♦

In this case, due to the similarity of the triangles $\Delta T^\tau\pi_A^0A^0$ and $\Delta T^\tau\pi_A^tA^t$ (setoff in yellow in Fig. 1), the coordinate z of the point A^t can be written as

$$z = \pi_A^tA^t = \nu\sin\lambda\cdot\frac{\tau - t}{\tau}.$$

The next step is to find the cosine of the angle $\angle OT^\tau\pi_A^0 = \omega$. To this end, we apply the cosine theorem for the triangles $\Delta OT^0\pi_A^0$ and $\Delta OT^\tau\pi_A^0$:

$$\left(O\pi_A^0\right)^2 = R^2 + \nu^2\cos^2\lambda - 2R\nu\cos\lambda\cos\gamma,$$

$$\left(O\pi_A^0\right)^2 = R^2 + v_A^2\tau^2 - \nu^2\sin^2\lambda - 2R\sqrt{v_A^2\tau^2 - \nu^2\sin^2\lambda}\cos\omega.$$

Equating the right-hand sides of the equalities yields

$$\cos\omega = \frac{v_A^2\tau^2 + 2R\nu\cos\lambda\cos\gamma - \nu^2}{2R\sqrt{v_A^2\tau^2 - \nu^2\sin^2\lambda}}.$$

Next, applying the cosine theorem for the triangle $\Delta OT^\tau\pi_A^t$, we derive the expression

$$\left(O\pi_A^t\right)^2 = r^2 = R^2 + v_A^2\left(\tau - t\right)^2 - z^2 - 2R\sqrt{v_A^2\left(\tau - t\right)^2 - z^2}\ \cos\omega.$$

Similarly, with the sine theorem applied to the same triangle,

$$\sin\varphi = \frac{\sqrt{v_A^2\left(\tau - t\right)^2 - z^2}}{r}\sin\omega,\ \text{where}\ \varphi = \angle T^\tau O\pi_A^t .$$

Thus, in the cylindrical frame, the point A^t is localized at

$$z = \nu\sin\lambda\cdot\frac{\tau - t}{\tau}, \qquad (2)$$

$$r^2 = R^2 + v_A^2\left(\tau - t\right)^2 - z^2 - 2R\sqrt{v_A^2\left(\tau - t\right)^2 - z^2}\ \cos\omega, \qquad (3)$$

$$\theta = \eta + \varphi = \eta + \arcsin\frac{\sqrt{v_A^2\left(\tau - t\right)^2 - z^2}}{r}\sin\omega. \qquad (4)$$

Remark. Also, there is another location of the straight line $O\pi_A^0$, different from that in Fig. 1 (when it intersects the segment T^0T^τ). With this configuration of the geometrical objects, the final coordinate angle θ will be expressed by the difference of angles (5) instead of the sum (4):

$$\theta = \eta - \varphi. \qquad (5)$$

This feature should be taken into account during simulation.

In addition, direct verification shows that changes in other values, e.g., the angle ζ, do not affect the final result. ♦

Thus, at the time instant *t*, the Defender must be at the point A^t with the coordinates (2)–(4). As is known [23], the square of absolute velocity in the cylindrical frame satisfies the equality

$$v_D^2 = \dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2, \qquad (6)$$

where the dot denotes the differentiation operator $\frac{d}{d\tau}$.

In view of $r = r(t(\tau), \tau)$, $\theta = \theta(t(\tau), \tau)$, and $z = z(t(\tau), \tau)$, substituting the expressions (2)–(4) into formula (6) gives the following dynamics equation for the intercept time $t(\tau)$:

$$A(t(\tau), \tau)\dot{t}^2 + B(t(\tau), \tau)\dot{t} + C(t(\tau), \tau) = 0. \qquad (7)$$

The coefficients of equation (7) are obtained by differentiating equations (2)–(4). The analytical expressions for these coefficients were obtained using the Maple symbolic computing package; in particular, $A(t(\tau), \tau) = v_A^2$. Due to the cumbersome form, the other coefficients are not presented here.

Equation (7), as a quadratic one in $\dot{t}(\tau)$, can be resolved with respect to the derivative:

$$\dot{t}(\tau) = \Big(-B(t(\tau), \tau) \pm \sqrt{B^2(t(\tau), \tau) - 4A(t(\tau), \tau)C(t(\tau), \tau)}\Big) \times \big(2A(t(\tau), \tau)\big)^{-1}.$$

As the initial condition we choose $t(0) = 0$. The reason is that, in order to intercept the Target at the time $\tau = 0$, the Attacker and Target must be at the same point of space. In other words, the Defender's position must coincide with those of the Target and Attacker at the same time instant, i.e., the intercept time is zero.

4. NUMERICAL SIMULATION

Let us select the following model parameters:

$$R = 1,\ v_A = 1,\ v_D = \frac{3}{4},\ v_T = \frac{1}{2},$$

$$\gamma = \frac{\pi}{6},\ \lambda = 0.05\pi,\ \tau \in [0, 3.5].$$

The numerical simulation results under these parameter values are shown in Fig. 2. The cylinder of radius *R*, built on the Target's evasion circle *O*, is set off in gray. The thick blue line (the arc $\smile T^0T^tT^\tau$) corresponds to the Target's path; the thick red line (the segment $\hat{A}^0A^t$), to the Attacker's path; the thick green line (T^0A^t), to the Defender's path. The dotted line is the part of the path lying inside the cylinder.

The straight line $T^0\hat{A}^0$ is the initial bearing line; the point $\hat{A}^0$ is the point from which the Attacker must start moving to intercept the Target at T^τ; the point π_A^0 is the projection of the point $\hat{A}^0$ on the plane of the circle *O*.

Let us increase the simulation time to 5, keeping the same parameter values. The simulation results are shown in Figs. 3 and 4.

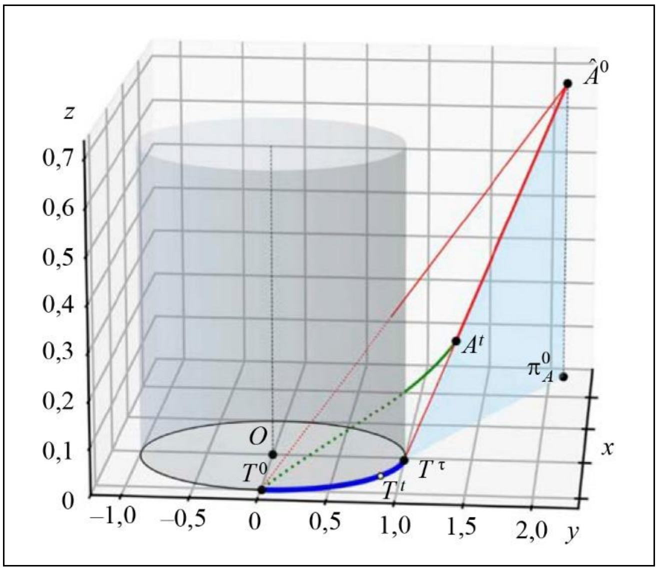

Fig. 2. Numerical simulation results for $\tau \in [0, 3.5]$.

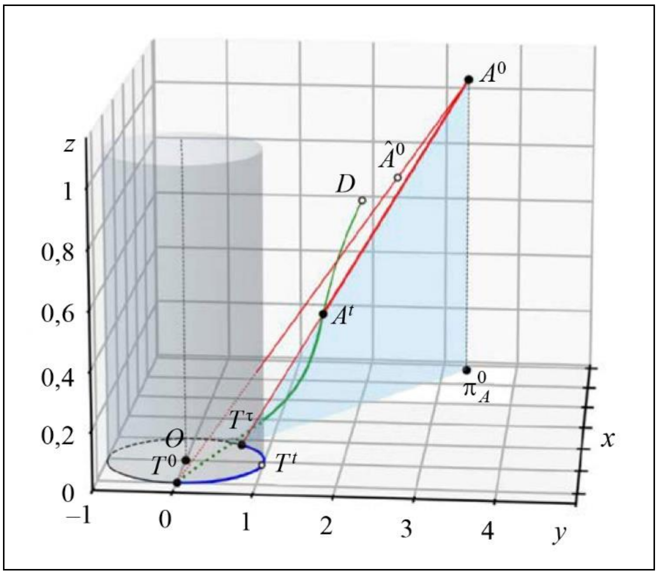

Fig. 3. Numerical simulation results for $\tau \in [0, 5]$.

Figures 3 and 4 have almost the same notations as Fig. 2; $\hat{A}^0$ is the initial point from which the Attacker must start moving to intercept the Target at the time instant $\tau = 3.5$ (i.e., it coincides with the point $\hat{A}^0$ in Fig. 2), and A^0 is the point from which the Attacker must start moving to intercept the Target at the point T^τ at the time instant $\tau = 5$.

If we extend the simulation horizon to $\tau = 7$, the Defender will continue moving along the curve A^tD after the point A^t up to some point D and further.

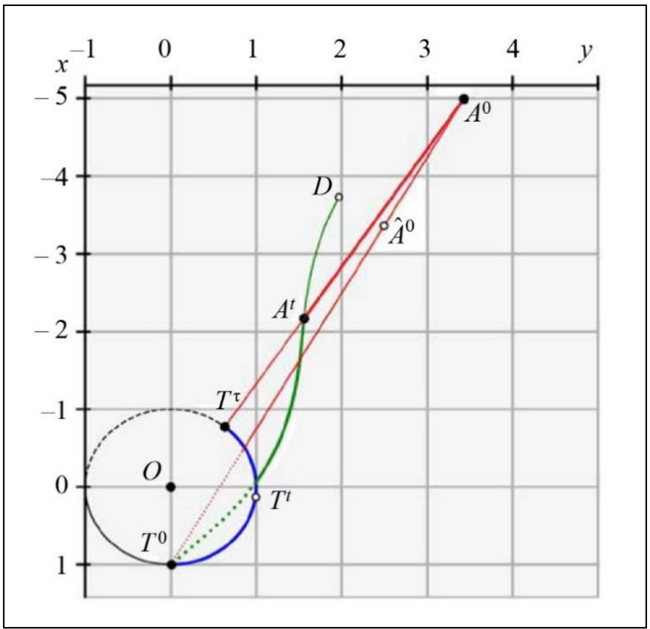

Fig. 4. Numerical simulation results for $\tau \in [0, 5]$ in the projection on the *xy* plane.

The case of two or more Defenders is also interesting. It makes no sense within the current problem statement: the possible interception point can be unambiguously reconstructed using the available information about the Attacker's motion. However, for example, let the Attacker's velocity be unknown but take finitely many values; then it is possible to build a set of possible interception points, which in turn can be distributed among the Defenders to apply the algorithm to each of them. Moreover, in such a formulation, it seems reasonable to distribute attacking objects by the degree of danger, which will be discussed later.

The range of the Attacker's velocity may also be known. In this case, it makes sense to pose an optimization problem: find the Defender's path that ensures the minimum probability of Target interception. It is natural to expect that using more than one Defender in such a problem may significantly increase the survivability of the defended object.

CONCLUSIONS

In this paper, we have considered an evasion maneuver of a Target with uniform flat circumferential movements from an attack by a uniformly and rectilinearly moving attacking player from the upper hemisphere of the circle. To disrupt the attack, the Target uses a mobile Defender with program movements; based on angular information about the Attacker at the pursuit start, the Defender builds its maneuver so as to intercept all the Attacker's paths dangerous for the Target (in the pointwise meeting sense).

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REFERENCES

1. Ivanov, M.N. and Maslov, E.P., On One Problem of Deviation, *Automation and Remote Control*, 1984, vol. 45, no. 8, pp. 1008–1014.
2. Zheleznov, V.S., Kryakovskii, B.S., and Maslov, E.P., On a Catch Problem, *Automation and Remote Control*, 1996, vol. 57, no. 8, pp. 1072–1078.
3. Rubinovich, E.Ya., Missile-Target-Defender Problem with Incomplete a Priori Information, *Dynamic Games and Applications*, 2019, vol. 9, pp. 851–857.
4. Boyell, R.L., Defending a Moving Target against Missile or Torpedo Attack, *IEEE Trans. Aerosp. Electron. Syst.*, 1976, vol. AES-12, no. 4, pp. 582–586.
5. Boyell, R.L., Counterweapon Aiming for Defence of a Moving Target, *IEEE Trans. Aerosp. Electron. Syst.*, 1980, vol. AES-16, no. 3, pp. 402–408.
6. Shneydor, N.A., Comments on "Defending a Moving Target against Missile or Torpedo Attack," *IEEE Trans. Aerosp. Electron. Syst.*, 1977, vol. AES-13, no. 3, pp. 321–321.
7. Garcia, E., Casbeer, D.W., Pham, K., and Pachter, M., Cooperative Aircraft Defense from an Attacking Missile, *Proc. of the 53rd IEEE Conference on Decision and Control (CDC)*, Los Angeles, USA, 2014, pp. 2926–2931.
8. Pachter, M., Garcia, M., and Casbeer, D.W., Active Target Defense Differential Game, *Proceedings of the 52nd Annual Allerton Conference on Communication, Control, and Computing*, Allerton, 2014, pp. 46–53.
9. Perelman, A., Shima, T., and Rusnak, I., Cooperative Differential Games Strategies for Active Aircraft Protection from a Homing Missile, *Journal of Guidance, Control, and Dynamics*, 2011, vol. 34, no. 3, pp. 761–773.
10. Rusnak, I., Weiss, H., and Hexner, G., Guidance Laws in Target-Missile-Defender Scenario with an Aggressive Defender, *Proceedings of the 18th IFAC World Congress*, Milano, Italy, 2011, pp. 9349–9354.
11. Rusnak, I., The Lady, the Bandits and the Body-Guards – a Two Team Dynamic Game, *Proceedings of the 16th IFAC World Congress*, Prague, Czech Republic, 2005, pp. 441–446.
12. Shima, T., Optimal Cooperative Pursuit and Evasion Strategies against a Homing Missile, *AIAA Journal of Guidance, Control, and Dynamics*, 2011, vol. 34, no 2, pp. 414–425.

13. Yamasaki, T., and Balakrishnan, S.N., Terminal Intercept Guidance and Autopilot for Aircraft Defense against an Attacking Missile via 3D Sliding Mode Approach, *Proceedings of the 2012 American Control Conference (ACC)*, Montreal, 2012, pp. 4631–4636.
14. Yamasaki, T., Balakrishnan, S.N., and Takano, H., Modified Command to Line-of-Sight Intercept Guidance for Aircraft Defense, *Journal of Guidance, Control, and Dynamics*, 2013, vol. 36, no. 3, pp. 898–902.
15. Liu, Y., Qi, N., and Shan, J., Cooperative Interception with Double-Line-of-Sight-Measuring, *Proceedings of the AIAA Guidance, Navigation, and Control (GNC) Conference*, Reston: American Institute of Aeronautics and Astronautics, 2013, art. no. AIAA 2013-5112.
16. Qi, N., Sun, Q., and Zhao, J., Evasion and Pursuit Guidance Law against Defended Target, *Chinese Journal of Aeronautics*, 2017, vol. 30, no. 6, pp. 1958–1973.
17. Weissyand, M., Shimazand, T., and Rusnak, I., Minimum Effort Intercept and Evasion Guidance Algorithms for Active Aircraft Defense, *Journal of Guidance, Control, and Dynamics*, 2016, vol. 39, no. 10, pp. 2297–2311.
18. Garcia, E., Casbeer, D.W., and Pachter, M., Active Target Defense Differential Game with a Fast Defender, *IET Control Theory and Applications*, 2017, vol.11, no. 17, pp. 2985–2993.
19. Garcia, E., Casbeer, D.W., and Pachter, M., The Complete Differential Game of Active Target Defense, *Journal of Optimization Theory and Applications*, 2021, vol. 191, pp. 1–25.
20. Gong, X., Chen, W., and Chen, Z., Intelligent Game Strategies in Target-Missile-Defender Engagement Using Curriculum-Based Deep Reinforcement Learning, *Aerospace*, 2023, vol. 10, no. 2, art. no. 133.
21. Galyaev, A.A., Samokhin, A.S., and Samokhina, M.A., Modeling of Target's Interception Delay in an ADT Game with One or Two Defenders, *Control Sciences*, 2024, no. 2, pp. 72–87.
22. Potapov, A.P. and Galyaev, A.A., Countermeasures Against the Attacker's Homing Algorithm in a Game of Three Players, *Mekhatronika, Avtomatizatsia, Upravlenie*, 2024, vol. 25, no. 11, pp. 575–584. (In Russian.)
23. Markeev, A.P., *Teoreticheskaya mekhanika* (Theoretical Mechanics), Moscow: CheRo, 1999. (In Russian.)

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