THE INCENTIVE-TARGETING PROBLEM IN A REFLEXIVE GAME WITH A POINT-TYPE AWARENESS STRUCTURE

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Abstract. This paper considers a collective behavior model of agents under strategic uncertainty and incomplete awareness. Decision-making is modeled by a reflexive game in which participants choose their actions based on a hierarchy of beliefs about the game parameters, beliefs about beliefs, and so on. The study is focused on reflexive games with a point-type awareness structure and the linear best response of players. As shown below, the informational equilibrium in such games is analogous to the Nash equilibrium in a game on a network. Explicit expressions for the equilibrium responses of players are derived and conditions for the existence and uniqueness of equilibria are established. An incentive-targeting problem similar to that in a corresponding game on a network is formulated: a relationship is obtained between the equilibria in the game with common knowledge and the game with incomplete awareness in which the Principal individually reports new incentives to the players.

Keywords: reflexive game, informational equilibrium, games on networks, networked control.

INTRODUCTION

Informational reflexion is the process and result of agent's thoughts about what the values of uncertain parameters are and what his/her opponents (other agents) know and think about these values [1]. Informational reflexion is often associated with insufficient mutual awareness, and its result is used in decision-making. Reflexive games are one way to model decision-making under incomplete awareness. In such games, agents' beliefs about each other are formalized by phantom agents. Introducing the concept of a phantom agent allows defining a reflexive game as that of real and phantom agents and an informational equilibrium as a generalization of Nash equilibrium for the reflexive game case. Within this framework, by assumption, each (real or phantom) agent uses his/her hierarchy of beliefs about objective and reflexive reality when calculating his/her subjective equilibrium (the equilibrium in the game he/she believes is being played).

The graph of a reflexive game is a convenient tool for studying its conceptual solution—informational equilibrium. The vertices of such a graph correspond to real and phantom agents, and each vertex-agent has incoming arcs from vertices-agents whose actions in a subjective equilibrium determine the payoff of this agent. The number of incoming arcs is one less than the number of real agents. Below, using a reflexive game with a point-type awareness structure as an example, we demonstrate that in some cases (when the best responses of players linearly depend on the response of the other agents related to them), informational equilibrium can be described explicitly in terms of the awareness structure.

For the class of games with the linear best response, the *incentive-targeting problem* was formulated in [2]. In this problem, the Principal tunes the parameter of the agents' utility function representing their marginal payoff independent of the actions of other agents. In a reflexive game with a point-type awareness structure, the parameter can be tuned in two ways, with different possible implementations: reporting the new parameter value to all agents simultaneously (which is equivalent to solving the original incentive-targeting problem) or the case in which only part of agents is informed about the value. In this paper, we provide an information revelation (message) mechanism used by the Principal when both cases—

the common knowledge and incomplete awareness of agents—produce the same equilibrium.

Section 1 describes the reflexive game model with a point-type awareness structure [3]. In Section 2, we demonstrate that informational equilibrium in such a game is analogous to Nash equilibrium in the game on a network with the linear best response [4] and can be formulated in terms of the awareness structure. Also, explicit expressions for the equilibrium responses of players are derived and conditions for the existence and uniqueness of equilibria are established. This formal statement allows considering an arbitrary structure of connections between players and obtaining the equilibrium in explicit form. In Section 3, we introduce an informational control problem similar to the incentive-targeting problem in the game on a network and obtain a relationship between the equilibria in the game with common knowledge and the game with incomplete awareness in which the Principal individually reports new incentives to the players.

1. REFLEXIVE GAMES AND THE AWARENESS STRUCTURE OF AGENTS

A reflexive game Γ_I with a finite awareness structure I is given by a tuple $\Gamma_I = \{N, (X_i)_{i \in N}, f_i(\cdot)_{i \in N}, \Theta, G_I\},$, where N denotes the set of real agents; X_i is the set of all feasible actions of real agent i; Θ is the set of possible values of an uncertain parameter (the "states of nature"); $f_i(\cdot)$: $\Theta \times X \to \Re^1$ is the payoff function of real agent i, $i \in N$; finally, G_I is the graph of this reflexive game.

Informally speaking, a reflexive game represents an interaction of agents reflexing about the parameter $\theta \in \Theta$. The agents have some beliefs about each other: true (then the corresponding agent is real) or false (then the agent is phantom). For example, there exists an instance ("image") of the first agent in the consciousness of the second one, either coinciding with the real first agent or not; in the latter case, the instance of the first agent is a phantom.

The graph of a reflexive game, in which the vertices correspond to the agents and the connections to their mutual beliefs about each other, consists of the following elements:

A_i, the set of vertices corresponding to possible instances of agent i, $I \in N = \{1, ..., n\}$; exactly one of them is real, and the others (if any) are phantom agents;

$A = A_1 \cup \ldots \cup A_n$, the set of all graph vertices representing all agents (we will not distinguish between agents and graph vertices);

$\theta(a)$, the state of nature from the viewpoint of agent a, where $a \in A$ and $\theta \in \Theta$;

$\nu(a)$, the set of all agents $i \neq a$ considered real by agent $a \in A$; this fact is reflected in the graph by arrows (directed connections) from such agents to a; for agent $a \in A_i$, the set $\nu(a) \subset A$ consists of $(n-1)$ elements belonging to the sets $A_1 \cup \ldots \cup A_{i-1} \cup A_{i+1} \cup \ldots \cup A_n$, respectively.

Assume that the payoff functions of all instances of the same real agent coincide: agent $a \in A_i$ seeks to maximize the function $f_i(\cdot)$. In this case, an informational equilibrium can be defined as a set of all actions of agents $x_a, a \in A$, such that

$$x_a \in \operatorname*{Argmax}_{x \in X_i} f_i(\theta(a),\ x, x_{-a}) \text{ for all } i \in N,\ a \in A_i,$$

where x_{-a} denotes the action profile of all agents from the set $\nu(a)$.

As one example, the reflexive game with the payoff function

$$f_i(\theta, x_1, x_2, x_3) = (\theta - x_1 - x_2 - x_3) x_i - \frac{x_i^2}{2}, \quad (1)$$

where $x_i \geq 0$, $i \in N = \{1, 2, 3\}$, and $\theta \in \Theta = \{1, 2\}$, was considered in [3]. Depending on the different awareness of players, the informational equilibrium was obtained by calculating the best responses of players from the first-order optimality conditions $\partial f_i / \partial x_i = 0$ and solving the resulting system of linear equations. Passing to an arbitrary number of agents, we can generalize the payoff function (1) of agent i to

$$f_a(\theta,\ x_a,\ x_{-a},\ G_I) = \left(\theta(a) - \beta \sum_b g_{ab} x_b\right) x_a - \frac{x_a^2}{2}, \quad (2)$$

where G_I is the graph of the reflexive game in which $g_{ab} = 1$ for $b \in \nu(a)$, $g_{ab} = 0$ otherwise, and $g_{aa} = 0$. The parameter $\beta > 0$ reflects the nature of dependence on the actions of neighbors. The system of the best responses of players with the payoff function (2) is written as

$$x_a = \theta(a) - \beta \sum_b g_{ab} x_b;$$

in matrix form, it becomes

$$x = \theta - \beta G_I x,$$

or

$$\left(I+\beta G_I\right)x=\theta,$$

where I denotes an identity matrix.[1] Using the results of [4], we formulate the following statement on the equilibrium in the reflexive game with a point-type awareness structure and the player's payoff function (2).

Proposition 1. *If* $\beta\lambda_{\min}\left(\frac{G_I+G_I^{\mathrm{T}}}{2}\right)<1$, *then the reflexive game* $\Gamma_I=\left\{N,\,(X_i)_{i\in N},\,v_i\left(\cdot\right)_{i\in N},\,G_I\right\}$ *with a point-type awareness structure and the agent's payoff function* (2) *has the unique informational equilibrium*

$$x^*=\left(I+\beta G_I\right)^{-1}\theta.$$

P r o o f is equivalent to that of Proposition 2 in [4]. It involves the property of this game to be potential: the game with the payoff function (2) possesses the potential function $\varphi\left(x\right)=x^{\mathrm{T}}1-\frac{1}{2}x^{\mathrm{T}}\left(I-\beta G_I\right)x$, whose maximum is a Nash equilibrium. A sufficient condition for the existence of a unique solution is the concavity of the potential function. The matrix of its second derivatives has the form $\nabla^2\varphi=-\left(I-\beta G_I\right)$, and the function φ is strictly concave if and only if the matrix $I-\beta G_I$ is positive definite, i.e., $y^{\mathrm{T}}\left(I-\beta G_I\right)y>0$ for any $y\neq 0$. This is equivalent to the condition $\beta\lambda_{min}\left(G_I\right)<1$, where $\lambda_{min}\left(G_I\right)$ denotes the smallest eigenvalue of the matrix G_I. Since the matrix G_I is asymmetric, the matter concerns the smallest eigenvalue of its Hermitian component, $\lambda_{\min}\left(\frac{G_I+G_I^{\mathrm{T}}}{2}\right)$. ♦

In this formulation, the game fully matches the local public goods game, which is quite popular among researchers [4–6]. More precisely, the Nash equilibrium in the local public goods game has the form $x^*=\left(I+\beta E\right)^{-1}\theta$, where the matrix $E=\left\{e_{ij}\right\}\in\left\{0,1\right\}^{n\times n}$, i.e., the mutual influence graph of agents, acts as the interaction network (e.g., the network of acquaintances, competition, etc.), and the informational equilibrium in the reflexive game is given by $x^*=\left(I+\beta G_I\right)^{-1}\theta$, where the graph G_I of the reflexive game acts as the interaction network. The equilibrium in such a game also exists for $\beta\lambda_{\min}\left(\frac{G_I+G_I^{\mathrm{T}}}{2}\right)>1$, but the analysis below will be restricted to the case of Proposition 1.

Similar models were considered in [7–9] with an additional parameter r_i characterizing the type of agent *i* (the efficiency or qualification of his/her activity). In the game with the payoff function $f_i\left(\theta,x_i\right)=\left(\theta_i-\sum_{j\in N}x_j\right)x_i-\frac{x_i^2}{2r_i}$ on an arbitrary graph G [7], the equilibrium can be represented as

$$x^*=\left(Ir^{-1}+\beta G\right)^{-1}b.$$

In the game where agents with the payoff function $f_i\left(\theta,x_i\right)=\left(\sum_{j\in N}x_j-h\right)x_i-\frac{x_i^2}{2r_i}$ apply efforts x_i to some joint action (with a positive contribution to their payoff functions if the total effort exceeds some threshold h) [8], the equilibrium on an arbitrary graph G can be represented as

$$x^*=\left(-Ir^{-1}+\beta G\right)^{-1}h.$$

2. THE INCENTIVE-TARGETING PROBLEM FOR THE REFLEXIVE GAME

For the local public goods game, the following incentive-targeting problem[2] was proposed in [2]: maximize the social welfare function by controlling the vector θ, i.e.,

$$W\left(\theta,\,G\right)=\sum_{i\in\mathrm{N}}f_i\to\max_{\theta}.$$

Consider an analogous problem for the reflexive game with a point-type awareness structure:

$$W\left(\theta,\,G_I\right)=\sum_{i\in\mathrm{N}}f_i\to\max_{\theta}.$$

Assume that it is possible to tune incentives only for real agents, but a central planner (control authority, the Principal) can differently inform real agents, thus

[1] As is easily verified, the model from the original paper [3] can be similarly represented as

$$f_i\left(\theta,\,x_a,x_{-a},\,G_I\right)=\left(\theta\left(a\right)-\sum_b g_{ab}x_b\right)x_a-x_a^2-\frac{x_a^2}{2},$$

and the system of the best responses takes the form $x_a=\frac{\theta\left(a\right)-\sum_b g_{ab}x_b}{3}$, or $\left(3I+G_I\right)x=\theta.$

[2] Formally, Galeotti et al [2] stated the incentive-targeting problem for the game with the equilibrium $x^*=\left(I-\beta G\right)^{-1}\theta$ [10], where the parameter β reflects the nature of dependence on the actions of neighbors: for $\beta>0$, the actions of players are *strategic complements*; for $\beta<0$, *strategic substitutes*. Thus, the problem considered here corresponds to the case of the game [2] with $\beta<0$.

changing the awareness structure and generating new phantom agents. (In fact, incentivizing phantom agents is informational control.) More precisely, we analyze the following multistep informational process evolving from the initial common knowledge of all agents: at each step, the Principal individually reports the change of the state $\theta_i \to \hat{\theta}_i$ to all real agents *i*.

This process has the following effect on the graph of the reflexive game. Suppose that at the initial step, the agents $N = \{1, 2, 3\}$ are connected by the mutual influence graph $E = \{e_{ij}\}$ (Fig. 1a, where arrows $i \sim j$ reflect the dependence of the agent *i*'s payoff on the agent *j*'s actions) and all three agents are equally informed: the state of nature is $\theta_i = \theta$ from the viewpoint of each agent *i*. After the Principal's first message about the new value of the parameter $\hat{\theta}_i$ sent to each player individually, the mutual influence graph remains unchanged; in the graph of the reflexive game, the real agents become phantom ones, and the new real agents are the vertices arising in this graph by adding vertex *i* and connections to the vertices of phantom agent i' in the mutual influence graph E. In addition, $\theta_i = \hat{\theta}$ for real agents and $\theta_{i'} = \theta$ for phantom ones. The procedure for creating new connections when changing the awareness structure $\theta_i \to \hat{\theta}_i$ can be represented as a bipartite graph: if vertices *i* and *j* are connected in the original graph E, then the new bipartite graph contains the original graph E as one lobe and E' as the other and there is a connection between vertices $i \sim j'$ (Fig. 1b).

Then at the initial step, the sets $A_1, \ldots, A_n$ are singletons, and the awareness structure G_I coincides with the mutual influence graph *E*. At each subsequent step, one instance $\hat{a}_i$ is added to each of the sets A_i, $i \in N$, and this instance becomes a real agent; in this case, $\theta(\hat{a}_i) = \hat{\theta}_i$, and the set $\nu(\hat{a}_i)$ consists of the instances of all agents real at the previous step. Formally, the equilibrium responses of players can be represented as follows: at the initial step (without Principal's messages), the equilibrium of real agents is

$$x^* = (I + \beta E)^{-1} \theta.$$

After the first Principal's message, the equilibrium responses of real agents take the form

$$x^{*(1)} = \hat{\theta} - \beta E x^*;$$

after the second,

$$x^{*(2)} = \hat{\theta} - \beta E x^{*(1)}.$$

The equilibrium responses of real agents after the *k*th Principal's message are

$$x^{*(k)} = \hat{\theta} - \beta E x^{*(k-1)}.$$

After the *k*th message, the graph of the reflexive game has kN vertices. The following result describes the Principal's control impact in terms of the initial mutual influence graph E.

Proposition 2. *If there exists a unique informational equilibrium in the reflexive game Γ_I with a point-type awareness structure, then the multistep procedure of changing the awareness structure G_I converges to the equilibrium in the game with the complete awareness Γ:*

$$x^{*(k)} \to (I + \beta E)^{-1} \hat{\theta} \quad \text{as} \quad k \to \infty.$$

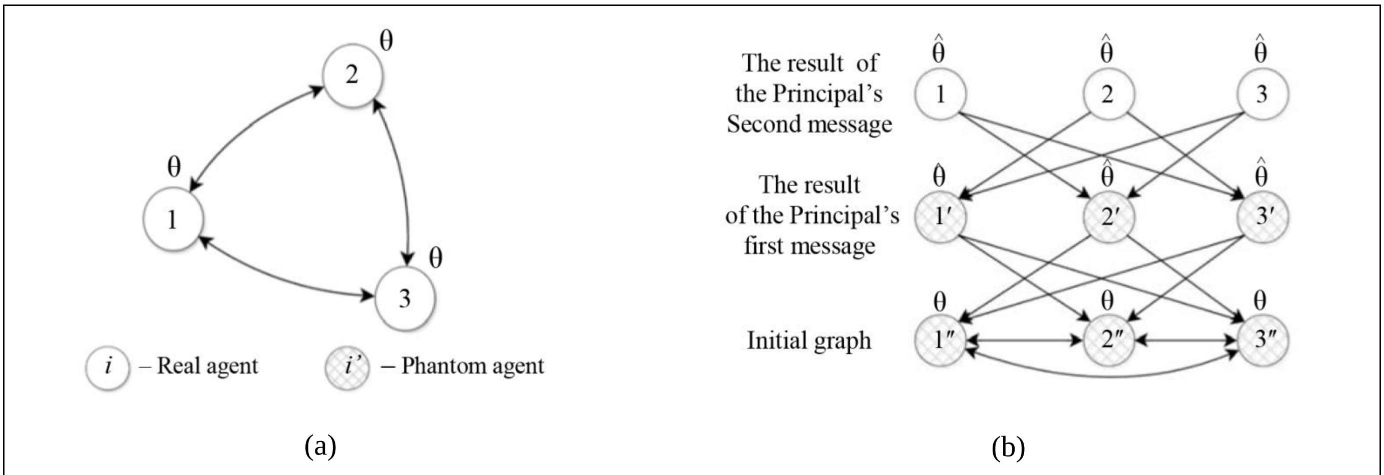

Fig. 1. The Principal's message procedure and changes in the graph of the reflexive game: (a) the initial mutual influence graph *E*, (b) the graph G_I of the reflexive game resulting from the Principal's messages (reporting the new value of the parameter θ_i to the agents).

P r o o f. For $k > 0$, the kth Principal's message results in the awareness structure G_I, generating the following equilibrium actions of agents:

$$x^{*(k)} = \sum_{j=0}^{k-1} (-1)^j \beta^j E^j \hat{\theta} + (-1)^k \beta^k E^k x^* .$$

This fact is easily verified by substitution. As $k \to \infty$, the second term $(-1)^k \beta^k E^k x^*$ vanishes due to the constraint $\beta\lambda_{\min}(E) < 1$. The expression $\sum_{j=0}^{k-1} (-1)^j \beta^j E^j$ in the first term is a Neumann series expansion of the matrix $(I + \beta E)^{-1}$ [11]. ♦

3. AN EXAMPLE

We illustrate the results with an example similar to that in [3]. Let the payoff function f_i of player i be

$$f_i = (\theta_i - x_1 - x_2 - x_3) x_i - \frac{x_i^2}{2}$$

for the case of three agents. The best responses of players are given by

$$\begin{cases} x_1 = \dfrac{\theta_i - x_2 - x_3}{3} \\ x_2 = \dfrac{\theta_i - x_1 - x_3}{3} \\ x_3 = \dfrac{\theta_i - x_1 - x_2}{3}. \end{cases}$$

Let us analyze the values of $W(U, \theta) = \sum_{i \in \{1,2,3\}} x_i$ depending on the number of Principal's messages (see the table and Fig. 2). For $\theta_{i \in \{1,2,3\}} = 1$, the equilibrium responses of players are $x^*_{i \in \{1,2,3\}} = 0.2$, and the social welfare function takes the value $W(\theta) = 0.6$. Suppose that the Principal changes the original values $\theta_{i \in \{1,2,3\}} = 1$ to the new ones $\hat{\theta}_{i \in \{1,2,3\}} = 2$. In the case of common knowledge regarding the change $\theta_i \to \hat{\theta}_i$, the equilibrium responses of players are $x^*_{i \in \{1,2,3\}} = 0.4$, and the social welfare function takes the value $W(\hat{\theta}) = 1.2$. However, when the agents are individually informed about the change $\theta_i \to \hat{\theta}_i$, at each iteration the Principal individually reports to each agent the situation of the previous step. As the result of these messages, the awareness structure changes; the equilibrium responses of players and the value of the social welfare function also change, reaching in the limit the values for the case of common knowledge.

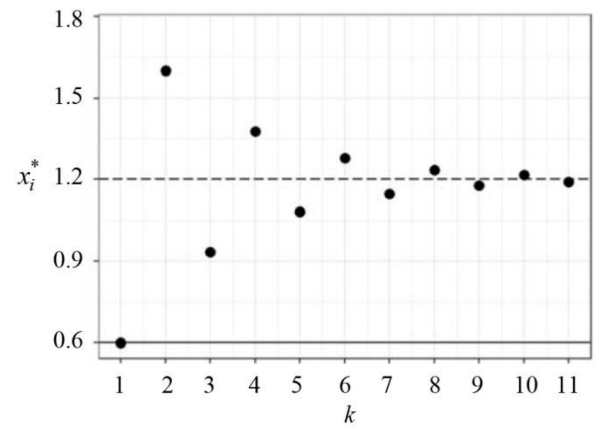

Fig. 2. The equilibrium responses of players depending on the number of Principal's messages *k*. Dots indicate the responses of player x_i^* depending on Principal's messages. The solid and dashed lines show the equilibrium responses of agents in the game with common knowledge and $\theta_{i \in \{1,2,3\}} = 1$ and $\hat{\theta}_{i \in \{1,2,3\}} = 2$, respectively.

CONCLUSIONS

This paper has considered a game-theoretic collective behavior model of agents under strategic uncertainty and incomplete awareness. For a reflexive game with a point-type awareness structure, it has been shown that if the best responses of players linearly depend on other connected agents, the informational equilibrium can be explicitly described in terms of the awareness structure. This approach allows interpreting the reflexive game from [3] as a local public goods game with an incomplete awareness of agents and describing in matrix form the equilibrium in the reflexive game with a point-type awareness structure and the linear best response.

The equilibrium responses of players depending on the number of Principal's messages

No.	0	1	2	3	4	5	6	7	8	9	10	11
θ	1	2	2	2	2	2	2	2	2	2	2	2
x_i^*	0.2	0.53	0.31	0.45	0.36	0.42	0.38	0.41	0.39	0.40	0.39	0.40
W	0.6	1.6	0.93	1.37	1.08	1.27	1.14	1.23	1.17	1.21	1.18	1.20

In addition, the results of this study can be used to compare the solutions of control problems for the cases of common knowledge and incomplete awareness of agents. As an example, we have formulated an informational control problem similar to the incentive-targeting problem in the local public goods game on a network; in this problem, the mechanism of Principal's messages yields in the limit the same response in the cases of the common knowledge and incomplete awareness of agents.

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