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AN APPROACH TO COMPARE ORGANIZATIONAL MODES OF ACTIVE AGENTS AND CONTROL METHODS

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Abstract. When interacting, active agents can behave independently, cooperate, or be connected by hierarchical relations. In turn, hierarchical impact may be exerted by administrative or economic methods with or without feedback. We systematically describe these organizational modes and control methods based on game-theoretic models with different information structures without considering uncertainty. It seems crucial to compare quantitatively the payoffs of individual agents and the whole set of them (social welfare) under these organizational modes and control methods. We propose a methodology for building the systems of social and private preferences in normal-form games and shares in the allocation of cooperative payoff. A system of relative efficiency indices is developed for detailed quantitative assessment. The proposed methodology is illustrated by several Cournot oligopoly models.

Keywords: inefficiency of equilibria, methods of control and resource allocation, organizational modes for active agents.

INTRODUCTION

At first glance, it seems obvious that cooperation is better than confrontation. Combining the efforts of active agents produces better results than their independent selfish behavior, much less hostility, and the additional coalition payoff can be allocated among all agents.

Unfortunately, the things are not so simple. For the society, the cooperative payoff is always at least not smaller than under the independent behavior of the active agents composing the society or hierarchical relations between them. But this is not true for each agent. For example, the payoff of an upper-level agent in the hierarchy can exceed its share in the uniform allocation under cooperation, even considering the additional effect. Also, it is not so easy to negotiate a division of the additional payoff, even if there is an agreement to cooperate in principle, and ensure the agreement's stability. Perhaps due to these considerations, there are many examples of abandoning cooperation in favor of conflict and competition for leadership in economics, public life, international relations, and other areas.

Therefore, a topical problem is to analyze in mathematical terms the conditions of profitable cooperation and compare the efficiency of different organizational modes of active agents, control methods, and allocation of the cooperative payoff. The fundamental foundations of such mathematical analysis are provided by the theory of active systems and control in organizations [1, 2], the information theory of hierarchical systems [3–7], and the theory of incentives and mechanism design [8]. The concept of sustainable management of active systems based on considering and coordinating the interests of active agents was proposed in [9, 10]. Game theory [11–15] is the main mathematical tool of the analysis. Complex dynamic problems of conflict control are solved using simulation modeling [16].

The problem of inefficiency of equilibria was analyzed in detail in [17–20]. The outcome of the rational behavior of independent selfish economic agents is usually worse for the society than that of centralized management or voluntary cooperation. An important question arises: how much worse is it? The price of anarchy is a common measure of the inefficiency of equilibria, defined as the ratio of the social payoff function value in the worst-case equilibrium to its value in the optimal outcome [21]. A wider set of indicators for dynamic games was proposed in [22]. The payoffs under different organizational modes of agents were compared in many papers on a game theory; for example, see [23–25].

However, the (in)efficiency of equilibria should be given a more general problem statement. First of all, the payoffs are to be compared in the basic organizational modes of active economic agents (equality, hierarchy, and cooperation) under different control methods specifying the rules of their interaction. In addition, and more importantly, the comparison should be made in terms of social welfare and the interests of individual agents. As mentioned, a game outcome beneficial for the society under some organizational mode may not be such for each agent.

Equality, hierarchy, and cooperation are the basic organizational models of interaction between active agents. Under equality, the agents (players) choose their actions simultaneously and independently, and Nash equilibrium is the solution of the arising normal-form game. In a hierarchical organization, two main control modes are possible. In the first one, the Leader chooses an intended action and informs one or several players of his choice; then the other players optimally respond to this action. As a result, the Germeier game Γ_1 arises, and Stackelberg equilibrium is considered its solution. (In English-language literature, it is also known as the Stackelberg game.) In the second control mode, the Leader chooses his strategy as a function of the expected actions of the Followers and informs them of his choice; then the Followers optimally respond to this strategy. As a result, the Germeier game Γ_2 arises, also called the inverse Stackelberg game in English-language literature. Its solution is calculated using Germeier's principle of guaranteed result [4]. It is also reasonable to distinguish between administrative (compulsion) and economic (impulsion) control methods. Compulsion is affecting the sets of admissible strategies of agents, whereas impulsion is affecting the payoff functions of agents [9, 10]. Another formalization of hierarchical relations is the extensive form games where the players act sequentially. Such games are not considered below. Finally, in cooperation, all players join together and maximize the total payoff function over all control variables. This interpretation corresponds to the utilitarian approach, as opposed to the egalitarian approach when the agents maximize the smallest payoff [26]. As a result, the original game is reduced to an optimization problem, in which the cooperative solution is Pareto-optimal. In this sense, the dynamic statements of conflict control problems (differential or difference games) do not fundamentally differ from the static ones [11–15].

Besides the payoff functions, which characterize the efficiency of agents' actions, game-theoretic models may contain additional constraints: coordination conditions [7] or sustainable development conditions [9, 10]. These conditions mean that the state of the controlled dynamic system should belong to some domain in the state space. In static models, these conditions are formulated as control constraints.

The contributions of this paper are as follows:

- We systematically describe the interaction modes of active agents and their control methods using game-theoretic models without uncertainty.
- We propose a comparative analysis methodology for the social and private efficiency of the control methods based on the agents' payoffs in normal-form games and their shares in the allocation of cooperative payoff in characteristic function games.
- We develop a system of relative efficiency indices for detailed quantitative assessment.
- We illustrate the proposed methodology by static and dynamic Cournot oligopoly models.

Section 1 describes a game-theoretic formalization of organizational modes for the interaction of active agents and their control methods. Comparing the agents' payoffs, we build the systems of social and private preferences. Note that comparative efficiency indices can be used for detailed quantitative characterization. Section 2 considers the comparative efficiency methodology based on game-theoretical models of conflict control. In Section 3, this methodology is illustrated by several Cournot oligopoly models. The results of this paper and further research are discussed in the Conclusions.

1. ORGANIZATIONAL MODES, CONTROL METHODS, AND SYSTEMS OF PREFERENCES

1.1. Organizational Modes and Control Methods

As noted, equality, hierarchy, and cooperation are the main organizational models of interaction between active agents. In a hierarchical organization, two main control methods are possible: compulsion (administrative mechanisms) and impulsion (economic mechanisms). These control mechanisms can be implemented with or without feedback.

Two questions arise during cooperation. First, how should the payoff of each coalition be defined? (How should the characteristic function be constructed?) Second, how should the total payoff be allocated among the players? (Which optimality principle should be chosen?) It is also natural to treat payoff allocation as a control problem.

Section 1 describes the interaction modes and control methods within static game-theoretic models.

The interaction of equal agents is modeled by a normal-form game of *n* players:

$$u_i(x_1,...,x_n) \to \max, x_i \in X_i, i \in N. \qquad (1.1)$$

Here $N = \{1,..., n\}$ denotes the set of players (active agents); X_i is the set of admissible actions of player *i*; x_i is a particular action chosen by player *i*; finally, $u_i : X \to R$ is the payoff function of player *i*. Players from the set N simultaneously and independently choose their actions x_i, resulting in a game outcome $x = (x_1,...,x_n) \in X = X_1 \times ... \times X_n$. Players can be of different nature. In economics, they are individual entrepreneurs, households, firms, regions, or countries. In politics, they are individual voters, political parties, movements and associations, and executive and legislative bodies. In organizational management, they are individual employees, departments and structural units, and entire organizations. What is important, the interests of each player are completely described by maximization of the payoff u_i (the postulate of economic rationality).

The solution of the game (1.1) is the set of Nash equilibria:

$$\mathrm{NE} = \{x^{\mathrm{NE}} \in X : \forall i \in N$$
$$\forall x_i \in X_i \; u_i(x^{\mathrm{NE}}) \ge u_i(x_i, x_{-i}^{\mathrm{NE}})\},$$
$$x_{-i} = (x_1,...,x_{i-1}, x_{i+1},...,x_n). \qquad (1.2)$$

In cooperation, players unite to maximize the total payoff (the utilitarian social welfare function) [26]

$$u(x) = \sum_{i \in N} u_i(x). \qquad (1.3)$$

The cooperative solution is given by

$$x^C \in X : u^C = u(x^C) = \max_{x \in X} u(x). \qquad (1.4)$$

Under hierarchical control, a dedicated player with subscript 0 (the Principal in the theory of active systems) is added to the set N. In the case of no feedback, the Principal chooses its action $x_0 \in X_0$ and informs the other players of it. Knowing the action x_0, the players choose their optimal responses. Anticipating this behavior, the Principal chooses the action $x_0 \in X_0$ to maximize its payoff on the set of optimal responses. Two cases are possible here.

- If the Principal expects the benevolence of the other agents, its payoff is given by

$$u_0^B = \sup_{x_0 \in X_0} \sup_{x \in R(x_0)} u_0(x_0, x), \qquad (1.5)$$

where $R(x_0)$ denotes the set of agents' optimal responses to the Principal's action x_0. How is this set determined for interconnected agents? The question has no obvious answer. According to a standard assumption, $R(x_0) = \mathrm{NE}(x_0)$ provided that $\forall x_0 \; \mathrm{NE}(x_0) \ne \varnothing$; otherwise, the set $R(x_0)$ should be defined separately for a particular model. The well-known monograph [13] gave the following definition of Stackelberg equilibrium for finite three-player games. (It can be easily generalized to the case $n > 3$.)

Consider a three-player game with one Leader (the first player) and two Followers (the second and third players). For player $i = 1, 2, 3$, we introduce the following notations: x_i and X_i are the strategy and the set of all admissible strategies of player *i*, respectively; $J_i(x_1, x_2, x_3)$ is the payoff of player *i* in the outcome (x_1, x_2, x_3). Then x_1^* is a hierarchical equilibrium strategy of the Leader if

$$\min_{(x_2, x_3) \in R(x_1^*)} J_1(x_1^*, x_2, x_3) = \max_{x_1 \in X_1} \min_{(x_2, x_3) \in R(x_1)} J_1(x_1, x_2, x_3),$$

where $R(x_1)$ is the set of optimal responses of the Followers. For each strategy $x_1 \in X_1$ of the Leader, this set is given by

$$R(x_1) = \{(y_2, y_3) \in X_2 \times X_3 : J_2(x_1, y_2, y_3) \ge$$
$$J_2(x_1, x_2, x_3) \wedge J_3(x_1, y_2, y_3) \ge J_3(x_1, x_2, x_3),$$
$$\forall x_2 \in X_2, x_3 \in X_3\}.$$

Any triplet $(x_1^*, x_2^*, x_3^*), (x_2^*, x_3^*) \in R(x_1^*)$, is a Stackelberg equilibrium [13, pp. 145 and 146].

- Otherwise (under the conscious or involuntary malevolence of the agents), the Principal's payoff is given by

$$u_0^{\mathrm{NB}} = \sup_{x_0 \in X_0} \inf_{x \in R(x_0)} u_0(x_0, x). \qquad (1.6)$$

Let ST denote the set of all Stackelberg equilibria (solutions of the hierarchical game between the Principal and the agents).

Remark 1. Many researchers suppose that Stackelberg equilibrium is defined by formula (1.5) only, referring formula (1.6) to Germeier's principle of guaranteed result. This is not true: in the widely known monograph [13], Stackelberg equilibrium was defined by formula (1.6).

Remark 2. In many applied models, there exists a unique optimal response of agents (e.g., a Nash equilibrium). In this case, the problem of agent's benevolence or malevolence does not arise.

In hierarchical control with feedback, the Principal chooses its strategy $\tilde{x}_0 \in \tilde{X}_0 = X_0^{\,X}$, i.e., $\tilde{x} : X \to X_0$, and informs the other players of it. Then the game has an information structure similar to the previous case with a natural modification. Knowing the strategy $\tilde{x}_0$, the players choose the optimal response. Anticipating this behavior, the Principal chooses the strategy $\tilde{x}_0 \in \tilde{X}_0$ to maximize its payoff on the set of optimal responses. If the Principal expects the benevolence of the agents, its payoff is given by

$$\tilde{u}_0^B = \sup_{\tilde{x}_0 \in \tilde{X}_0} \sup_{x \in R(\tilde{x}_0)} u_0(\tilde{x}_0(x), x). \quad (1.7)$$

Otherwise (under the conscious or involuntary malevolence of the agents), the Principal's payoff is given by

$$\tilde{u}_0^{\text{NB}} = \sup_{\tilde{x}_0 \in \tilde{X}_0} \inf_{x \in R(\tilde{x}_0)} u_0(\tilde{x}_0(x), x). \quad (1.8)$$

Let IST denote the set of solutions of the hierarchical game between the Principal and the agents under control with feedback.

Hierarchical control involves administrative methods (compulsion) or economic methods (impulsion) [9, 10]. We formalize these concepts in the case of control without feedback for unfavorable agents. The hierarchical game has the form

$$u_0(p, q, x) \to \max, \ p \in P, \ q \in Q; \quad (1.9)$$

$$u_i(p_i, x) \to \max, \ x_i \in X_i(q_i), \ i \in N. \quad (1.10)$$

Here $p = (p_1, ..., p_n)$ is the vector of the Principal's economic controls applied to the agents' payoff functions; $q = (q_1, ..., q_n)$ is the vector of the Principal's administrative controls applied to the sets of agents' admissible actions.

The set of compulsion equilibria in the game (1.9), (1.10) is the set of outcomes $\text{COMP} = \{(x_0^{\text{COMP}}, x^{\text{COMP}}):\ u_0(x_0^{\text{COMP}}, x^{\text{COMP}}) = u_0^{\text{COMP}}\}$, where

$$u_0^{\text{COMP}} = \sup_{q \in Q} \inf_{x \in R(q)} u_0(q, x) \quad (1.11)$$

with a fixed value p.

The set of impulsion equilibria in the game (1.9), (1.10) is the set of outcomes $\text{IMP} = \{(x_0^{\text{IMP}}, x^{\text{IMP}}): u_0(x_0^{\text{IMP}}, x^{\text{IMP}}) = u_0^{\text{IMP}}\}$, where

$$u_0^{\text{IMP}} = \sup_{p \in P} \inf_{x \in R(p)} u_0(p, x) \quad (1.12)$$

with a fixed value q. The case of control with feedback is formalized by analogy.

Remark 3. Other information structures are known, e.g., the Germeier game Γ_3 [7]. Therefore, the proposed classification does not claim to be exhaustive: it covers the main organizational modes of active agents.

Games in characteristic function form (cooperative games) [11, 12] are a reasonable framework to describe the allocation of the cooperative payoff (1.4). A characteristic function is a mapping $v: 2^N \to R$, and its value $v(K)$ gives the payoff of a coalition $K \subseteq N$. The most common example is the von Neumann–Morgenstern characteristic function [27]:

$$v^{\text{NM}}(K) = \text{val}(K, N \setminus K) = \sup_{x_i, i \in K} \inf_{x_j, j \in N \setminus K} \sum_{i \in K} u_i(x_1, ..., x_n). \quad (1.13)$$

Also, the Petrosyan–Zaccour [28]

$$v^{\text{PZ}}(K) = \sup_{x_i, i \in K} \sum_{i \in K} u_i(x_K, x_{N \setminus K}^{\text{NE}}) \quad (1.14)$$

and the Gromova–Petrosyan [29]

$$v^{\text{PG}}(K) = \inf_{x_j, j \in N \setminus K} \sum_{i \in K} u_i(x_K^C, x_{N \setminus K}) \quad (1.15)$$

characteristic functions were proposed with the following notations: x_K is the set of strategies of all players from a coalition K, and $x_{N \setminus K}$ is the set of strategies of all players from the anti-coalition $N \setminus K$. (The superscripts NE and C indicate Nash equilibrium and the cooperative solution, respectively.) Note that for all characteristic functions (1.13)–(1.15),

$$v^{\text{NM}}(N) = v^{\text{PZ}}(N) = v^{\text{PG}}(N) = \sup_{x_1, ..., x_n} \sum_{i \in N} u_i(x_1, ..., x_n) = u^C.$$

In other words, the payoff of the maximal coalition always coincides with the cooperative payoff (1.4). The Shapley value [30] is a convenient solution of cooperative games: it always exists and is unique. The components of the Shapley value are given by

$$\Phi_i(v) = \sum_{i \in K} \gamma_n(k)[v(K) - v(K \setminus \{i\})], \ i \in N,$$

$$\gamma_n(k) = \frac{(n-k)!(k-1)!}{n!}, \ k = |K|, \ n = |N|. \quad (1.16)$$

The player's share in the cooperative payoff allocation based on the Shapley value shows his contribution to all his coalitions considering their power.

1.2. Systems of Preferences and Relative Efficiency Indices

The efficiency of different organizational modes of active agents, control methods, and cooperative payoff allocation should be compared from two points of view: the society and individual agents. For the system of social preferences, indicators are the total payoffs (1.3). For the sake of convenience, assume that $N = \{0, 1, ..., n\}$, and player 0 does not differ from other players in the cases of equality and cooperation.

The social payoffs are as follows:

– under equality,

$$u^{\text{NE}} = \min_{x \in \text{NE}} u(x); \quad (1.17)$$

– under cooperation,

$$u^C = \max_{x \in X} u(x); \quad (1.18)$$

– under hierarchical control without feedback,

$$u^{\mathrm{ST}} = \sum_{i \in N} u_i(x^{\mathrm{ST}}); \qquad (1.19)$$

– under hierarchical control with feedback,

$$u^{\mathrm{IST}} = \sum_{i \in N} u_i(x^{\mathrm{IST}}). \qquad (1.20)$$

Formula (1.5) or (1.6) can be used to calculate the social payoff (1.19) and formula (1.7) or (1.8) to calculate the payoff (1.20), depending on the agent's benevolence (malevolence) assumption. Also, we can take the set COMP (1.11), the set IMP (1.12), or their analogs (1.7), (1.8) under control with feedback instead of IST as the solution of the hierarchical game instead of ST. As a result, we obtain the social payoffs u^{COMP}, u^{IMP}, u^{ICOMP}, and u^{IIMP}, respectively.

According to definition (1.4), the social payoff *under cooperation is always not smaller than under any* other organizational mode or control method. To assess the losses (the inefficiency of equilibria), we introduce the social efficiency indices

$$K^{\mathrm{NE}} = \frac{u^{\mathrm{NE}}}{u^C}, K^{\mathrm{ST}} = \frac{u^{\mathrm{ST}}}{u^C}, K^{\mathrm{IST}} = \frac{u^{\mathrm{IST}}}{u^C}, \ K^{\mathrm{COMP}} = \frac{u^{\mathrm{COMP}}}{u^C},$$

$$K^{\mathrm{IMP}} = \frac{u^{\mathrm{IMP}}}{u^C}, \ K^{\mathrm{ICOMP}} = \frac{u^{\mathrm{ICOMP}}}{u^C},$$

$$K^{\mathrm{IIMP}} = \frac{u^{\mathrm{IIMP}}}{u^C}. \qquad (1.21)$$

Remark 4. The indices (1.21) assume that all payoffs are positive; in this case, all these fractions do not exceed 1. This standard assumption in the theory of (in)efficiency of equilibria [17, *p. 444*] restricts the universality of the proposed approach. In fact, the initial values of the payoffs can be used for comparative efficiency analysis. The indices (1.21) serve for additional quantitative characterization when necessary.

For the system of individual preferences of agents $i \in N$, we select the following indicators:

- under equality,

$$u_i^{\mathrm{NE}} = \min_{x \in \mathrm{NE}} u_i(x) \qquad (1.22)$$

(the player's payoff in the worst-case Nash equilibrium, the principle of optimality for this organizational mode);

- under cooperation,

$$u_i^C = \frac{u^C}{|N|}, \qquad (1.23)$$

or the Shapley value $\Phi_i(v)$ for the characteristic function (1.13), (1.14), or (1.15);

- under hierarchical control without feedback, u_i^{ST};
- under hierarchical control with feedback, u_i^{IST}.

The sets ST and IST can be replaced by their analogs COMP and ICOMP (compulsion) IMP and IIMP (impulsion). For a detailed quantitative comparative assessment, we propose the individual efficiency indices

$$K_i^{\mathrm{NE}} = \frac{u_i^{\mathrm{NE}}}{u_i^C}, \ K_i^{\mathrm{ST}} = \frac{u_i^{\mathrm{ST}}}{u_i^C}, \ K_i^{\mathrm{IST}} = \frac{u_i^{\mathrm{IST}}}{u_i^C},$$

$$K_i^{\mathrm{NM}} = \frac{\Phi_i^{\mathrm{NM}}}{u_i^C}, \ K_i^{\mathrm{PZ}} = \frac{\Phi_i^{\mathrm{PZ}}}{u_i^C} \ \ K_i^{\mathrm{PG}} = \frac{\Phi_i^{\mathrm{PG}}}{u_i^C},$$

$$K_i^{\mathrm{COMP}} = \frac{u_i^{\mathrm{COMP}}}{u_i^C}, \ K_i^{\mathrm{IMP}} = \frac{u_i^{\mathrm{IMP}}}{u_i^C}, \ K_i^{\mathrm{ICOMP}} = \frac{u_i^{\mathrm{ICOMP}}}{u_i^C},$$

$$K_i^{\mathrm{IIMP}} = \frac{u_i^{\mathrm{IIMP}}}{u_i^C}, \ i \in N. \qquad (1.24)$$

The social and individual efficiency indicators and the corresponding indices are combined in Table 1.

Table 1

Social and individual efficiency: indicators and indices

	Equality	Cooperation	Hierarchical control without feedback	Hierarchical control with feedback
Social efficiency indicators	u^{NE}	$u^C = v(N)$	u^{ST}, u^{COMP}, u^{IMP}	u^{IST}, u^{ICOMP}, u^{IIMP}
Individual efficiency indicators, $i \in N$	u_i^{NE}	u_i^C, Φ_i^{NM}, Φ_i^{PZ}, Φ_i^{PG}	u_i^{ST}, u_i^{COMP}, u_i^{IMP}	u_i^{IST}, u_i^{ICOMP}, u_i^{IIMP}
Social efficiency indices	$K^{\mathrm{NE}} = \frac{u^{\mathrm{NE}}}{u^C}$	–	$K^{\mathrm{ST}} = \frac{u^{\mathrm{ST}}}{u^C}$	$K^{\mathrm{IST}} = \frac{u^{\mathrm{IST}}}{u^C}$
Individual efficiency indices, $i \in N$	$K_i^{\mathrm{NE}} = \frac{u_i^{\mathrm{NE}}}{u_i^C}$	$K_i^{\mathrm{NM}} = \frac{\Phi_i^{\mathrm{NM}}}{u_i^C}$, $K_i^{\mathrm{PZ}} = \frac{\Phi_i^{\mathrm{PZ}}}{u_i^C}$, $K_i^{\mathrm{PG}} = \frac{\Phi_i^{\mathrm{PG}}}{u_i^C}$	$K_i^{\mathrm{ST}} = \frac{u_i^{\mathrm{ST}}}{u_i^C}$, $K_i^{\mathrm{COMP}} = \frac{u_i^{\mathrm{COMP}}}{u_i^C}$, $K_i^{\mathrm{IMP}} = \frac{u_i^{\mathrm{IMP}}}{u_i^C}$	$K_i^{\mathrm{IST}} = \frac{u_i^{\mathrm{IST}}}{u_i^C}$, $K_i^{\mathrm{ICOMP}} = \frac{u_i^{\mathrm{ICOMP}}}{u_i^C}$, $K_i^{\mathrm{IIMP}} = \frac{u_i^{\mathrm{IIMP}}}{u_i^C}$

The conditions of coordination (sustainable development) have the form

$$u \in U^*, \qquad (1.25)$$

where U^* is a given set. These conditions can supplement any model considered.

2. AN APPROACH TO COMPARE EFFICIENCY

A comparative analysis of the efficiency of organizational modes of active agents, control methods, and allocation of the cooperative payoff includes the following steps.

1. The set of active agents (players) $N = \{0, 1, ..., n\}$ is introduced.

2. Under equality, agent 0 does not differ from the others. The normal-form game (1.1) is constructed, and the set of Nash equilibria (1.2) is found. Finally, the indicators (1.17) and (1.22) are calculated.

3. Under cooperation, agent 0 does not differ from the others as well. The optimization problem (1.4) is solved, and the indicators (1.18) and (1.23) are calculated.

4. Under hierarchical control, agent 0 acts as the Principal. (Any of the initially equal players can claim this role.) For the information structure with control without feedback, the payoffs (1.5) and (1.6) are calculated, and the corresponding sets ST are found. Finally, the indicators (1.19) and u_i^{ST} are calculated.

5. For the information structure with control without feedback, the payoffs (1.7) and (1.8) are calculated, and the corresponding sets IST are found. Finally, the indicators (1.20) and u_i^{IST} are calculated.

6. For the compulsion mode in the hierarchical game (1.9), (1.10), the set COMP (1.11) is found. Then the analogs of the indicators (1.19) and u_i^{ST} are calculated.

7. For the impulsion mode in the hierarchical game (1.9), (1.10), the set IMP (1.12) is found. Then the analogs of the indicators (1.20) and u_i^{IST} are calculated.

8. The games in characteristic function form (1.13)–(1.15) are constructed based on the normal-form game (1.1). Then the Shapley value (1.16) is calculated for these games.

9. The games in characteristic function form (1.13)–(1.15) are constructed based on the hierarchical game without or with feedback. In this case, three types of coalitions are possible: the agents only; the Principal only; the Principal and at least one agent (including the maximal coalition). Finally, the Shapley value (1.16) is calculated for the constructed cooperative games.

10. The additional constraints (1.25) are considered.

11. The system of social preferences is built by ordering the indicators (1.17), (1.19), and (1.20) and the values $u^{\text{COMP}}, u^{\text{IMP}}, u^{\text{ICOMP}}$, and u^{IIMP}. In addition, the losses due to the inefficiency of equilibria are assessed using the indices (1.21).

12. The system of private preferences is built by ordering the indicators (1.22) and (1.23). The comparative efficiency is assessed using the indices (1.24).

Remark 5. In most cases, game-theoretic problems of conflict control can be solved only numerically. Then, the comparison involves the average values of all indicators over the set of computational experiments for different input datasets.

Remark 6. Some steps may be omitted depending on the problem statement and research capabilities.

3. COURNOT OLIGOPOLY MODELS

As an illustrative example, we compare the efficiency of several Cournot oligopoly models.

Example 1. The Cournot oligopoly with symmetric agents.

Let $N = \{1, ..., n\}$ be the set of equal symmetric agents (firms). For fixed costs including tax, the model has the form

$$u_i(x) = (1-p)[(D-\bar{x})x_i - cx_i] \to \max,$$
$$0 \le x_i \le 1/n, \ i \in N.$$

Here x_i denotes the production output of firm i; D is the demand volume; c is the specific costs of each firm; the parameter $p \in [0, 1]$ specifies a fixed tax rate; finally, $\bar{x} = x_1 + ... + x_n$. For the sake of definiteness, assume that $D = 1, \ c = 1/n$. (In the paper [31], this parametrization was used for $n = 2$.) Then

$$u_i(x) = (1-p)\left(\frac{n-1}{n} - \bar{x}\right)x_i \to \max, \qquad (3.1)$$
$$0 \le x_i \le 1/n, \ i \in N.$$

In addition, we denote $\bar{u}(x) = u_1(x) + ... + u_n(x)$. The first-order optimality conditions yield

$$\frac{\partial u_i}{\partial x_i} = 0 = \frac{n-1}{n} - 2x_i - \sum_{j \ne i} x_j, \ i \in N;$$

$$2x_i + \sum_{j \ne i} x_j = \frac{n-1}{n}, \ i \in N.$$

Hence, in the Nash equilibrium,

$$x_i^{\text{NE}} = x^{NE} = \frac{n-1}{n(n+1)}; \ \bar{x}^{\text{NE}} = \frac{n-1}{n+1};$$
$$u_i^{\text{NE}} = u^{\text{NE}} = \frac{(1-p)(n-1)^2}{n^2(n+1)^2}; \ \bar{u}^{\text{NE}} = \frac{(1-p)(n-1)^2}{n(n+1)^2}. \qquad (3.2)$$

Now let the agents from the set N cooperate. The model takes the form

$$\bar{u}(x) = (1-p)\left(\frac{n-1}{n} - \bar{x}\right)\bar{x} \to \max,$$
$$0 \le x_i \le 1/n,\ i \in N.$$

Obviously, in this case, $\forall i \in N\ x_i = x$ and

$$\bar{u}(x) = (1-p)\left(\frac{n-1}{n} - nx\right)nx .$$

The first-order optimality condition reduces to

$$\frac{\partial \bar{u}}{\partial x} = 0 = n-1-2nx .$$

Hence, the solution of the cooperative optimization problem and the corresponding payoffs are:

$$x_i^C = x^C = \frac{n-1}{2n^2};\ \bar{x}^C = \frac{n-1}{2n};$$
$$u_i^C = u^C = \frac{(1-p)(n-1)^2}{4n^3};\ \bar{u}^C = \frac{(1-p)(n-1)^2}{4n^2}.$$

Suppose that agent 1 becomes the Principal: first chooses x_1 and informs the other agents of it. Then, each agent solves the problem

$$u_i = (1-p)\left(\frac{n-1}{n} - x_1 - \sum_{j=2}^{n} x_j\right)x_i \to \max,$$
$$0 \le x_i \le 1/n,\ \ i = 2,\dots, n.$$

The first-order optimality conditions

$$\frac{\partial u_i}{\partial x_i} = 0 = \frac{n-1}{n} - x_1 - 2x_i - \sum_{j=2, j\ne i}^{n} x_j\ ,\ i = 2,\dots, n\ ,$$

yield the best response of each agent:

$$x_i^{\mathrm{BR}} = x^{\mathrm{BR}} = \frac{n-1-nx_1}{n^2},\ i = 2,\dots,n .$$

The Principal's problem takes the form

$$u_1(x_1) = (1-p)\left(\frac{n-1}{n} - x_1 - \frac{(n-1)(n-1-nx_1)}{n^2}\right)x_1 =$$
$$\frac{1-p}{n^2}(n-1-nx_1)x_1 \to \max,\ 0 \le x_1 \le 1/n.$$

The first-order optimality conditions

$$\frac{\partial u_1}{\partial x_1} = 0 = n-1-2nx_1$$

lead to the Stackelberg equilibrium

$$x_1^{\mathrm{ST}} = \frac{n-1}{2n};\ x_i^{\mathrm{ST}} = \frac{n-1}{2n^2},\ i = 2,\dots, n,$$

the total output $\bar{x}^{\mathrm{ST}} = \dfrac{(n-1)(2n-1)}{2n^2}$ and the payoffs

$$u_1^{\mathrm{ST}} = \frac{(1-p)(n-1)^2}{4n^3};$$
$$u_i^{\mathrm{ST}} = \frac{(1-p)(n-1)^2}{4n^4}, i = 2,\dots,n ;$$
$$\bar{u}^{\mathrm{ST}} = \frac{(1-p)(n-1)(n^2-n+1)}{4n^4}.$$

Now consider the following case: the Principal is an additional non-production agent 0 that assigns the tax rate p. Then the Principal's problem can be written as

$$u_0 = \left(\frac{n-1}{n} - \bar{x}\right)\bar{x}p - ap^2 \to \max,\ 0 \le p \le 1\ , \qquad (3.3)$$

where $a > 0$ is the coefficient of tax collection costs.

The optimal response of the agents is the Nash equilibrium in their game; see formula (3.2). The Principal's problem takes the form

$$u_0 = \frac{(n-1)^2}{n(n+1)^2}\, p - ap^2 \to \max,\ 0 \le p \le 1 . \qquad (3.4)$$

As a result, $p^{\mathrm{ST}} = \dfrac{(n-1)^2}{2an(n+1)^2}$, and the Stackelberg (impulsion) equilibrium is

$$\mathrm{ST{=}IMP} = \left(\frac{(n-1)^2}{2an(n+1)^2}, \frac{n-1}{n(n+1)}, \dots, \frac{n-1}{n(n+1)}\right).$$

The payoffs of the Principal and agents are given by

$$u_0^{\mathrm{IMP}} = \frac{(n-1)^2[(n-1)^2 - an(n+1)^2]}{2an^2(n+1)^2};$$
$$u_i^{\mathrm{IMP}} = \frac{(n-1)^2[2an(n+1)^2 - (n-1)^2]}{2an^2(n+1)^4},\ i = 1,\dots, n\ ;$$
$$\bar{u}^{\mathrm{IMP}} = \frac{(n-1)^2[2an(n+1)^2 - (n-1)^2]}{2an(n+1)^4}.$$

Next, we impose an environmental constraint of the form

$$d\bar{x} \le P_{\max}\ , \qquad (3.5)$$

where the coefficient d characterizes the ratio of the volume of pollutant emissions to the total output, and $P_{\max}$ is the maximum permissible limit of emissions. This constraint expresses a sustainable development condition to be ensured by the Principal (an additional constraint in the optimization problem (3.4)). In the Nash equilibrium, this condition reduces to

$$\frac{n-1}{n+1} \le \frac{P_{\max}}{d}\ ; \qquad (3.6)$$

under cooperation, to

$$\frac{n-1}{2n} \le \frac{P_{\max}}{d}\ . \qquad (3.7)$$

If inequalities (3.6) or (3.7) hold, the equal or cooperative behavior of agents is compatible with the sustainable development conditions.

Otherwise, the Principal can use the impulsion mechanism to stimulate sustainable development among equal agents:

$$\tilde{p}(x) = \begin{cases} p^+ \text{ for } x \le \dfrac{P_{\max}}{dn}, \\ p^- \text{ otherwise.} \end{cases}$$

To examine the allocation of the cooperative payoff, we first construct the von Neumann–Morgenstern characteristic function (1.13). Obviously, $x_i = 1/n,\ i \in N \setminus K,$ for any coalition K. Therefore,

$$v^{\rm NM}(i) = (1-p)\max_{x_i}\left(\frac{n-1}{n} - \frac{n-1}{n} - x_i\right)x_i = 0,\ i \in N\ ;$$

$$v^{\rm NM}(K) = (1-p)\max_{x_i, i\in K}\left(\frac{n-1}{n} - \frac{n-k}{n} - \bar{x}_K\right)\bar{x}_K,\ k = |K|.$$

Since $\forall i \in K\ x_i = x$, we have $v^{\rm NM}(K) = k(1-p)\times$ $\times \max_x \left(\frac{k-1}{n} - kx\right)x$.

The first-order optimality condition $\frac{k-1}{n} - 2kx = 0$ yields $x^* = \frac{k-1}{2kn}, \bar{x}^* = \frac{k-1}{2n}$, and consequently,

$$v^{\rm NM}(K) = k(1-p)\left(\frac{k-1}{n} - \frac{k(k-1)}{2kn}\right)\frac{k-1}{2kn} = (1-p)\frac{(k-1)^2}{4n^2}.$$

Hence,

$$v^{\rm NM}(N) = \frac{(1-p)(n-1)^2}{4n^2} = \bar{u}^C.$$

By definition, the Petrosyan–Zaccour characteristic function (1.14) here coincides with the von Neumann–Morgenstern function. We construct the Gromova–Petrosyan characteristic function (1.15):

$$v^{\rm GP}(i) = (1-p)\left(\frac{n-1}{n} - \frac{n-1}{n} - \frac{n-1}{2n^2}\right)\frac{n-1}{2n^2} = -\frac{(1-p)(n-1)^2}{4n^4},\ i \in N;$$

$$v^{\rm GP}(K) = (1-p)\left(\frac{n-1}{n} - \frac{n-k}{n} - \frac{k(n-1)}{2n^2}\right)\frac{k(n-1)}{2n^2} = (1-p)\frac{k(n-1)(kn-2n+k)}{4n^4};$$

$$v^{\rm GP}(N) = \frac{(1-p)(n-1)^2}{4n^2} = \bar{u}^C.$$

Due to the symmetry of cooperative games, for all characteristic functions,

$$\Phi^{\rm NM} = \Phi^{\rm PZ} = \Phi^{\rm GP} = \left(\frac{(1-p)(n-1)^2}{4n^3}, \ldots, \frac{(1-p)(n-1)^2}{4n^3}\right).$$

Obviously, in models with symmetric agents, social and private preferences coincide under equality and cooperation; the Shapley value-based allocations of the cooperative payoff are always the same for all characteristic functions.

Note that in this model, $u^C = u_1^{\rm ST} > u_i^{\rm ST}, i = 2, \ldots, n$. Thus, cooperation is more beneficial than hierarchical control for all agents except the Principal (who does not care). In this case,

$$\bar{u}^C - \bar{u}^{\rm ST} = \frac{(1-p)(n-1)}{4n^2}\left(n - 1 - \frac{n^2-n+1}{n^2}\right) = \frac{(1-p)(n-1)}{4n^4}(n^2(n-2)+n-1) > 0,$$

i.e., cooperation is more beneficial than hierarchical control for the society. The indices

$$K^{\rm NE} = \frac{4n}{(n+1)^2} \underset{n\to\infty}{\to} 0,\ K^{\rm ST} = \frac{n^2-n+1}{n^2(n-1)} \underset{n\to\infty}{\to} 0$$

show that the benefit of cooperation compared to equality and hierarchical control will grow with the number of agents.

Example 2. The Cournot duopoly with asymmetric agents.

The model has the form

$$u_i = (1-p)(1-c_i-x_1-x_2)x_i \to \max,$$
$$0 \le x_i \le 1/2,\ i = 1, 2.$$

Compared to formula (3.1), $c_i \in (0, 1/2)$ is the costs of firm *i*. If the agents behave equally, the Nash equilibrium is found from the system of equations

$$\frac{\partial u_i}{\partial x_i} = 0, i = 1,2.$$

As is easily verified,

$$x_i^{\rm NE} = (1 + c_j - 2c_i)/3,\ i,\ j = 1, 2;$$

$$\bar{u}^{\rm NE} = (2 - c_1 - c_2)/3, \qquad (3.8)$$

and the payoffs are:

$$u_i^{\rm NE} = (1-p)(1+c_j-2c_i)^2/9, i = 1,2;$$

$$\bar{u}^{\rm NE} = (1-p)(2-2c_1-2c_2+5c_1^2+5c_2^2-8c_1c_2)/9.$$

The cooperation of agents leads to the optimization problem

$$\bar{u} = (1-p)[(1-\bar{x})\bar{x} - c_1x_1 - c_2x_2] \to \max,$$
$$0 \le x_i \le 1/2,\ i = 1, 2.$$

The system of equations $\partial\bar{u}/\partial x_i = 0,\ i = 1, 2,$ reduces to

$$\begin{cases} x_1 + x_2 = (1-c_1)/2, \\ x_1 + x_2 = (1-c_2)/2. \end{cases}$$

Under the assumption $c_1 \ne c_2$, it has no solution. On the boundary of the set of admissible controls, the total payoff function takes the following values:

$$\bar{u}(0, 0) = 0;\ \bar{u}(1/2, 1/2) = -(1-p)(c_1+c_2) < 0;$$
$$\bar{u}(1/2, 0) = (1-p)(1/4 - c_1/2);$$
$$\bar{u}(0, 1/2) = (1-p)(1/4 - c_2/2).$$

Thus, the solution to the cooperative problem and the corresponding payoffs are given by

$$x^C = \begin{cases} (1/2, 0),\ c_1 < c_2 \Rightarrow \bar{u}^C = (1-p)(1-2c_1)/4, \\ \quad u_i^C = (1-p)(1-2c_1)/8,\ i = 1, 2; \\ (0, 1/2),\ c_1 > c_2 \Rightarrow \bar{u}^C = (1-p)(1-2c_2)/4, \\ \quad u_i^C = (1-p)(1-2c_2)/8,\ i = 1, 2. \end{cases}$$

In both cases, the total cooperative output is $\bar{x}^C = 1/2$.

Suppose that agent 1 becomes the Principal: first chooses x_1 and informs the other agents of it. Similarly to Example 1, we arrive at the Stackelberg equilibrium

$$x_1^{\rm ST} = (1-2c_1+c_2)/2;\ x_2^{\rm ST} = (1+2c_1-3c_2)/4.$$

In this case, the total output and payoffs are:

$$\bar{x}^{\text{ST}} = (3-2c_1-c_2)/4\,;$$

$$u_1^{\text{ST}} = (1-p)(1-2c_1+c_2)/8\,;$$

$$u_2^{\text{ST}} = (1-p)(1+2c_1-3c_2)/16;$$

$$\bar{u}^{\text{ST}} = (1-p)(3-4c_1-2c_2+12c_1^2-20c_1c_2+11c_2^2)/16.$$

Now consider the following case: the Principal is an additional non-production agent 0 that assigns the tax rate p. By analogy with (3.3), the Principal's problem can be written as

$$u_0 = (1-c_1-c_2-\bar{x})\bar{x}p - ap^2 \to \max,\ 0 \le p \le 1.$$

The optimal response of agents to the Principal's strategy p is the Nash equilibrium in their game (3.8). Similarly to Example 1, we obtain the Principal's impulsion strategy

$$p^{\text{IMP}} = \frac{(1-2c_1-2c_2)(2-c_1-c_2)}{18a}$$

and the impulsion equilibrium $\text{IMP} = (p^{\text{IMP}},\ x_1^{\text{NE}},\ x_2^{\text{NE}})$.

Finally, we introduce the environmental constraint (sustainable development condition) (3.5). In the Nash equilibrium, this condition takes the form

$$d(2-c_1-c_2) \le 3P_{\max}\,;$$

under cooperation,

$$d \le 2P_{max}\,.$$

For $n = 2$, constructing the game in characteristic function form is unreasonable.

CONCLUSIONS

An obvious main result of this paper consists in the following. In deterministic models, cooperation is not worse for the society than any other organizational mode of the interaction of active agents because it leads to a nonnegative cooperative effect. The collective losses due to rejecting cooperation can be assessed using various indices (the classical problem of the inefficiency of equilibria).

However, individual agents may benefit more from seizing the leadership or keeping independence. The rules for allocating the cooperative payoff among agents are not obvious as well. Therefore, along with the social preferences, it is necessary to consider the private ones (generally, unequal for different agents). Here, comparative efficiency indices can be also used.

Even in simple models, it is not easy to calculate the payoffs of individual agents and the society and compare them analytically. This paper has illustrated the proposed comparative analysis methodology in the case of symmetric agents. Further research will focus on a numerical study of the comparative efficiency of organizational modes, control methods, and allocations of the cooperative payoff for several static and dynamic Cournot oligopoly models.

REFERENCES

1. Burkov, V.N. and Novikov, D.A., *Teoriya aktivnykh sistem: sostoyanie i perspektivy* (Theory of Active Systems: State and Prospects), Moscow: Sinteg, 1999. (In Russian.)
2. Novikov, D.A., *Theory of Control in Organizations*, New York: Nova Science Publishers, 2013.
3. Germeier, Yu.B., *Vvedenie v teoriyu issledovaniya operatsii* (Introduction to Operations Research), Moscow: Nauka, 1971. (In Russian.)
4. Germeier, Yu.B., *Non-Antagonistic Games*, Dordrecht: Springer, 1986.
5. Moiseev, N.N., *Elementy teorii optimal'nykh sistem* (Elements of the Theory of Optimal Systems), Moscow: Nauka, 1975. (In Russian.)
6. Kukushkin, N.S. and Morozov, V.V., *Teoriya neantagonisticheskikh igr* (Theory of Non-Antagonistic Games), Moscow: Moscow State University, 1984. (In Russian.)
7. Gorelik, V.A., Gorelov, M.A., and Kononenko, A.F., *Analiz konfliktnykh situatsii v sistemakh upravleniya* (Analysis of Conflict Situations in Control Systems), Moscow: Radio i Svyaz', 1991. (In Russian.)
8. Laffont, J.-J. and Martimort, D., *The Theory of Incentives: The Principal-Agent Model*, Princeton University Press, 2002.
9. Ougolnitsky, G.A., *Upravlenie ustoichivym razvitiem aktivnykh sistem* (Sustainable Management of Active Systems), Rostov-on-Don: Southern Federal University, 2016. (In Russian.)
10. Ougolnitsky, G.A., Methodology and Applied Problems of the Sustainable Management in Active Systems, *Control Sciences*, 2019, no. 2, pp. 19–29.
11. Mazalov, V.V., *Mathematical Game Theory and Applications*, Wiley, 2014.
12. Petrosyan, L.A., Zenkevich, N.A., and Shevkoplyas, E.V., *Teoriya igr* (Game Theory), St. Petersburg: BKhV- Peterburg, 2011. (In Russian.)
13. Basar, T. and Olsder, G.Y., *Dynamic Non-Cooperative Game Theory*, SIAM, 1999.
14. Dockner, E., Jorgensen, S., Long, N.V., and Sorger, G., *Differential Games in Economics and Management Science*, Cambridge: Cambridge University Press, 2000.
15. Gorelov, M.A. and Kononenko, A.F., Dynamic Models of Conflicts. III. Hierarchical Games, *Automation and Remote Control*, 2015, vol. 76, no. 2, pp. 264–277.
16. Ougolnitsky, G.A. and Usov, A.B., Computer Simulations as a Solution Method for Differential Games, in *Computer Simulations: Advances in Research and Applications*, Pfeffer, M.D. and Bachmaier, E., Eds., New York: Nova Science Publishers, 2018.
17. *Algorithmic Game Theory*, Nisan, N., Roughgarden, T., Tardos, E., and Vazirany, V., Eds., Cambridge: Cambridge University Press, 2007.
18. Dubey, P., Inefficiency of Nash Equilibria, *Math. Operations Research*, 1986, no. 11(1), pp. 1–8.
19. Johari, R. and Tsitsiklis, J.N., Efficiency Loss in a Network Resource Allocation Game, *Math. Operations Research*, 2004, no. 29(3), pp. 407–435.
20. Roughgarden, T., *Selfish Routing and the Price of Anarchy*, MIT Press, 2005.
21. Papadimitriou, C.H., Algorithms, Games, and the Internet, *Proc. 33rd Symp. Theory of Computing*, 2001.
22. Basar, T. and Zhu, Q., Prices of Anarchy, Information, and Cooperation in Differential Games, *Dynamic Games and Applications*, 2011, no. 1(1), pp. 50–73.

23. Dahmouni, I., Vardar, B., and Zaccour, G., A Fair and Time-Consistent Sharing of the Joint Exploitation Payoff of a Fishery, *Natural Resource Modeling*, 2019, vol. 32, article no. e12216.
24. Zhang, W., Zhao, S., and Wan, X., Industrial Digital Transformation Strategies Based on Differential Games, *Applied Mathematical Modeling*, 2021, vol.98, pp. 90–108.
25. Sharma, A. and Jain, D., Game-Theoretic Analysis of Green Supply Chain under Cost-Sharing Contract with Fairness Concerns, *International Game Theory Review*, 2021, vol. 23, no.2, pp. 1–32.
26. Moulin, H., Axioms of Cooperative Decision Making, Cambridge: Cambridge University Press, 1988.
27. von Neumann, J. and Morgenstern, O., *Theory of Games and Economic Behavior*, Princeton: Princeton University Press, 1953.
28. Petrosyan, L. and Zaccour, G., Time-Consistent Shapley Value Allocation of Pollution Cost Reduction, *Journal of Economic Dynamics and Control*, 2003, vol. 27, no. 3, pp. 381–398.
29. Gromova, E.V. and Petrosyan, L.A., On an Approach to Constructing a Characteristic Function in Cooperative Differential Games, *Automation and Remote Control*, 2017, vol. 78, pp. 1680–1692.
30. Shapley, L., A Value for *n*-person Games, Santa Monica, CA: The RAND Corporation, 1952.
31. Moulin, H., Game Theory for the Social Sciences (Studies in Game Theory and Mathematical Economics), New York: New York University Press, 1986.

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