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MEAN VALUES: A MULTICRITERIA APPROACH. PART III

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Abstract. A new approach to defining mean values based on the ideas of multicriteria optimization was proposed and developed previously; see the papers [4] and [5]. The distances between the current point and the sample points were treated as components of a vector estimate. The conventional approach to defining mean values involves the scalarization of vector estimates: they are replaced, e.g., by the sums of their squared components. On the contrary, we proceeded from comparing vector estimates by preference. Several types of mean values corresponding to different amounts of information about preferences were considered. The properties of such mean values were investigated, and computational methods for constructing them were given. However, in the case of equally important criteria, the method turns out to be approximate and rather computationally intensive. In this paper, we present an exact and efficient numerical method for constructing a set of mean values of the specified type. The method is illustrated by a computational example.

Keywords: mean values, multicriteria choice problems, preference relations, criteria importance theory.

INTRODUCTION

Mean values are widely used in management, economics, engineering, and other fields of science and practice (for example, see the books [1, 2]). However, according to the preface of [3], there is no possibility of finding some universal formula that would exhaust the concept of a mean value and possess constructive advantages. Therefore, it is topical to suggest a general conceptualization of a mean value with an appropriate concretization for different situations.

This paper is a direct continuation of the previous publications [4, 5], where a new approach to defining mean values as points non-dominated with respect to special preference relations was proposed and developed. Also, methods were presented to construct the sets of such mean values for preference relations corresponding to different types of information about criteria. In particular, a method was proposed for the case of equally important criteria (the information E). Unfortunately, this method is approximate and requires much machine time even for moderate sample sizes. In this paper, we introduce an exact and efficient method for constructing the set of mean values $G^E(X)$.

1. NECESSARY BACKGROUND

For the reader's convenience, we briefly describe the preliminaries from the paper [4] that will be used below.

Consider a set X consisting of $n \geq 2$ real numbers, hereafter called data (or points), that are the results of measuring the intensity of some feature:

$$X = \{x_1, x_2, \ldots, x_n\}. \quad (1)$$

These data are homogeneous in the sense that the measurements were made on the same scale, no less perfect than the interval scale [6, 7]. The sets rearranged in non-decreasing and nonincreasing order have the form

$$X_{\uparrow} = \langle x_{(1)}, x_{(2)}, \ldots, x_{(n)} \rangle \quad \text{and} \quad X_{\downarrow} = \langle x_{[1]}, x_{[2]}, \ldots, x_{[n]} \rangle, \quad (2)$$

respectively, where $x_{(1)} \leq x_{(2)} \leq \ldots \leq x_{(n)}$ and $x_{[1]} \geq x_{[2]} \geq \ldots \geq x_{[n]}$ are obtained from the set (1) using appropriate permutations.

Let x be an arbitrary fixed number, i.e., a point on the real line Re. Its remoteness from a point x_i of the set X can be estimated by the distance $y_i = |x - x_i|$. Then the remoteness of x from the set of all points of the set X is characterized by the vector $y = (y_1, y_2, \ldots, y_n)$ composed of such distances. It can be interpreted as the value of a vector criterion $f(x) = (f_1(x), f_2(x), \ldots, f_n(x))$, where $f_i(x) = |x - x_i|$. The value set Z of this vector criterion is the positive orthant $\mathrm{Re}^n_+ = [0, +\infty)^n$, i.e., the set of all n-dimensional vectors with nonnegative components. The value $f(x) = (f_1(x), f_2(x), \ldots, f_n(x))$ of the vector criterion f is called the vector estimate of the point x. For brevity, we also denote $y = (y_1, y_2, \ldots, y_n)$, where $y_i = f_i(x)$, $i \in N = \{1, 2, \ldots, n\}$.

Let a preference relation (a strict partial order) P^Γ, where Γ means information about the preferences of a decision-maker (DM), be defined on the set Z as follows: if $y'P^\Gamma y''$, then the point $y' = f(x')$ is closer to the value set of the vector criterion $Y = \{y \in Z | \ y = f(x), x \in X\}$ than the point $y'' = f(x'')$. The relation P^Γ induces a relation P_Γ on the real line with a similar meaning: $x'P_\Gamma x'' \Leftrightarrow y'P^\Gamma y''$, where $y' = f(x')$ and $y'' = f(x'')$. Only the points non-dominated with respect to P_Γ can be candidates for those that are closest to X and represent the entire set X. (A point x is non-dominated with respect to P_Γ if there exists no point x' such that $x'P_\Gamma x$.) If the set $G^\Gamma(X)$ of such points is externally stable (for every dominated point x, there exists a non-dominated point x' such that $x'P_\Gamma x$), then all of them are called mean values with respect to P_Γ.

A natural assumption is that the preferences decrease with increasing values of the criteria f_i. (In other words, the criteria have to be minimized.) In the absence of other information about the preferences, they are described on the set Z by the Pareto relation $P^\varnothing$ defined as follows: $yP^\varnothing z \Leftrightarrow (y_i \le z_i,\ i = 1, 2, \ldots, n$, with at least one of the inequalities being strict). The relation $P^\varnothing$ induces the Pareto relation $P_\varnothing$ on Re: $xP_\varnothing x' \Leftrightarrow yP^\varnothing y'$. As it turns out, the mean values with respect to $P_\varnothing$ are all points of the segment with the endpoints $x_{(1)} = \min_{i \in N} x_i$ and $x_{(n)} = \max_{i \in N} x_i$, i.e., $G^\varnothing(X) = \bar{X} = [x_{(1)}, x_{(n)}]$.

Let all criteria have equal importance (the information $\Gamma = E$). Let Π be the set of all permutations $\pi = \langle\pi(1), \pi(2), \ldots, \pi(n)\rangle$ of the set $\{1, 2, \ldots, n\}$. The criteria $f_1, f_2, \ldots, f_n$ are said to be equally important if any vector estimate y is identical by preference (indifferent) to its permutation $\pi(y) = (y_{\pi(1)}, y_{\pi(2)}, \ldots, y_{\pi(n)})$, where $\pi \in \Pi$. The non-strict preference relation R^E on Z is defined as follows:

$$yR^E z \Leftrightarrow [\text{There exist } \pi, \rho \in \Pi \text{ such that } y_{\pi(1)} \le z_{\rho(1)}, y_{\pi(2)} \le z_{\rho(2)}, \ldots, y_{\pi(n)} \le z_{\rho(n)}]. \quad (3)$$

The remoteness of a point x from the set X is estimated using the relation R_E on Re. It is induced by the relation R^E on Re^n, which is defined by each of the two equivalent decision rules:

$$yP^E z \Leftrightarrow [y_{(1)} \le z_{(1)}, y_{(2)} \le z_{(2)}, \ldots, y_{(n)} \le z_{(n)}, \text{ with at least one of the inequalities being strict}], \quad (4)$$

$$yP^E z \Leftrightarrow [y_{[1]} \le z_{[1]}, y_{[2]} \le z_{[2]}, \ldots, y_{[n]} \le z_{[n]}, \text{ with at least one of the inequalities being strict}]. \quad (5)$$

Here, the mean values with respect to P_E, forming the set $G^E(X)$, are the points of the real line that are non-dominated with respect to P_E. The set of such points is externally stable. Since $P_\varnothing \subseteq P_E$, we have $G^E(X) \subseteq G^\varnothing(X)$.

2. A METHOD FOR CONSTRUCTING THE SET $G^E(X)$

This method is based on the two propositions below that characterize the important properties of mean values with respect to P_E.

Proposition 1. *Let all initial points of a set be located in the nodes of a uniform grid. Then the belonging of any grid node to the mean values with respect to* R_E *can be checked by comparing its vector estimate with those of the other grid nodes.*

Proposition 2. *Let the points of the set X be located in the nodes of a uniform grid with a step of* $h = 2\xi$. *Then either the entire interval* $(k\xi, k\xi + \xi)$, *where k is an integer, belongs to the set of mean values* $G^E(X)$ *or no point of this interval belongs to* $G^E(X)$.

Proposition 1 was established in [4]. The proof of Proposition 2 is provided in the Appendix.

First of all, we emphasize that a uniform grid satisfying the conditions of Propositions 1 and 2 can always be constructed if all points in the set X are rational numbers. In applications, a typical case is when these numbers are integers or finite decimal fractions. Since the set $G^\varnothing(X)$ is externally stable and $P_\varnothing \subseteq P_E$, it suffices to consider the grid only on the segment $\bar{X} = [x_{(1)}, x_{(n)}]$. Indeed, let $uP_E x$ for the point $x \in \bar{X}$ under consideration and some point $u \in \mathrm{Re} \setminus \bar{X}$. Due to $G^\varnothing(X) = \bar{X}$ and the external stability of this set, there exists a point $x^* \in \bar{X}$ such that $x^* P_\varnothing u$. Then we have $x^* P_E u$ and, by transitivity, $x^* P_E x$.

According to Proposition 2, the set $G^E(X)$ is the union of the intervals between the grid nodes consisting of all non-dominated points with respect to P_E and the non-dominated nodes of this grid (the limits of these intervals).

The method is as follows. On the segment $\bar{X}$, a grid is constructed with a step of $\frac{1}{2}\xi = \frac{1}{4}h$:

$$\{x_{(1)}, (x_{(1)} + \tfrac{1}{4}h), (x_{(1)} + \tfrac{1}{2}h), \ldots, (x_{(n)} - \tfrac{1}{4}h), x_{(n)}\}. \quad (6)$$

Its non-dominated nodes—some points of the grid (6)—are selected by their pairwise comparisons with respect to P_E using any of the decision rules (4) or (5). According to Proposition 1, they belong to the set $G^E(X)$.

Next, the intervals of length $\xi = \frac{1}{2}h$ with limits located in grid nodes are considered:

$$(x_{(1)}, x_{(1)} + \tfrac{1}{2} h), (x_{(1)} + \tfrac{1}{2} h, x_{(1)} + h), \ldots, (x_{(n)} - \tfrac{1}{2} h, x_{(n)});$$

their midpoints are grid nodes of the form

$$x_{(1)} + \tfrac{1}{4} h, x_{(1)} + \tfrac{3}{4} h, x_{(1)} + h, \ldots, x_{(n)} - \tfrac{1}{4} h, x_{(n)}.$$

Among the intervals described above, those are selected whose midpoints are non-dominated with respect to P_E.

Finally, the selected intervals are united, and their non-dominated limits (the grid nodes with a step of ½ h) are also added.

The proposed algorithmic method for constructing the set $G^E(X)$ is exact and efficiently implementable. The next section provides a computational example to illustrate this method.

3. AN EXAMPLE OF CONSTRUCTING THE SET $G^E(X)$

Let X = {1, 2, 5, 9, 11}. Since all numbers in the set X are natural, we use a grid with a step of 0.25 covering the segment $\bar{X}$ = [1, 11].

According to the pairwise comparisons with respect to P_E, the following points located in the nodes of this grid are non-dominated with respect to P_E:

$$\begin{gathered}1.5, 1.75, 2, 2.25, 2.5, 2.75, 3, 3.25, 3.5,\\ 3.75, 4, 4.25, 4.5, 4.75, 5, 5.25, 5.5, 5.75,\\ 6, 6.25, 6.5, 6.75, 7, 7.25, 8.75, 9, \text{and } 9.25.\end{gathered} \qquad (7)$$

Due to Proposition 1, these points belong to the set $G^E(X)$. The other grid nodes are dominated points. For example, point 4.5 dominates points 7.5 and 8.5, and point 2.5 dominates point 9.5.

Since point 1.25 is dominated, the interval (1, 1.5) does not intersect with the set $G^E(X)$ by Proposition 2. Point 1.75 is non-dominated, so $(1.5, 2) \subset G^E(X)$. By analogy, we establish that the intervals

$$\begin{gathered}(2, 2.5), (2.5, 3), (3, 3.5), (3.5, 4),\\ (4, 4.5), (4.5, 5), (5, 5.5), (5.5, 6), (6, 6.5),\\ (6.5, 7), (7, 7.5), (8.5, 9), \text{and } (9, 9{,}5)\end{gathered} \qquad (8)$$

are included in the set $G^E(X)$, and the intervals

$$(7.5, 8), (8, 8.5), (9.5, 10), (10, 10.5), \text{and } (10.5, 11)$$

do not intersect with the set $G^E(X)$.

Considering the results (7) and (8) and the inclusion $(1.5, 2) \subset G^E(X)$, we arrive at $G^E(X) = [1.5, 7.5) \cup (8.5, 9.5)$.

CONCLUSIONS

In this paper, we have proposed an algorithmic method for constructing a set of mean values with equally important criteria. The method is exact and can be easily implemented on a PC. Also, a computational example has been provided.

Thus, for all types of mean values introduced and studied in [4, 5], exact computational methods are now available. They can be effectively used in practical data analysis.

APPENDIX

Proof of Proposition 2. We begin with proving the following result: if an arbitrary point $(k\xi + \varepsilon)$, where k is an integer and $0 < \varepsilon < \xi$, from the interval $(k\xi, k\xi + \xi)$ is dominated with respect to P_E on Re, then any other point $(k\xi + \lambda)$, $0 < \lambda < \xi$, from this interval is dominated as well. There are two possibilities: the point $(k\xi + \varepsilon)$ is dominated by a point $m\xi$ lying on the grid with a step of ξ or by a point $(m\xi + \eta)$ outside the grid, where m is an integer and $0 < \eta < \xi$.

Let $(m\xi)\ \mathrm{P}_{\mathrm{E}}\ (k\xi + \varepsilon)$. Then, according to (3), we have the inequalities

$$f_{\pi(i)}(m\xi) \le f_{\rho(i)}(k\xi + \varepsilon),\ i = 1, 2, ..., n, \qquad \text{(A1)}$$

with at least one inequality being strict. Here, $\pi = \{\pi(1), \pi(2), ..., \pi(n)\}$ and $\rho = \{\rho(1), \rho(2), ..., \rho(n)\}$ are permutations of the component numbers $i = 1, 2, ..., n$ in the vectors $f_{\uparrow}(m\xi)$ and $f_{\uparrow}(k\xi + \varepsilon)$, respectively. Since $f_{\pi(i)}(m\xi)$ are the distances between grid points, they are multiples of ξ, whereas $f_{\rho(i)}(k\xi + \varepsilon)$ are not. Therefore, all inequalities (A1) are strict, and for each $i = 1, 2, ..., n$,

$$f_{\pi(i)}(m\xi) \le f_{\rho(i)}(k\xi) = f_{\rho(i)}(k\xi + \varepsilon) - \varepsilon \quad \text{if } x_{\rho(i)} \le k\xi < k\xi + \varepsilon,$$

$$f_{\pi(i)}(m\xi) \le f_{\rho(i)}(k\xi + \xi) = f_{\rho(i)}(k\xi + \varepsilon) - (\xi - \varepsilon) \quad \text{if } k\xi + \varepsilon < k\xi + \xi \le x_{\rho(i)}.$$

However, for $x_{\rho(i)} \le k\xi$, we obtain $x_{\rho(i)} < k\xi + \lambda$ and $f_{\rho(i)}(k\xi + \lambda) = f_{\rho(i)}(k\xi) + \lambda > f_{\pi(i)}(m\xi)$; for $k\xi + \xi \le x_{\rho(i)}$, we have $k\xi + \lambda < x_{\rho(i)}$ and $f_{\rho(i)}(k\xi + \lambda) = f_{\rho(i)}(k\xi + \xi) + (\xi - \lambda) > f_{\pi(i)}(m\xi)$. Hence,

$$f_{\pi(i)}(m\xi) < f_{\rho(i)}(k\xi + \lambda), \quad i = 1, ..., n,$$

i.e., the point $(k\xi + \lambda)$ turns out to be dominated by the same point $m\xi$.

Now, let $(m\xi + \eta) P_E\ (k\xi + \varepsilon)$. According to the relation (3), we arrive at the inequalities

$$f_{\pi(i)}(m\xi + \eta) \le f_{\rho(i)}(k\xi + \varepsilon), \quad i = 1, 2, ..., n, \qquad \text{(A2)}$$

with at least one inequality being strict. Here, $\pi = \{\pi(1), \pi(2), ..., \pi(n)\}$ is a permutation of the component numbers $i = 1, 2, ..., n$ in the vector $f_{\uparrow}(m\xi + \eta)$.

The same interval $(m\xi, m\xi + \xi)$ contains a point $(m\xi + \theta)$, $0 < \theta < \xi$, dominating the point $(k\xi + \lambda)$. To demonstrate this fact, we recall that the points of the set X are located in nodes of a coarse grid with a step of $h = 2\xi$. For definiteness, let these nodes correspond to the nodes of a fine grid with a step of ξ and even numbers. If k is even, the left limit of the interval ($k\xi$, $k\xi + \xi$) will adjoin to the node $k\xi$ of the coarse grid; if k is odd, the right limit will adjoin to the node ($k\xi + \xi$) of the coarse grid. There are four combinations of parity for the numbers k and m.

1. Let k and m be even. Consider the ith inequality in (A2). There are four possible arrangements of the points $x_{\pi(i)}$ and $x_{\rho(i)}$ with respect to the intervals.

1.1. $x_{\pi(i)} \le m\xi$, $x_{\rho(i)} \le k\xi$. Then

$$f_{\pi(i)}(m\xi + \eta) = f_{\pi(i)}(m\xi) + \eta,\ \ f_{\pi(i)}(m\xi + \theta) = f_{\pi(i)}(m\xi) + \theta,$$
$$f_{\rho(i)}(k\xi + \varepsilon) = f_{\rho(i)}(k\xi) + \varepsilon,\ \ f_{\rho(i)}(k\xi + \lambda) = f_{\rho(i)}(k\xi) + \lambda.$$

From (A2) it follows that $(f_{\pi(i)}(m\xi) < f_{\rho(i)}(k\xi)) \vee (f_{\pi(i)}(m\xi) = f_{\rho(i)}(k\xi) \wedge \eta \le \varepsilon)$.

If $f_{\pi(i)}(m\xi) < f_{\rho(i)}(k\xi)$, we have $f_{\pi(i)}(m\xi + \theta) < f_{\rho(i)}(k\xi + \lambda)$ $\forall\theta\in(0, \xi)$.

If $f_{\pi(i)}(m\xi) = f_{\rho(i)}(k\xi)$, we have $f_{\pi(i)}(m\xi + \theta) \le f_{\rho(i)}(k\xi + \lambda)$ $\forall\theta\in(0, \lambda]$.

Moreover, if the ith inequality in (A2) holds as equality, then $\eta = \varepsilon$ and θ can also be chosen equal to λ.

1.2. $x_{\pi(i)} \le m\xi$, $x_{\rho(i)} > k\xi$. Then

$$f_{\pi(i)}(m\xi + \eta) = f_{\pi(i)}(m\xi) + \eta,\ \ f_{\pi(i)}(m\xi + \theta) = f_{\pi(i)}(m\xi) + \theta,$$
$$f_{\rho(i)}(k\xi + \varepsilon) = f_{\rho(i)}(k\xi + 2\xi) + 2\xi - \varepsilon,$$
$$f_{\rho(i)}(k\xi + \lambda) = f_{\rho(i)}(k\xi + 2\xi) + 2\xi - \lambda.$$

From (A2) it follows that $f_{\pi(i)}(m\xi) < f_{\rho(i)}(k\xi + 2\xi)$. As a result, we obtain $f_{\pi(i)}(m\xi + \theta) < f_{\rho(i)}(k\xi + \lambda)$ $\forall\theta\in(0, \xi)$.

1.3. $x_{\pi(i)} > m\xi$, $x_{\rho(i)} \le k\xi$. Then

$$f_{\pi(i)}(m\xi + \eta) = f_{\pi(i)}(m\xi + 2\xi) + 2\xi - \eta,$$
$$f_{\pi(i)}(m\xi + \theta) = f_{\pi(i)}(m\xi + 2\xi) + 2\xi - \theta,$$
$$f_{\rho(i)}(k\xi + \varepsilon) = f_{\rho(i)}(k\xi) + \varepsilon,\ \ f_{\rho(i)}(k\xi + \lambda) = f_{\rho(i)}(k\xi) + \lambda.$$

From (A2) it follows that $f_{\pi(i)}(m\xi + 2\xi) < f_{\rho(i)}(k\xi)$. As a result, we obtain $f_{\pi(i)}(m\xi + \theta) < f_{\rho(i)}(k\xi + \lambda)$ $\forall\theta\in(0, \xi)$.

1.4. $x_{\pi(i)} > m\xi$, $x_{\rho(i)} > k\xi$. Then

$$f_{\pi(i)}(m\xi + \eta) = f_{\pi(i)}(m\xi + 2\xi) + 2\xi - \eta,$$
$$f_{\pi(i)}(m\xi + \theta) = f_{\pi(i)}(m\xi + 2\xi) + 2\xi - \theta,$$
$$f_{\rho(i)}(k\xi + \varepsilon) = f_{\rho(i)}(k\xi + 2\xi) + 2\xi - \varepsilon,$$
$$f_{\rho(i)}(k\xi + \lambda) = f_{\rho(i)}(k\xi + 2\xi) + 2\xi - \lambda.$$

From (A2) it follows that $(f_{\pi(i)}(m\xi + 2\xi) < f_{\rho(i)}(k\xi + 2\xi)) \vee (f_{\pi(i)}(m\xi + 2\xi) = f_{\rho(i)}(k\xi + 2\xi) \wedge \eta \ge \varepsilon)$.

If $f_{\pi(i)}(m\xi + 2\xi) < f_{\rho(i)}(k\xi + 2\xi)$, we have $f_{\pi(i)}(m\xi + \theta) < f_{\rho(i)}(k\xi + \lambda)$ $\forall\theta\in(0, \xi)$.

If $f_{\pi(i)}(m\xi + 2\xi) = f_{\rho(i)}(k\xi + 2\xi)$, we have $f_{\pi(i)}(m\xi + \theta) \le f_{\rho(i)}(k\xi + \lambda)$ $\forall\theta\in[\lambda, \xi)$.

Moreover, if the ith inequality in (A2) holds as equality, then $\eta = \varepsilon$ and θ can also be chosen equal to λ.

If cases 1.1 and 1.4 occur in different inequalities of (A2), simultaneously implying $\eta \le \varepsilon$ and $\eta \ge \varepsilon$, then obviously $\eta = \varepsilon$ and $\theta = \lambda$. As a result, we obtain

$$f_{\pi(i)}(m\xi + \theta) \le f_{\rho(i)}(k\xi + \lambda),\quad i = 1, 2, ..., n; \qquad \text{(A3)}$$

if (A2) has a strict inequality, the corresponding inequality in (A3) will be strict as well. Thus, given even numbers k and m, there exists θ such that $(m\xi + \theta)\mathrm{P_E}\,(k\xi + \lambda)$.

2. Let k be even and m be odd. Consider the ith inequality in (A2). There are four possible arrangements of the points $x_{\pi(i)}$ and $x_{\rho(i)}$ with respect to the intervals.

2.1. $x_{\pi(i)} \le m\xi$, $x_{\rho(i)} \le k\xi$. Then

$$f_{\pi(i)}(m\xi + \eta) = f_{\pi(i)}(m\xi - \xi) + \xi + \eta,$$
$$f_{\pi(i)}(m\xi + \theta) = f_{\pi(i)}(m\xi - \xi) + \xi + \theta,$$
$$f_{\rho(i)}(k\xi + \varepsilon) = f_{\rho(i)}(k\xi) + \varepsilon,\ \ f_{\rho(i)}(k\xi + \lambda) = f_{\rho(i)}(k\xi) + \lambda.$$

From (A2) it follows that $f_{\pi(i)}(m\xi - \xi) < f_{\rho(i)}(k\xi)$. As a result, we obtain $f_{\pi(i)}(m\xi + \theta) < f_{\rho(i)}(k\xi + \lambda)$ $\forall\theta\in(0, \xi)$.

2.2. $x_{\pi(i)} \le m\xi$, $x_{\rho(i)} > k\xi$. Then

$$f_{\pi(i)}(m\xi + \eta) = f_{\pi(i)}(m\xi - \xi) + \xi + \eta,$$
$$f_{\pi(i)}(m\xi + \theta) = f_{\pi(i)}(m\xi - \xi) + \xi + \theta,$$
$$f_{\rho(i)}(k\xi + \varepsilon) = f_{\rho(i)}(k\xi + 2\xi) + 2\xi - \varepsilon,$$
$$f_{\rho(i)}(k\xi + \lambda) = f_{\rho(i)}(k\xi + 2\xi) + 2\xi - \lambda.$$

From (A2) it follows that $(f_{\pi(i)}(m\xi - \xi) < f_{\rho(i)}(k\xi + 2\xi)) \vee (f_{\pi(i)}(m\xi - \xi) = f_{\rho(i)}(k\xi + 2\xi) \wedge \eta \le (\xi - \varepsilon))$.

If $f_{\pi(i)}(m\xi - \xi) < f_{\rho(i)}(k\xi + 2\xi)$, we have $f_{\pi(i)}(m\xi + \theta) < f_{\rho(i)}(k\xi + \lambda)$ $\forall\theta\in(0, \xi)$.

If $f_{\pi(i)}(m\xi - \xi) = f_{\rho(i)}(k\xi + 2\xi)$, we have $f_{\pi(i)}(m\xi + \theta) \le f_{\rho(i)}(k\xi + \lambda)$ $\forall\theta\in(0, (\xi - \lambda)]$.

Moreover, if the ith inequality in (A2) holds as equality, then $\eta = (\xi - \varepsilon)$ and θ can also be chosen equal to $(\xi - \lambda)$.

2.3. $x_{\pi(i)} > m\xi$, $x_{\rho(i)} \le k\xi$. Then

$$f_{\pi(i)}(m\xi + \eta) = f_{\pi(i)}(m\xi + \xi) + \xi - \eta,$$
$$f_{\pi(i)}(m\xi + \theta) = f_{\pi(i)}(m\xi + \xi) + \xi - \theta,$$
$$f_{\rho(i)}(k\xi + \varepsilon) = f_{\rho(i)}(k\xi) + \varepsilon,\ \ f_{\rho(i)}(k\xi + \lambda) = f_{\rho(i)}(k\xi) + \lambda.$$

From (A2) it follows that $(f_{\pi(i)}(m\xi + \xi) < f_{\rho(i)}(k\xi)) \vee (f_{\pi(i)}(m\xi + \xi) = f_{\rho(i)}(k\xi) \wedge \eta \ge (\xi - \varepsilon))$.

If $f_{\pi(i)}(m\xi + \xi) < f_{\rho(i)}(k\xi)$, we have $f_{\pi(i)}(m\xi + \theta) < f_{\rho(i)}(k\xi + \lambda)$ $\forall\theta\in(0, \xi)$.

If $f_{\pi(i)}(m\xi + \xi) = f_{\rho(i)}(k\xi)$, we have $f_{\pi(i)}(m\xi + \theta) \le f_{\rho(i)}(k\xi + \lambda)$ $\forall\theta\in[(\xi - \lambda), \xi]$.

Moreover, if the ith inequality in (A2) holds as equality, then $\eta = (\xi - \varepsilon)$ and θ can also be chosen equal to $(\xi - \lambda)$.

2.4. $x_{\pi(i)} > m\xi$, $x_{\rho(i)} > k\xi$. Then

$$f_{\pi(i)}(m\xi + \eta) = f_{\pi(i)}(m\xi + \xi) + \xi - \eta,$$
$$f_{\pi(i)}(m\xi + \theta) = f_{\pi(i)}(m\xi + \xi) + \xi - \theta,$$
$$f_{\rho(i)}(k\xi + \varepsilon) = f_{\rho(i)}(k\xi + 2\xi) + 2\xi - \varepsilon,$$
$$f_{\rho(i)}(k\xi + \lambda) = f_{\rho(i)}(k\xi + 2\xi) + 2\xi - \lambda.$$

From (A2) it follows that $f_{\pi(i)}(m\xi + \xi) < f_{\rho(i)}(k\xi + 2\xi)$. As a result, we obtain $f_{\pi(i)}(m\xi + \theta) < f_{\rho(i)}(k\xi + \lambda)$ $\forall\theta\in(0, \xi)$.

If cases 2.2 and 2.3 occur in different inequalities of (A2), simultaneously implying $\eta \le (\xi - \varepsilon)$ and $\eta \ge (\xi - \varepsilon)$, then obviously $\eta = (\xi - \varepsilon)$ and $\theta = (\xi - \lambda)$. As a result, we obtain inequalities (A3); if (A2) has a strict inequality, the corresponding inequality in (A3) will be strict as well. Thus, given an even number k and an odd number m, there exists θ such that $(m\xi + \theta)P_E(k\xi + \lambda)$.

For the other combinations (odd k and odd m; odd k and even m), the considerations are similar. Well, we have established the first part of Proposition 2: if an arbitrary point from the interval $(k\xi, k\xi + \xi)$ is dominated with respect to P_E on Re, then the same property holds for any other point from this interval.

Finally, we show that if an arbitrary point $(k\xi + \varepsilon)$, where k is an integer and $0 < \varepsilon < \xi$, from the interval $(k\xi, k\xi + \xi)$ is non-dominated with respect to P_E on Re, then the same property holds for any other point $(k\xi + \lambda)$, $0 < \lambda < \xi$, from this interval. Assume on the contrary that a point $(k\xi + \lambda)$ is dominated with respect to P_E on Re. In this case, due to the arguments above, the point $(k\xi + \varepsilon)$ is dominated with respect to P_E. This contradiction completes the proof of the second part of Proposition 2.

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