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A NUMERICAL AGGREGATION METHOD FOR FINITE-STATE MACHINES USING ALGEBRAIC OPERATIONS

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Abstract. This paper considers the problem of synthesizing finite-state machines (FSMs) based on algebraic methods. The aggregation operations of FSMs are numerically implemented using symbolic matrices that describe their functioning. An algebra is defined for these matrices as follows: the carriers are matrix elements and special symbols, and the signature includes two operations serving to determine actions over these symbols. As a result, it becomes possible to define an algebra of symbolic matrices whose signature includes three operations. The classical operations over FSMs are represented in matrix form based on the algebra of symbolic matrices. Next, special operations over FSMs are constructed involving classical operations over them. Special operations are constructed considering the constraints and requirements of the subject area. A numerical example of FSM synthesis—the joint activity of two functional groups in an emergency zone—is provided.

Keywords: synthesis of automata, algebra of automata, symbolic matrices.

INTRODUCTION

The theory of finite-state machines (FSMs, also termed finite automata in the literature) is an effective apparatus for modeling the functioning of objects and systems [1]. Presently, an urgent problem is to describe the functioning of several interacting objects or systems of different nature. This problem can be solved by aggregating the corresponding FSMs [2–4]. Therefore, it is necessary to introduce operations on the set of FSMs [2, 3, 5] in order to aggregate FSMs under the constraints and requirements of the subject area. Among them, we mention the following possibilities:

1) the functioning of the general FSM with synchronous state changing if the aggregated FSMs simultaneously change the states;

2) the functioning of the general model with asynchronous state change if the state change instants do not coincide;

3) the initiation of state change of one FSM model by the output action of another FSM model;

4) the elimination of certain combinations of states from the general model since the objects or systems modeled cannot simultaneously be in these states.

An example of a subject area with such constraints is the process of emergency response, which involves joint actions of several functional groups [6, 7].

Aggregation operations allow synthesizing FSMs, yielding an algebra [8] of FSMs $\mathcal{A} = \langle \mathcal{M}, \mathcal{S} \rangle$, where $\mathcal{M}$ and $\mathcal{S}$ denote the set of FSMs (the carrier) and the set of operations over them (the signature), respectively.

Synthesis of FSMs is a traditional problem of automata theory. As a rule, parallel and (or) sequential composition operations are used [2–5, 9]. However, when implementing these operations in practice, one should consider the constraints of the subject area.

To solve modeling problems using an algebra $\mathcal{A}$, it is necessary to develop a numerical method for the software implementation of FSM aggregation operations considering additional requirements. For this purpose, the matrix representation of FSMs [10] can be adopted; as a result, the operations over FSMs are reduced to matrix operations.

In the works [11–14], FSM synthesis procedures were described by tables or transition matrices with additional conditions for identifying their elements to ensure the commutativity of aggregation operations. This fact increases the computational complexity of the numerical synthesis method of an FSM.

In this regard, we develop below a numerical method with the sets of input and output symbols of given FSMs as the input and output symbols of the general FSM synthesized from them. In this case, the matrices describing the functioning of FSMs consist of sets. Such matrices are further called symbolic matrices; modern programming languages allow handling their elements.

In addition, the FSM models of objects and systems in many subject areas are synthesized only with the joint use of parallel and sequential composition operations involving the same actions.

The numerical method proposed in this paper excludes duplication when executing aggregation operations considering the requirements of the subject area.

1. THE REPRESENTATION OF FINITE-STATE MACHINES BY SYMBOLIC MATRICES

Let a Mealy FSM $A=(X, Y, Q, \lambda, \mu)$ be given with the following notations: X is an input alphabet; Y is an output alphabet; Q is the set of states; $\lambda: X\times Q\to Q$ is the transition function; finally, $\mu: X\times Q\to Y$ is the output function. The functions λ and μ can be fully characterized by the operator F of the form

$$\left\{Fq^i=\left\{q^{i_1}\left(x^{j_1}/y^{k_1}\right),\ldots, q^{i_l}\left(x^{j_l}/y^{k_l}\right),\ldots,\right.\right.$$
$$\left.\left.q^{i_{n_i}}\left(x^{j_{n_i}}/y^{k_{n_i}}\right)\right\}, i=\overline{1,|Q|}\right\}.$$

The expression $q^{i_l}\left(x^{j_l}/y^{k_l}\right)\in Fq^i$ means that if the FSM is in a state q^i and the input symbol is x^{j_l}, it will pass to a state q^{i_l} with an output symbol y^{k_l}.

The functions λ and μ can be described by two square symbolic matrices

$$R_A^X=\left(r_{ij}^x\right)_{i,j=\overline{1,|Q|}} \text{ and } R_A^Y=\left(r_{ij}^y\right)_{i,j=\overline{1,|Q|}},$$

respectively, where

$$r_{ij}^x=\begin{cases} x \text{ if } q^j(x/y)\in Fq^i \\ \theta \text{ otherwise,}\end{cases}$$

and

$$r_{ij}^y=\begin{cases} y \text{ if } q^j(x/y)\in Fq^i \\ \theta \text{ otherwise.}\end{cases}$$

Accordingly, the operator F can be described by a unique connection matrix $R_A=\left(R_A^X/R_A^Y\right)$ expressed through the matrices R_A^X and R_A^Y as follows:

$$R_A=\begin{pmatrix} x_A^{11}/y_A^{11} & \cdots & x_A^{1|Q|}/y_A^{1|Q|} \\ \vdots & \ddots & \vdots \\ x_A^{|Q|1}/y_A^{|Q|1} & \cdots & x_A^{|Q||Q|}/y_A^{|Q||Q|}\end{pmatrix}. \quad (1)$$

The symbol θ in these matrices characterizes the impossibility of transition from a state q^i to a state q^j if the FSM A is partial. The elements of the matrices R_A^X and R_A^Y are sets, whereas the elements of the matrix R_A are the ordered pairs of sets.

Let us illustrate the matrix representation of FSMs by a simple example.

Consider FSMs $A_1=\left(X_{A_1}, Y_{A_1}, Q_{A_1}, F_{A_1}\right)$ and $A_2=\left(X_{A_2}, Y_{A_2}, Q_{A_2}, F_{A_2}\right)$, where:

· The input alphabets are $X_{A_1}=\left\{x_{A_1}^1, x_{A_1}^2\right\}$ and $X_{A_2}=\left\{x_{A_2}^1, x_{A_2}^2\right\}$.

· The output alphabets are $Y_{A_1}=\left\{y_{A_1}^1, y_{A_1}^2\right\}$ and $Y_{A_2}=\left\{y_{A_2}^1, y_{A_2}^2\right\}$.

· The sets of states are $Q_{A_1}=\left\{q_{A_1}^1, q_{A_1}^2\right\}$ and $Q_{A_2}=\left\{q_{A_2}^1, q_{A_2}^2\right\}$.

· The operators are given by

$$F_{A_1}=\left\{\begin{matrix} F_{A_1}q_{A_1}^1=\left\{q_{A_1}^1\left(x_{A_1}^1/y_{A_1}^1\right), q_{A_1}^2\left(x_{A_1}^2/y_{A_1}^2\right)\right\}, \\ F_{A_1}q_{A_1}^2=\varnothing\end{matrix}\right\}$$

$$\text{and } F_{A_2}=\left\{\begin{matrix} F_{A_2}q_{A_2}^1=\left\{q_{A_2}^2\left(x_{A_2}^1/y_{A_2}^1\right)\right\}, \\ F_{A_2}q_{A_2}^2=\left\{q_{A_2}^2\left(x_{A_2}^2/y_{A_2}^2\right)\right\}\end{matrix}\right\}.$$

The matrices of input and output connections have the form

$$R_{A_1}^X=\begin{pmatrix} x_{A_1}^1 & x_{A_1}^2 \\ \theta & \theta\end{pmatrix},\ R_{A_2}^X=\begin{pmatrix} \theta & x_{A_2}^1 \\ \theta & x_{A_2}^2\end{pmatrix},$$

$$R_{A_1}^Y=\begin{pmatrix} y_{A_1}^1 & y_{A_1}^2 \\ \theta & \theta\end{pmatrix},\ R_{A_2}^Y=\begin{pmatrix} \theta & y_{A_2}^1 \\ \theta & y_{A_2}^2\end{pmatrix}.$$

According to (1), the connection matrices of the FSMs A_1 and A_2 are given by

$$R_{A_1}=\begin{pmatrix} x_{A_1}^1/y_{A_1}^1 & x_{A_1}^2/y_{A_1}^2 \\ \theta & \theta \end{pmatrix},\ R_{A_2}=\begin{pmatrix} \theta & x_{A_2}^1/y_{A_2}^1 \\ \theta & x_{A_2}^2/y_{A_2}^2 \end{pmatrix}.$$

To introduce actions over symbolic matrices, it is necessary to define operations over the elements of these matrices (symbols), which are arbitrary sets.

2. ALGEBRAS OF SYMBOLS AND SYMBOLIC MATRICES

We denote by $\mathcal{M}$ the set of square symbolic matrices and by M the set of their elements. Assume that the set M additionally includes a special element ε, an analog of the unit element, which cannot be an input or output symbol of FSMs by definition.

We define an algebra $\mathcal{A}_1=\langle M,\cdot,\vee\rangle$ as follows.

Let $c_1, c_2\in M$; the operations $\cdot$ and $\vee$ are executed according to the following rules:

$$\forall c_1, c_2\notin\{\theta,\varepsilon\}\ c_1\cdot c_2=\{c_1,c_2\};$$
$$\forall c_1\ \theta\cdot c_1=c_1\cdot\theta=\theta;$$
$$\forall c_1, c_2\neq\theta\ \varepsilon\cdot c_1=c_1\cdot\varepsilon=c_1;$$
$$\forall c_1\ \theta\vee c_1=c_1\vee\theta=c_1.$$

The expression $c_1\vee c_2$ means that:

– If one of the input symbols c_1 or c_2 arrives at the FSM input, it is used for transition.

– If both symbols arrive simultaneously, the operation $c_1\cdot c_2$ is executed.

Assume that the set $\mathcal{M}$ contains an identity matrix of dimensions $k\times k$:

$$E_k=\begin{pmatrix} \varepsilon & \cdots & \theta \\ \vdots & \ddots & \vdots \\ \theta & \cdots & \varepsilon \end{pmatrix}.$$

We define an algebra $\mathcal{A}_2=\langle\mathcal{M},\cdot,\times,\cup\rangle$ as follows.

Let $c_1\in M$ and $V,W\in\mathcal{M}$ be given elements such that

$$V=\left(v^{ij}\right)_{i,j=\overline{1,n}} \text{ and } W=\left(w^{ij}\right)_{i,j=\overline{1,m}}.$$

Then the operations $\cdot$, $\vee$, and $\times$ are executed according to the following rules:

$$c_1\cdot V=\begin{pmatrix} c_1\cdot v_{11} & \cdots & c_1\cdot v_{1n} \\ \vdots & \ddots & \vdots \\ c_1\cdot v_{n1} & \cdots & c_1\cdot v_{nn} \end{pmatrix};\qquad(2)$$

$$V\times W=\begin{pmatrix} v_{11}\cdot W & \cdots & v_{1n}\cdot W \\ \vdots & \ddots & \vdots \\ v_{n1}\cdot W & \cdots & v_{nn}\cdot W \end{pmatrix};\qquad(3)$$

if $n=m$, then

$$V\cup W=\begin{pmatrix} v_{11}\vee w_{11} & \cdots & v_{1n}\vee w_{1n} \\ \vdots & \ddots & \vdots \\ v_{n1}\vee w_{n1} & \cdots & v_{nn}\vee w_{nn} \end{pmatrix}.\qquad(4)$$

The identity matrix is necessary to define the sum operation over FSMs, so it has been introduced separately.

The algebra $\mathcal{A}_2$ is convenient for handling input and output connection matrices R_A^X and R_A^Y of FSMs. However, to simplify the description of certain operations over FSMs, we extend the operations $\times$ and $\cup$ of the algebra $\mathcal{A}_2$ to the connection matrices R_{A_1} and R_{A_2}.

Let $*\in\{\times,\cup\}$; then

$$R_{A_1}*R_{A_2}=\left(\left(R_{A_1}^X*R_{A_2}^X\right)/\left(R_{A_1}^Y*R_{A_2}^Y\right)\right).$$

We illustrate the execution of the signature operations $\{\cdot,\times,\cup\}$ on an example of the FSMs A_1 and A_2. Since the operation $\cdot$ is executed when implementing the operation $\times$, we calculate $R_{A_1}^X\times R_{A_2}^X$ and $R_{A_1}^X\cup R_{A_2}^X$ only.

Due to (2) and (3), the matrix of output connections $R_{A_1}^X\times R_{A_2}^X$ has the form

$$R_{A_1}^X\times R_{A_2}^X=\begin{pmatrix} x_{A_1}^1 & x_{A_1}^2 \\ \theta & \theta \end{pmatrix}\times\begin{pmatrix} \theta & x_{A_2}^1 \\ \theta & x_{A_2}^2 \end{pmatrix}$$

$$=\begin{pmatrix} x_{A_1}^1\cdot\begin{pmatrix} \theta & x_{A_2}^1 \\ \theta & x_{A_2}^2 \end{pmatrix} & x_{A_1}^2\cdot\begin{pmatrix} \theta & x_{A_2}^1 \\ \theta & x_{A_2}^2 \end{pmatrix} \\ \theta\cdot\begin{pmatrix} \theta & x_{A_2}^1 \\ \theta & x_{A_2}^2 \end{pmatrix} & \theta\cdot\begin{pmatrix} \theta & x_{A_2}^1 \\ \theta & x_{A_2}^2 \end{pmatrix} \end{pmatrix}$$

$$=\begin{pmatrix} \theta & \{x_{A_1}^1, x_{A_2}^1\} & \theta & \{x_{A_1}^2, x_{A_2}^1\} \\ \theta & \{x_{A_1}^1, x_{A_2}^2\} & \theta & \{x_{A_1}^2, x_{A_2}^2\} \\ \theta & \theta & \theta & \theta \\ \theta & \theta & \theta & \theta \end{pmatrix}.$$

In view of (4), the connection matrix $R_{A_1}^X \cup R_{A_2}^X$ has the form

$$R_{A_1}^X \cup R_{A_2}^X = \begin{pmatrix} x_{A_1}^1 & x_{A_1}^2 \\ \theta & \theta \end{pmatrix} \cup \begin{pmatrix} \theta & x_{A_2}^1 \\ \theta & x_{A_2}^2 \end{pmatrix}$$

$$= \begin{pmatrix} x_{A_1}^1 \vee \theta & x_{A_1}^2 \vee x_{A_2}^1 \\ \theta \vee \theta & \theta \vee x_{A_2}^2 \end{pmatrix} = \begin{pmatrix} x_{A_1}^1 & x_{A_1}^2 \vee x_{A_2}^1 \\ \theta & x_{A_2}^2 \end{pmatrix}.$$

Similarly, calculating $R_{A_1}^Y \times R_{A_2}^Y$ and $R_{A_1}^Y \cup R_{A_2}^Y$ and using (1), we obtain

$$R_{A_1} \times R_{A_2}$$

$$= \begin{pmatrix} \theta & \{x_{A_1}^1, x_{A_2}^1\}/\{y_{A_1}^1, y_{A_2}^1\} & \theta & \{x_{A_1}^2, x_{A_2}^1\}/\{y_{A_1}^2, y_{A_2}^1\} \\ \theta & \{x_{A_1}^1, x_{A_2}^2\}/\{y_{A_1}^1, y_{A_2}^2\} & \theta & \{x_{A_1}^2, x_{A_2}^2\}/\{y_{A_1}^2, y_{A_2}^2\} \\ \theta & \theta & \theta & \theta \\ \theta & \theta & \theta & \theta \end{pmatrix},$$

$$R_{A_1} \cup R_{A_2} = \begin{pmatrix} x_{A_1}^1 / y_{A_1}^1 & x_{A_1}^2 / y_{A_1}^2 \\ \theta & \theta \end{pmatrix}$$

$$\cup \begin{pmatrix} \theta & x_{A_2}^1 / y_{A_2}^1 \\ \theta & x_{A_2}^2 / y_{A_2}^2 \end{pmatrix}$$

$$= \begin{pmatrix} x_{A_1}^1 / y_{A_1}^1 & x_{A_1}^2 / y_{A_1}^2 \vee x_{A_2}^1 / y_{A_2}^1 \\ \theta & x_{A_2}^2 / y_{A_2}^2 \end{pmatrix}.$$

3. THE MATRIX REPRESENTATION OF CLASSICAL OPERATIONS OVER FINITE-STATE MACHINES

Consider Mealy FSMs $A_1 = (X_{A_1}, Y_{A_1}, Q_{A_1}, F_{A_1})$ and $A_2 = (X_{A_2}, Y_{A_2}, Q_{A_2}, F_{A_2})$ and the corresponding connection matrices R_{A_1} and R_{A_2}.

The following operations over FSMs are used to aggregate FSM models. Their matrix description is based on the operations of the algebra $\mathcal{A}_2$ introduced above.

We define the connection matrix R_Π of the FSM $\Pi = A_1 \times A_2$ as follows:

$$R_\Pi = R_{A_1} \times R_{A_2}. \qquad (5)$$

The FSM Π describes a parallel synchronous change in the states of the FSMs A_1 and A_2 since its matrix is composed of the pairs of element sets $\{x_{A_1}^{k_1}, x_{A_2}^{k_2}\}/\{y_{A_1}^{l_1}, y_{A_2}^{l_2}\}$ such that one element of each set belongs to A_1 and the other to A_2. Consequently, each such pair describes a simultaneous change in the states of the FSMs A_1 and A_2 that make up the FSM Π.

We define the connection matrix R_Σ of the FSM $\Sigma = A_1 + A_2$ as follows:

$$R_\Sigma = \left(R_{A_1} \times E_{|Q_{A_2}|}\right) \cup \left(E_{|Q_{A_1}|} \times R_{A_2}\right). \qquad (6)$$

The FSM Σ describes an asynchronous change in the states of the FSMs A_1 and A_2 since its matrix is composed of the elements $x_{A_i}^k / y_{A_i}^l$, $i = 1, 2$: they describe a change in the state of only one component (A_1 or A_2) of the FSM Σ.

The composition $\circ$ of FSMs is used to initiate the functioning of one FSM by means (output symbols) of another FSM. The connection matrix R_K of the FSM $K = A_1 \circ A_2$ is defined as follows:

$$R_{A_1} \circ R_{A_2} = \left(r_{ps}\right)_{p,s=\overline{1,|Q_{A_1}|\cdot|Q_{A_2}|}}, \qquad (7)$$

where $r_{p,s} = \begin{cases} \{x_{A_1}, x_{A_2}\}/\{y_{A_1}, y_{A_2}\} \\ \quad \text{if } y_{A_1} = x_{A_2} \text{ or } y_{A_2} = x_{A_1} \\ \theta \text{ otherwise.} \end{cases}$

The union $\cup$ of FSMs is necessary to describe their functioning in different modes. The connection matrix R_C of the FSM $C = A_1 \cup A_2$ is defined as follows:

$$R_C = R_{A_1} \cup R_{A_2}. \qquad (8)$$

The aggregation of FSM models involves several operations over automata; for example, it is necessary to construct an FSM model that represents both synchronous and asynchronous changes in the states of FSMs. Note that these operations are executed with the same actions (e.g., constructing the set of states). Therefore, it is reasonable to combine classical operations over FSMs into groups to avoid any duplication of these actions.

Consider the operations $+$ and $\circ$ on an example of the FSMs A_1 and A_2 introduced in Section 1. (The operations $\times$ and $\cup$ have been described in Section 2; see formulas (5) and (8).)

Due to (6), the connection matrix describing the FSM $A_1 + A_2$ has the form

$$\left(R_{A_1} \times E_{|Q_{A_2}|}\right) \cup \left(E_{|Q_{A_1}|} \times R_{A_2}\right)$$

$$= \left(\begin{pmatrix} x_{A_1}^1 / y_{A_1}^1 & x_{A_1}^2 / y_{A_1}^2 \\ \theta & \theta \end{pmatrix} \times \begin{pmatrix} \varepsilon & \theta \\ \theta & \varepsilon \end{pmatrix}\right)$$

$$\cup \left(\begin{pmatrix} \varepsilon & \theta \\ \theta & \varepsilon \end{pmatrix} \times \begin{pmatrix} \theta & x_{A_2}^1 / y_{A_2}^1 \\ \theta & x_{A_2}^2 / y_{A_2}^2 \end{pmatrix}\right)$$

$$= \begin{pmatrix} x_{A_1}^1 / y_{A_1}^1 & \theta & x_{A_1}^2 / y_{A_1}^2 & \theta \\ \theta & x_{A_1}^1 / y_{A_1}^1 & \theta & x_{A_1}^2 / y_{A_1}^2 \\ \theta & \theta & \theta & \theta \\ \theta & \theta & \theta & \theta \end{pmatrix}$$

$$\cup \begin{pmatrix} \theta & x_{A_2}^1 / y_{A_2}^1 & \theta & \theta \\ \theta & x_{A_2}^2 / y_{A_2}^2 & \theta & \theta \\ \theta & \theta & \theta & x_{A_2}^1 / y_{A_2}^1 \\ \theta & \theta & \theta & x_{A_2}^2 / y_{A_2}^2 \end{pmatrix}$$

$$= \begin{pmatrix} x_{A_1}^1 / y_{A_1}^1 & x_{A_2}^1 / y_{A_2}^1 & x_{A_1}^2 / y_{A_1}^2 & \theta \\ \theta & x_{A_1}^1 / y_{A_1}^1 \vee x_{A_2}^2 / y_{A_2}^2 & \theta & x_{A_1}^2 / y_{A_1}^2 \\ \theta & \theta & \theta & x_{A_2}^1 / y_{A_2}^1 \\ \theta & \theta & \theta & x_{A_2}^2 / y_{A_2}^2 \end{pmatrix}.$$

For a more visual representation of the connection matrix describing the FSM $A_1 \circ A_2$, we impose an additional condition, i.e., $y_{A_1}^1 = x_{A_2}^1$, $y_{A_1}^2 = x_{A_2}^1$, and $y_{A_1}^1 = x_{A_2}^2$. Then, according to (7), the connection matrix of the FSM $A_1 \circ A_2$ has the form

$$R_{A_1} \circ R_{A_2}$$

$$= \begin{pmatrix} \theta & \left\{x_{A_1}^1, x_{A_2}^1\right\} / \left\{y_{A_1}^1, y_{A_2}^2\right\} & \theta & \left\{x_{A_1}^2, x_{A_2}^1\right\} / \left\{y_{A_1}^2, y_{A_2}^1\right\} \\ \theta & \left\{x_{A_1}^1, x_{A_2}^2\right\} / \left\{y_{A_1}^1, y_{A_2}^2\right\} & \theta & \theta \\ \theta & \theta & \theta & \theta \\ \theta & \theta & \theta & \theta \end{pmatrix}.$$

4. THE MATRIX REPRESENTATION OF COMBINED OPERATIONS OVER FINITE-STATE MACHINES

In applications, under the requirements of the subject area, it is often necessary to use the aggregation operations of FSMs jointly. For example, the following combined operations describe requirements 1–3 (see the Introduction) and serve to model emergency response processes.

We define the operation $\otimes$ over FSMs A_1 and A_2 as follows:

$$\Psi = A_1 \otimes A_2 = \left(A_1 \times A_2\right) \cup \left(A_1 + A_2\right).$$

Its matrix representation has the form

$$R_\Psi = \left(R_{A_1} \times R_{A_2}\right) \cup \left[\left(R_{A_1} \times E_{|Q_{A_2}|}\right) \cup \left(E_{|Q_{A_1}|} \times R_{A_2}\right)\right]. \quad (9)$$

The elements of the matrix R_Ψ are pairs of the form $\left\{x_{A_1}^{k_1}, x_{A_2}^{k_2}\right\} / \left\{y_{A_1}^{l_1}, y_{A_2}^{l_2}\right\}$ or $x_{A_i}^k / y_{A_i}^l$, $i = 1, 2$. In other words, they characterize both synchronous and asynchronous changes in the states of the FSMs A_1 and A_2.

We define the operation $\odot$ over FSMs A_1 and A_2 as follows:

$$\Phi = A_1 \odot A_2 = \left(A_1 \circ A_2\right) \cup \left(A_1 + A_2\right).$$

Its matrix representation has the form

$$R_\Phi = \left(R_{A_1} \circ R_{A_2}\right) \cup \left[\left(R_{A_1} \times E_{|Q_{A_2}|}\right) \cup \left(E_{|Q_{A_1}|} \times R_{A_2}\right)\right]. \quad (10)$$

The elements of the matrix R_Φ are pairs of the form $\left\{x_{A_1}^{k_1}, x_{A_2}^{k_2}\right\} / \left\{y_{A_1}^{l_1}, y_{A_2}^{l_2}\right\}$, where $y_{A_1}^{l_1} = x_{A_2}^{k_2}$ or $y_{A_2}^{l_2} = x_{A_1}^{k_1}$, or of the form $x_{A_i}^k / y_{A_i}^l$, $i = 1, 2$. In other words, they characterize both the initiation of the functioning of one FSM by means (output symbols) of another FSM and an asynchronous change in the states of the FSMs A_1 and A_2.

Let an FSM $A_3 = \left(X_{A_3}, Y_{A_3}, Q_{A_3}, F_{A_3}\right)$ be obtained from FSMs A_1 and A_2 by transformations using the operations $\otimes$ and $\odot$. We denote by Ξ_{A_3} the set of all inadmissible states of the FSM A_3. The filtering operation ∇ of the FSM A_3 over the set Ξ_{A_3}, ∇A_3, is defined as follows:

$$Q_{A_3} = Q_{A_3} \setminus \Xi_{A_3}.$$

To derive the matrix representation of the filtering operation, we write the connection matrix of the FSM A_3:

$$R_{A_3} = \left(r_{ij}\right)_{i,j=\overline{1,|Q_{A_3}|}}.$$

Then the filtering operation has the form

$$R_{\nabla A_3} = M_{i_1,\ldots,i_n}^{i_1,\ldots,i_n}, \tag{11}$$

where $M_{i_1,\ldots,i_n}^{i_1,\ldots,i_n}$ is a submatrix of the matrix R_{A_3}; $i_1,\ldots, i_n$ specify the rows and columns excluded from the matrix R_{A_3}; $i_k = i$ if $q_{A_3}^i \in \Xi_{A_3}$ (i.e., this state belongs to the set of inadmissible states Ξ_{A_3}). The filtering operation can be also used to reduce the sets X_{A_3} and Y_{A_3}: if some input symbols in the set X_{A_3} do not belong to the matrix $R_{\nabla A_3}$ after the filtering operation, they are removed from the set X_{A_3} and the result is the set X_{A_3}. A similar procedure is performed for the set Y_{A_3}, yielding the set Y_{A_3}. The operator F_{A_3} also turns into the operator F_{A_3}, which is defined by the matrix $R_{\nabla A_3}$.

Here is an example of the filtering operation over FSMs A_1 and A_2. According to (9), the matrix describing the functioning of the FSM $A_1 \otimes A_2$ has the form

$$R_{A_1} \otimes R_{A_2} = \begin{pmatrix} \theta & \{x_{A_1}^1, x_{A_2}^1\}/\{y_{A_1}^1, y_{A_2}^1\} & \theta & \{x_{A_1}^2, x_{A_2}^1\}/\{y_{A_1}^2, y_{A_2}^1\} \\ \theta & \{x_{A_1}^1, x_{A_2}^2\}/\{y_{A_1}^1, y_{A_2}^2\} & \theta & \{x_{A_1}^2, x_{A_2}^2\}/\{y_{A_1}^2, y_{A_2}^2\} \\ \theta & \theta & \theta & \theta \\ \theta & \theta & \theta & \theta \end{pmatrix}$$

$$\cup \begin{pmatrix} x_{A_1}^1/y_{A_1}^1 & x_{A_2}^1/y_{A_2}^1 & x_{A_1}^2/y_{A_1}^2 & \theta \\ \theta & x_{A_1}^1/y_{A_1}^1 \vee x_{A_2}^2/y_{A_2}^2 & \theta & x_{A_1}^2/y_{A_1}^2 \\ \theta & \theta & \theta & x_{A_2}^1/y_{A_2}^1 \\ \theta & \theta & \theta & x_{A_2}^2/y_{A_2}^2 \end{pmatrix}$$

$$= \begin{pmatrix} x_{A_1}^1/y_{A_1}^1 & \{x_{A_1}^1, x_{A_2}^1\}/\{y_{A_1}^1, y_{A_2}^2\} \vee x_{A_2}^1/y_{A_2}^1 & x_{A_1}^2/y_{A_1}^2 & \{x_{A_1}^2, x_{A_2}^1\}/\{y_{A_1}^2, y_{A_2}^1\} \\ \theta & \{x_{A_1}^1, x_{A_2}^2\}/\{y_{A_1}^1, y_{A_2}^2\} \vee x_{A_1}^1/y_{A_1}^1 \vee x_{A_2}^2/y_{A_2}^2 & \theta & \{x_{A_1}^2, x_{A_2}^2\}/\{y_{A_1}^2, y_{A_2}^2\} \vee x_{A_1}^2/y_{A_1}^2 \\ \theta & \theta & \theta & x_{A_2}^1/y_{A_2}^1 \\ \theta & \theta & \theta & x_{A_2}^2/y_{A_2}^2 \end{pmatrix}.$$

Then, due to (11), the connection matrix of the FSM $\nabla\left(A_1 \otimes A_2\right)$ is given by

$$\nabla\left(R_{A_1} \otimes R_{A_2}\right) = \begin{pmatrix} x_{A_1}^1/y_{A_1}^1 & \{x_{A_1}^1, x_{A_2}^1\}/\{y_{A_1}^1, y_{A_2}^2\} \vee x_{A_2}^1/y_{A_2}^1 & \{x_{A_1}^2, x_{A_2}^1\}/\{y_{A_1}^2, y_{A_2}^1\} \\ \theta & \{x_{A_1}^1, x_{A_2}^2\}/\{y_{A_1}^1, y_{A_2}^2\} \vee x_{A_1}^1/y_{A_1}^1 \vee x_{A_2}^2/y_{A_2}^2 & x_{A_1}^2/y_{A_1}^2 \\ \theta & \theta & x_{A_2}^2/y_{A_2}^2 \end{pmatrix}.$$

5. NUMERICAL EXAMPLE

As an example, we consider the joint functioning of a police station and a radiation, chemical, and biological monitoring post in an emergency zone. The activity of these units includes the following responsibilities:

- The police station organizes the interaction and general governance of subordinate forces and means; its actions are modeled by an FSM $A_1 = \left(X_{A_1}, Y_{A_1}, Q_{A_1}, F_{A_1} \right)$;
- The radiation, chemical, and biological monitoring post detects the factual contamination of objects and terrain; its actions are modeled by an FSM $A_2 = \left(X_{A_2}, Y_{A_2}, Q_{A_2}, F_{A_2} \right)$.

As an emergency evolves, these units may be in the following states (Table 1).

Table 1

The states of the FSMs A_1 and A_2 (the actions of functional groups)

Notation	Description
$q^1_{A_1}$	Daily activities
$q^2_{A_1}$	Report to the management, refinement of the incoming information about an emergency, and staff preparation for appropriate measures
$q^3_{A_1}$	Control of the units carrying out their activities
$q^4_{A_1}$	Actions in accordance with the current situation
$q^5_{A_1}$	Measures to finalize the emergency
$q^1_{A_2}$	Waiting for the post to be formed
$q^2_{A_2}$	Measurements in the controlled area
$q^3_{A_2}$	Report to the police station, measures to localize the center of contamination and eliminate the consequences of radiation, chemical, and biological effect

The operators F_{A_1} and F_{A_2} are described in Tables 2 and 3, respectively.

Problem. It is required to determine an FSM describing the joint functioning of the police station and the radiation, chemical, and biological monitoring post. The following combinations of states are inadmissible: $q^1_{A_1}$ and $q^2_{A_2}$; $q^1_{A_1}$ and $q^3_{A_2}$; $q^2_{A_1}$ and $q^2_{A_2}$; $q^2_{A_1}$ and $q^3_{A_2}$; $q^3_{A_1}$ and $q^1_{A_2}$; $q^3_{A_1}$ and $q^3_{A_2}$; $q^4_{A_1}$ and $q^1_{A_2}$; $q^5_{A_1}$ and $q^2_{A_2}$; $q^5_{A_1}$ and $q^3_{A_2}$. The FSMs can initiate the functioning of each other as follows: the output symbols of the FSM A_1 coincide with the input symbols of the FSM A_2, i.e., $y^2_{A_1} = x^1_{A_2}$, $y^4_{A_1} = x^4_{A_2}$, and $y^5_{A_1} = x^5_{A_2}$; the output symbol of the FSM A_2 coincides with the input symbol of the FSM A_1, i.e., $y^3_{A_2} = x^3_{A_1}$.

Table 2

The operator F_{A_1}

F_{A_1}	$q^1_{A_1}$	$q^2_{A_1}$	$q^3_{A_1}$	$q^4_{A_1}$	$q^5_{A_1}$
$q^1_{A_1}$	–	$x^1_{A_1} / y^1_{A_1}$	–	–	–
$q^2_{A_1}$	–	–	$x^2_{A_1} / y^2_{A_1}$	–	–
$q^3_{A_1}$	–	–	–	$x^3_{A_1} / y^3_{A_1}$	–
$q^4_{A_1}$	–	–	$x^4_{A_1} / y^4_{A_1}$	–	$x^5_{A_1} / y^5_{A_1}$
$q^5_{A_1}$	$x^6_{A_1} / y^6_{A_1}$	–	–	–	–

Table 3

The operator F_{A_2}

F_{A_2}	$q^1_{A_2}$	$q^2_{A_2}$	$q^3_{A_2}$
$q^1_{A_2}$	–	$x^1_{A_2} / y^1_{A_2}$	–
$q^2_{A_2}$	$x^5_{A_2} / y^5_{A_2}$	–	$x^2_{A_2} / y^2_{A_2} \vee x^3_{A_2} / y^3_{A_2}$
$q^3_{A_2}$	–	$x^4_{A_2} / y^4_{A_2}$	–

Based on the problem statement, we obtain the set $\Xi = \left\{ \left\{ q^1_{A_1}, q^2_{A_2} \right\}, \left\{ q^1_{A_1}, q^3_{A_2} \right\}, \left\{ q^2_{A_1}, q^2_{A_2} \right\}, \left\{ q^2_{A_1}, q^3_{A_2} \right\}, \left\{ q^3_{A_1}, q^1_{A_2} \right\}, \left\{ q^3_{A_1}, q^3_{A_2} \right\}, \left\{ q^4_{A_1}, q^1_{A_2} \right\}, \left\{ q^5_{A_1}, q^2_{A_2} \right\}, \left\{ q^5_{A_1}, q^3_{A_2} \right\} \right\}$. Since $X_{A_1} \cap Y_{A_2} \neq \varnothing$, the operation $\odot$ is used: the desired FSM has the form $A_3 = A_1 \odot A_2$.

The connection matrices of the FSMs A_1 and A_2 are given by

$$R_{A_1} = \begin{pmatrix} \theta & x^1_{A_1} / y^1_{A_1} & \theta & \theta & \theta \\ \theta & \theta & x^2_{A_1} / y^2_{A_1} & \theta & \theta \\ \theta & \theta & \theta & x^3_{A_1} / y^3_{A_1} & \theta \\ \theta & \theta & x^4_{A_1} / y^4_{A_1} & \theta & x^5_{A_1} / y^5_{A_1} \\ x^6_{A_1} / y^6_{A_1} & \theta & \theta & \theta & \theta \end{pmatrix},$$

$$R_{A_2} = \begin{pmatrix} \theta & x^1_{A_2} / y^1_{A_2} & \theta \\ x^5_{A_2} / y^5_{A_2} & \theta & x^2_{A_2} / y^2_{A_2} \vee x^3_{A_2} / y^3_{A_2} \\ \theta & x^4_{A_2} / y^4_{A_2} & \theta \end{pmatrix}.$$

Due to the cumbersome calculations, we will provide only the final result of the operations. After filtering, the set of states has the form

$$Q_{A_3} = \{\{q^1_{A_1}, q^1_{A_2}\}, \{q^2_{A_1}, q^1_{A_2}\}, \{q^3_{A_1}, q^2_{A_2}\},$$
$$\{q^2_{A_1}, q^3_{A_2}\}, \{q^4_{A_1}, q^3_{A_2}\}, \{q^4_{A_1}, q^2_{A_2}\}, \{q^5_{A_1}, q^1_{A_2}\}\}.$$

For the sake of simplification, let us introduce the notations $\{q^i_{A_1}, q^j_{A_2}\} = q^{ij}_{A_3}$, $\{x^i_{A_1}, x^j_{A_2}\} = x^{ij}_{A_3}$, and $\{y^i_{A_1}, y^j_{A_2}\} = y^{ij}_{A_3}$. Then, according to (10) and (11),

$$R_{A_3} = \nabla\left(R_{A_1} \odot R_{A_2}\right)$$
$$= \begin{pmatrix} \theta & x^1_{A_1} & \theta & \theta & \theta & \theta \\ \theta & \theta & x^{21}_{A_3} / y^{21}_{A_3} & \theta & \theta & \theta \\ \theta & \theta & \theta & x^{33}_{A_3} / y^{33}_{A_3} & x^3_{A_1} / y^3_{A_1} & \theta \\ \theta & \theta & x^{44}_{A_3} / y^{44}_{A_3} & \theta & x^4_{A_2} / y^4_{A_2} & \theta \\ \theta & \theta & x^4_{A_1} / y^4_{A_1} & \theta & \theta & x^{55}_{A_3} / y^{55}_{A_3} \\ x^6_{A_1} & \theta & \theta & \theta & \theta & \theta \end{pmatrix}.$$

Consequently, the operator F_{A_3} takes the form presented in Table 4.

Table 4

The operator F_{A_3}

F_{A_3}	$q^{11}_{A_3}$	$q^{21}_{A_3}$	$q^{32}_{A_3}$	$q^{43}_{A_3}$	$q^{42}_{A_3}$	$q^{51}_{A_3}$
$q^{11}_{A_3}$	–	$x^1_{A_1}$	–	–	–	–
$q^{21}_{A_3}$	–	–	$x^{21}_{A_3} / y^{21}_{A_3}$	–	–	–
$q^{32}_{A_3}$	–	–	–	$x^{33}_{A_3} / y^{33}_{A_3}$	$x^3_{A_1} / y^3_{A_1}$	–
$q^{43}_{A_3}$	–	–	$x^{44}_{A_3} / y^{44}_{A_3}$	–	$x^4_{A_2} / y^4_{A_2}$	–
$q^{42}_{A_3}$	–	–	$x^4_{A_1} / y^4_{A_1}$	–	–	$x^{55}_{A_3} / y^{55}_{A_3}$
$q^{51}_{A_3}$	$x^6_{A_1}$	–	–	–	–	–

Thus, the resulting FSM A_3 interprets the joint activity of functional groups involved in emergency response. It includes an asynchronous change in the states of the FSMs A_1 and A_2 and the initiation of the functioning of A_1 by A_2: if the FSM A_2 passes from one state to another when receiving a certain symbol at the input, corresponding to the output symbol that coincides with the input symbol of the FSM A_1, then A_1 makes a state transition. The FSM A_2 is initiated by the FSM A_1 in a similar way.

Note that if the operations $\circ$ and $+$ are used separately, we have to calculate the sets of states for each operation and then filter each of them. The operation $\odot$ allows avoiding duplication when calculating the set of states and when executing the filtering operation ∇.

CONCLUSIONS

This paper has considered the representation of an FSM by a symbolic matrix. This representation method allows reducing operations over FSMs to operations over the corresponding symbolic matrices. A new algebra of symbols included in such matrices has been introduced. The carrier of the algebra has been supplemented with the special symbols θ and ε; due to their properties, the synthesis of FSMs is described correctly. Also, a new algebra of symbolic matrices has been considered to construct operations over FSMs in matrix form.

For a large number of FSM synthesis problems, parallel and sequential composition operations are performed jointly. In view of this fact, a new operation combining the two types of composition has been proposed to avoid certain duplication of actions. Special operations have been introduced to aggregate an FSM model considering possible constraints of the subject area. These operations have been given the matrix representation as well.

A numerical example of FSM aggregation has been provided. The corresponding FSM describes the joint actions of functional groups in an emergency situation.

The matrix representation of operations derived above will be further used in a computational experiment: this representation simplifies the software implementation of the FSM synthesis procedure.

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