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ANALYSIS OF INFORMATION INCONSISTENCY IN BELIEF FUNCTION THEORY. PART II: INTERNAL CONFLICT[1]

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Abstract. Part II of the survey considers the measure of internal conflict in a body of evidence within belief function theory (the Dempster–Shafer theory of evidence). The concepts of non-conflict focal elements in a body of evidence and the basic requirements applied to measures of internal conflict are discussed. Some axiomatics of a measure of internal conflict based on strengthening desirable properties is studied. The general forms of measures of internal conflict that satisfy this system of axioms are presented and analyzed. Different methods for estimating internal conflict are considered: an entropy approach, methods based on auto-conflict calculation and contour function maximization, and metric and decompositional approaches. The decompositional approach assumes that the information source for a body of evidence with great internal conflict could be heterogeneous. This approach is considered in detail. Many illustrative examples are provided.

Keywords: belief function theory, combining rules, inconsistency of bodies of evidence, measure of internal conflict.

INTRODUCTION

This paper is a direct continuation of the publication [1], in which the main methods for analyzing the inconsistency of information between bodies of evidence in belief function theory (the Dempster–Shafer theory of evidence) were considered. However, one body of evidence can also provide inconsistent information. In this case, we speak of an internal conflict. An example of such evidence with a large internal conflict is as follows: the value of the company's stocks tomorrow will be in the range [0, 10] or [30, 35] with equal weights.

Different concepts describing internal conflicts in bodies of evidence (or the corresponding belief functions) were discussed by several authors in the 1980–1990s. Meanwhile, the idea of distinguishing between external and internal conflicts of evidence goes back to [2, 3].

This survey deals with the axiomatics and basic methods for estimating internal conflict in a body of evidence. The remainder of this paper is organized as follows. Section 1 introduces a basic background on belief function theory. For details, see part I in [1]. In Section 2, we discuss non-conflict focal elements and the basic requirements to measures of internal conflict. Section 3 presents the general forms of measures satisfying a given system of axioms. In Section 4, different methods for estimating internal conflict are considered: an entropy approach (Section 4.1), methods based on auto-conflict calculation and contour function maximization (Section 4.2), and metric and decompositional approaches (Sections 4.3 and 4.4, respectively). In the Conclusions, we summarize some findings of this study.

1. BASIC BACKGROUND ON BELIEF FUNCTION THEORY

For convenience, we recall in brief the Dempster–Shafer theory of evidence [4, 5]. For details, see Sections 1 and 2 of the paper [1].

Let $X=\{x_1,...,x_n\}$ be a finite set, and 2^X be the set of all subsets from *X.* In the Dempster–Shafer theory of evidence, a *basic belief assignment* (BBA, also termed a mass function) is a set function *m:*

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$2^X \to [0, 1]$ that satisfies the condition $\sum_{A\in 2^X} m(A) = 1$.

A subset $A \subseteq X$ is called a focal element of a BBA m if $m(A) > 0$. A pair $F = (\mathcal{A}, m)$ composed of the set of all focal elements $\mathcal{A} = \{A\}$ and a corresponding BBA $m(A)$, $A \in \mathcal{A}$, is called a body of evidence. We denote by $\mathcal{F}(X)$ the set of all bodies of evidence on X and by $\mathcal{P}(X)$ the set of all probability measures on X.

A body of evidence $F = (\mathcal{A}, m)$ can be bijectively described by the *belief function* $Bel(A) = \sum_{B\subseteq A} m(B)$ and the *plausibility function* $Pl(A) = 1 - Bel(A^c) = \sum_{A:B\cap A\neq\varnothing} m(B)$, where A^c indicates the complement of the set A. The function $Pl(x) = \sum_{A\in\mathcal{A}:x\in A} m(A)$, $x \in X$, is called the contour function of a body of evidence. The belief and plausibility functions will be denoted by Bel_F and Pl_F, respectively, whenever their dependence on the body of evidence $F = (\mathcal{A}, m)$ should be emphasized.

An order relation can be defined on the set of set functions *g:* $2^X \to \mathbb{R}$ as follows: $g_1 \le g_2$ if $g_1(A) \le g_2(A)$ $\forall A \in 2^X$.

A belief function (and the corresponding body of evidence) is said to be:

– *categorical* if it has only one focal element; the corresponding body of evidence will be denoted by $F_A = (A, 1)$;

– *vacuous* if the entire set X is the only focal element of this function, $F_X = (X, 1)$;

– *consonant* if its focal elements are nested, i.e., $\forall A, B \in \mathcal{A}$: $A \subseteq B$ or $B \subseteq A$;

– *simple* if the BBA has no more than two focal elements and, in the case of two focal elements, X is one of them;

– *dogmatic* if $X \notin \mathcal{A}$ (i.e., $m(X) = 0$).

Any body of evidence $F = (\mathcal{A}, m)$ can be represented as $F = \sum_{A\in\mathcal{A}} m(A) F_A$. A simple body of evidence can be represented as $F_A^\omega = (1-\omega)F_A + \omega F_X$, where $\omega \in [0, 1]$.

A body of evidence $F' = (\mathcal{A}', m')$ is called a specialization of a body of evidence $F'' = (\mathcal{A}'', m'')$ (and denoted by $F' \sqsubseteq F''$) if there exists a partition $\mathcal{A}' = \mathcal{A}'_1 \cup \ldots \cup \mathcal{A}'_k$, $\mathcal{A}'_i \cap \mathcal{A}'_j = \varnothing$ $\forall i \neq j$, $k = |\mathcal{A}''|$ such that $\bigcup_{A\in\mathcal{A}'_i} A \subseteq B_i$ and $\sum_{A\in\mathcal{A}'_i} m'(A) = m''(B_i)$, $\forall B_i \in \mathcal{A}''$, $i = 1, \ldots, k$. In other words, a body of evidence F' refines (specializes) a body of evidence F''. The latter body is called a generalization of a body of evidence F'.

The amount of ignorance in the information contained in a body of evidence $F = (\mathcal{A}, m)$ can be estimated using the so-called imprecision indices [6]. An example of such an index is the normalized generalized Hartley measure [7, 8] $H_0(F) = \sum_{A\in\mathcal{A}} m(A) \log_{|X|} |A|$, mostly used below.

Belief function theory provides well-developed tools to combine bodies of evidence. A combining rule is understood as a certain operation $\otimes: \mathcal{F}(X) \times \mathcal{F}(X) \to \mathcal{F}(X)$. The most widespread rules include the following [9]:

- The unnormalized Dempster rule $\otimes_{ND}$: $m_{ND}(A) = \sum_{B\cap C=A} m_1(B) m_2(C)$ $\forall A \in 2^X$.

The canonical measure of (external) conflict is the value $K = K(F_1, F_2) = m_{ND}(\varnothing) = \sum_{B\cap C=\varnothing} m_1(B) m_2(C) \in [0, 1]$. It characterizes the degree of conflict of information sources described by the bodies of evidence F_1 and F_2: the greater this value is, the more inconsistent information the sources will provide.

- The Dempster rule [4] $\otimes_D : m_D(A) = \dfrac{m_{ND}(A)}{1-K}$ $\forall A \in 2^X \setminus \varnothing$. If $K = 1$ (complete conflict), the Dempster combining rule becomes inapplicable.
- The disjunctive consensus rule $\otimes_\cup$ [10]:

$$m_\cup(A) = \sum_{B\cup C=A} m_1(B) m_2(C), \quad A \in 2^X. \qquad (1)$$

2. THE CONCEPT OF NON-CONFLICT FOCAL ELEMENTS AND MEASURE OF INTERNAL CONFLICT

The measure of internal conflict in a body of evidence is understood as a certain functional $Con_{int} : \mathcal{F}(X) \to [0, 1]$ that achieves maximum under complete conflict between its focal elements and minimum without any conflict.

By analogy with bodies of evidence (see Section 3 of [1]), the following degrees of non-conflict are considered for the focal elements of a given body of evidence $F = (\mathcal{A}, m)$:

1) strong non-conflict: $\bigcap_{A\in\mathcal{A}} A \neq \varnothing$,

2) (simple) non-conflict: $A \cap B \neq \varnothing$ $\forall A, B \in \mathcal{A}$.

(The corresponding information sources are said to be conflict-free.)

Remark 1. Focal elements satisfying condition 1) are said to be logically consistent [11], and those satisfying condition 2) are said to be pairwise consistent [12]. Logical consistency implies pairwise consistency; however, the converse fails. The paper [12] considered some properties of bodies of evidence satisfying the general *s*-consistency condition: $\bigcap_{i=1}^{s} A_i \neq \varnothing$ $\forall A_1,..., A_s \in \mathcal{A}$, where $2 \leq s \leq |\mathcal{A}|$.

Generally, the measure of internal conflict Con_{int} should satisfy the following conditions:

I1: $Con_{int}(F) = 0$ if the focal elements of a body of evidence F are logically (or weakly pairwise) consistent.

Under condition I1, we particularly have:

- $Con_{int}(F) = 0$ if F is a categorical body of evidence.
- $Con_{int}(F) = 0$ if F is a consonant body of evidence.
- $Con_{int}(F) = 0$ if F is a simple body of evidence.

I2: $Con_{int}(F_1) \geq Con_{int}(F_2)$ $\forall F_1, F_2 \in \mathcal{F}(X)$, $F_1 \sqsubseteq F_2$ (antimonotonicity with respect to specialization).

In addition, the measure of internal conflict of a "complex" body of evidence should not be smaller than the minimum measure of external conflict between its components ("elementary" bodies of evidence). This requirement can be formulated for the class $\mathcal{OR}$ of so-called optimistic combining rules for bodies of evidence.

A combining rule $\otimes$ is said to be optimistic (pessimistic) with respect to an imprecision index f if $f(F_1 \otimes F_2) \leq f(F_i)$ ($f(F_1 \otimes F_2) \geq f(F_i)$, respectively), where $i = 1, 2$; see [6].

The following statement is true; for example, see the paper [13].

Proposition 1. $H_0(F_1 \otimes_{ND} F_2) \leq H_0(F_i)$ *and* $H_0(F_1 \otimes_{\cup} F_2) \geq H_0(F_i)$, $i = 1, 2,$ $\forall F_1, F_2 \in \mathcal{F}(X)$.

Thus, the unnormalized Dempster rule $\otimes_{ND}$ is optimistic, and the disjunctive consensus rule is pessimistic with respect to the normalized generalized Hartley measure H_0. The same result holds for any linear strict imprecision index [6].

I3: $Con_{int}(F_1 \otimes F_2) \geq \min\{Con_{int}(F_1), Con_{int}(F_2)\}$ $\forall F_1, F_2 \in \mathcal{F}(X)$ $\forall \otimes \in \mathcal{OR}$.

For measures of internal conflict, desirable properties also include independence from the ordering of alternatives of the set X or some generalization of this property. Let $\varphi: X \to Y$ be a bijective mapping. Then we can consider the image of a body of evidence $F = (\mathcal{A}, m)$ under the mapping φ: $F^{\varphi} = (\mathcal{A}^{\varphi}, m^{\varphi})$, where $\mathcal{A}^{\varphi} = \{\varphi(A): \ A \in \mathcal{A}\} \subseteq 2^Y$ and $m^{\varphi}(B) = \sum_{A:\varphi(A)=B} m(A)$ $\forall B \in \mathcal{A}^{\varphi}$.

I4: For any bijective mapping φ, $Con_{int}(F^{\varphi}) = Con_{int}(F)$ $\forall F \in \mathcal{F}(X)$.

3. AN AXIOMATICS FOR A MEASURE OF INTERNAL CONFLICT

An implicit-form axiomatics for measures of internal conflict appears when axiomatizing the so-called uncertainty measures within belief function theory [14] and imprecise probability theory [15].

An explicit-form axiomatics for measures of internal conflict was considered in [16]. More specifically, a system of axioms based on strengthening conditions I1–I4 was investigated:

B1: $Con_{int}(F) = 0 \Leftrightarrow$ the focal elements of a body of evidence $F = (\mathcal{A}, m)$ are in strong non-conflict, i.e., $\bigcap_{A \in \mathcal{A}} A \neq \varnothing$.

B2: $Con_{int}(F_1) \geq Con_{int}(F_2)$ $\forall F_1, F_2 \in \mathcal{F}(X)$, $Bel_{F_1} \geq Bel_{F_2}$.

B3: $Con_{int}(\alpha F_1 + (1-\alpha) F_2) \geq \alpha Con_{int}(F_1) + (1-\alpha) Con_{int}(F_2)$ $\forall \alpha \in [0, 1]$, $\forall F_1, F_2 \in \mathcal{F}(X)$.

B4: For any mapping $\varphi: X \to Y$, $Con_{int}(F^{\varphi}) \leq Con_{int}(F)$ $\forall F \in \mathcal{F}(X)$; for an injective mapping φ, $Con_{int}(F^{\varphi}) = Con_{int}(F)$.

Axiom B2 strengthens property I2 since $Bel_{F'} \geq Bel_{F''}$ implies $F' \sqsubseteq F''$. However, the converse fails [10].

Axiom B3 strengthens property I3 for the case of linear combining rules since $\alpha Con_{int}(F_1) + (1-\alpha) \times Con_{int}(F_2) \geq \min\{Con_{int}(F_1, \ Con_{int}(F_2)\}$. (These rules are simultaneously optimistic and pessimistic with respect to the linear imprecision index.)

The axiom B4 strengthens property I4 for the case of non-injective mappings φ: if the images of different elements from the set X are the same element from the set Y, the measure of internal conflict in the body of evidence F^{φ} will not exceed that of the body of evidence F.

Then the measure of internal conflict can be extended from the set of probability measures $\mathcal{P}(X)$ to the set of all bodies of evidence $\mathcal{F}(X)$.

Theorem 1 [16]. *If a functional* $Con:\mathcal{P}(X)\to[0,1]$ *satisfies axioms* B1, B3, *and* B4 *on the set* $\mathcal{P}(X)$, *then the functional*

$$Con_{int}(F)=\inf\left\{Con(P):P\in\mathcal{P}_{Bel_F}\right\}$$

satisfies axioms B1–B4 *on the set* $\mathcal{F}(X)$, *where* Bel_F *is a belief function corresponding to the body of evidence* F, *and* $\mathcal{P}_{Bel_F}=\{P\in\mathcal{P}(X): Bel_F(A)\le P(A)$ $\forall A\subseteq X\}$ *is the set of probability measures agreed with* Bel_F.

Theorem 1 allows determining the measure of internal conflict on the set $\mathcal{F}(X)$ given this measure on the set $\mathcal{P}(X)$. Since $P=\sum_{i=1}^{n}P(\{x_i\})F_{\{x_i\}}$,

$$Con(P)=f(P(\{x_1\}),...,\ P(\{x_n\})), \qquad (2)$$

where $f(t_1,...,t_n)$ is some function with $n=|X|$. The paper [16] found necessary and sufficient conditions on the function f under which the functional (2) satisfies axioms B1–B4 on the set $\mathcal{P}(X)$. In particular, the following proposition describes a wide class of such functions.

Proposition 2 [16]. *Assume that a function* $g:[0,1]\to[0,+\infty)$ *is concave,* $g(0)=g(1)=0$, *and* g *is strictly decreasing at the point* $t=1$. *Then the function* $f(t_1,...,t_n)=\sum_{i=1}^{n}g(t_i)$ *defines by formula* (2) *a measure of internal conflict on the set* $\mathcal{P}(X)$, *and this measure satisfies axioms* B1–B4.

Examples of the function g (a generating function for a measure of conflict on the set $\mathcal{P}(X)$) are:

– $g(t)=\begin{cases}-t\ln t, & t\in(0,1],\\ 0, & t=0,\end{cases}$ (in this case, $Con(P)=-\sum_{i=1}^{n}P(\{x_i\})\ln P(\{x_i\})$ is the Shannon entropy),

– $g(t)=t-t^2$, $t\in[0,1]$ (in this case, $Con(P)=E_I(P)$, where E_I is the entropy functional from the representation (6) in the paper [1]).

4. METHODS TO ESTIMATE INTERNAL CONFLICT

4.1. Entropy approach

In this case, the measure of internal conflict in a body of evidence $F=(\mathcal{A},m)$ should reflect the distribution of its mass function values on conflicting focal elements, i.e., on those focal elements that are not in strong or weak non-conflict. This definition of internal conflict was investigated in the early 1980s as a generalization of the Shannon entropy in the Dempster–Shafer theory [17]. As a rule, the entropy functional is the average value of the distribution of focal elements with respect to some conflict function:

$$\sum_{A\in\mathcal{A}}m(A)\theta\left(\psi(A)\right),$$

where $\theta:[0,1]\to[0,+\infty]$ is an increasing and convex function such that $\theta(0)=0$ (e.g., $\theta(t)=-\log_2(1-t)$ for the functionals considered below), and $\psi:2^X\to[0,1]$ is a set function whose values $\psi(A)$, $A\in2^X$, characterize the total mass of all focal elements in conflict with the set A. In particular, the following functionals are often considered:

- the *measure of dissonance* [18]

$$E(F)=-\sum_{A\in\mathcal{A}}m(A)\log_2 Pl(A)=-\sum_{A\in\mathcal{A}}m(A)\log_2\left(1-K(A)\right),$$

where $K(A)=\sum_{A\cap B=\varnothing}m(B)$ is the total mass of all focal elements in conflict with the set A by the non-intersection relation;

- the *measure of confusion* [19] as the average value of conflicting focal elements by the non-inclusion relation:

$$C(F)=-\sum_{A\in\mathcal{A}}m(A)\log_2 Bel(A)=-\sum_{A\in\mathcal{A}}m(A)\log_2\left(1-L(A)\right),$$

where $L(A)=\sum_{B\not\subseteq A}m(B)$ is the total mass of all focal elements in conflict with the set A by the non-inclusion relation;

- the *measure of discord* [20]

$$D(F)=-\sum_{A\in\mathcal{A}}m(A)\log_2\left(1-Conf(A)\right),$$

where $Conf(A)=\sum_{B\in\mathcal{A}}m(B)\dfrac{|B\setminus A|}{|B|}$ is the total weighted mass of all focal elements in conflict with the set A; obviously, $K(A)\le Conf(A)\le L(A)$;

- the *measure of strife* [21]

$$ST(F)=-\sum_{A\in\mathcal{A}}m(A)\log_2\left(1-CONF(A)\right),$$

where $CONF(A)=\sum_{B\in\mathcal{A}}m(B)\dfrac{|A\setminus B|}{|A|}$ is the total weighted mass of all focal elements in conflict with the set A.

Each entropy functional characterizes a certain type of conflict of focal elements. Only the entropy measure of dissonance satisfies conditions I1 and I2. Generally speaking, the other entropy measures under consideration do not satisfy these conditions.

Example 1. Let $X=\{x_1,...,x_5\}$. We find entropy measures of conflict for the bodies of evidence

$F_i(\alpha) = \alpha F_A + (1-\alpha)F_{B_i}$, $\alpha \in [0,1]$, $i = 1,2,3$, and different mutual arrangements of the focal elements $A \in 2^X$ and $B_i \in 2^X$, $i = 1,2,3$. In all cases, we consider $A = \{x_1, x_2\}$ and $|B_i| = 3$, $i = 1,2,3$.

1) $B_1 = \{x_1, x_2, x_3\}$. In this case, $A \subseteq B_1$. Then $K(A) = K(B_1) = 0$, $L(A) = 1-\alpha$, $L(B_1) = 0$, $Conf(A) = \frac{1}{3}(1-\alpha)$, $Conf(B_1) = 0$, $CONF(A) = 0$, and $CONF(B_1) = \frac{1}{3}\alpha$. Hence, we obtain the entropy measures $E(F_1(\alpha)) = 0$, $C(F_1(\alpha)) = -\alpha\log_2\alpha$, $D(F_1(\alpha)) = -\alpha\log_2\left(\frac{2}{3}+\frac{1}{3}\alpha\right)$, and $ST(F_1(\alpha)) = -(1-\alpha)\log_2\left(1-\frac{1}{3}\alpha\right)$.

2) $B_2 = \{x_2, x_3, x_4\}$. In this case, $A \cap B_2 \neq \varnothing$, but $A \nsubseteq B_2$ and $B_2 \nsubseteq A$. Then $K(A) = K(B_2) = 0$, $L(A) = 1-\alpha$, $L(B_2) = \alpha$, $Conf(A) = \frac{2}{3}(1-\alpha)$, $Conf(B_2) = \frac{1}{2}\alpha$, $CONF(A) = \frac{1}{2}(1-\alpha)$, and $CONF(B_2) = \frac{2}{3}\alpha$. Hence, we obtain the entropy measures $E(F_2(\alpha)) = 0$, $C(F_2(\alpha)) = -\alpha\log_2\alpha - (1-\alpha)\log_2(1-\alpha)$, $D(F_2(\alpha)) = -\alpha\log_2\left(\frac{1}{3}+\frac{2}{3}\alpha\right) - (1-\alpha)\log_2\left(1-\frac{1}{2}\alpha\right)$, and $ST(F_2(\alpha)) = -\alpha\log_2\left(\frac{1}{2}+\frac{1}{2}\alpha\right) - (1-\alpha)\log_2\left(1-\frac{2}{3}\alpha\right)$.

Thus, the measure of dissonance is uninformative ($E(F_i) \equiv 0$, $i = 1,2$) in the first two cases: it only considers the non-intersection relation of focal elements, absent in these cases.

3) $B_3 = \{x_3, x_4, x_5\}$. In this case, $A \cap B_3 = \varnothing$. Then $K(A) = L(A) = Conf(A) = CONF(A) = 1-\alpha$ and $K(B_3) = L(B_3) = Conf(B_3) = CONF(B_3) = \alpha$. Hence, $E(F_3(\alpha)) = C(F_3(\alpha)) = D(F_3(\alpha)) = ST(F_3(\alpha)) = -\alpha\log_2\alpha - (1-\alpha)\times \log_2(1-\alpha)$. In this case, all entropy measures of conflict coincide, achieving maximum for any fixed $\alpha \in [0, 1]$.

Note that $D(F_i(1-\alpha)) = ST(F_i(\alpha))$, $i = 1,2,3$, in all cases considered. In addition, the entropy conflict by any measure increases pointwise with increasing the “degree of non-intersection” of the focal elements (for example, $D(F_1(\alpha)) \le D(F_2(\alpha)) \le D(F_3(\alpha))$ $\forall \alpha \in [0, 1]$). ♦

4.2. Methods based on auto-conflict calculation and contour function maximization

A body of evidence $F = (\mathcal{A}, m)$ can be considered internally non-conflict if it has no conflict with itself by some measure of external conflict; see part I in [1]. For example, a body of evidence F can be non-conflict with itself by the canonical measure of conflict K: $K(F, F) = 0$. The value $K(F, F)$ can be treated as a measure of internal conflict. The paper [22] introduced the so-called *auto-conflict* of order s: $Con_{aut,s}(F) = K(\underbrace{F,..., F}_{s})$. For $s = 2$, such a measure will be simply called auto-conflict: $Con_{aut}(F) = Con_{aut,2}(F)$. The measure of auto-conflict Con_{aut} satisfies conditions I1 (in the case of simple non-conflict of focal elements) and conditions I2, I4, and I3 if $\mathcal{OR} = \{\otimes_{ND}\}$.

Another approach involves the strong non-conflict of focal elements. Clearly,

$$\Omega_{\mathcal{A}} = \bigcap\nolimits_{A\in\mathcal{A}} A \neq \varnothing \quad \Leftrightarrow$$
$$\exists x \in X : Pl(x) = \sum\nolimits_{A\in\mathcal{A}: x\in A} m(A) = 1.$$

In other words, the logical consistency of a body of evidence ($\Omega_{\mathcal{A}} \neq \varnothing$) is equivalent to $\max_{x\in X} Pl(x) = 1$ (the contour function achieves maximum equal to 1). Note that if $|\mathcal{A}| = s$ and $\Omega_{\mathcal{A}} \neq \varnothing$, then $Con_{aut,s}(F) = 0$. This fact was used in [23] to introduce the measure of internal conflict $Con_{pl}(F) = 1 - \max\{Pl(x) : x \in X\}$. In this case, the maximum of the contour function, $\max\{Pl(x) : x \in X\} = 1 - Con_{pl}(F)$, is a measure of non-conflict. The measure Con_{pl} satisfies conditions I1–I4 (and condition I3 if $\mathcal{OR} = \{\otimes_{ND}\}$). Other properties of this measure were investigated in [2, 23].

Remark 2. As shown in [16], the measure of internal conflict $Con_{pl}(F)$ can be obtained by extending the measure of conflict (2) to the set $\mathcal{F}(X)$, where $f(t_1,..., t_n) = \min\{1-t_1,...,1-t_n\}$ and $n = |X|$; see Theorem 1. The resulting measure satisfies axioms B1–B4.

Example 2. For the bodies of evidence $F_i(\alpha)$, $i = 1,2,3,$ from Example 1, we obtain $Con_{pl}(F_1(\alpha)) = Con_{pl}(F_2(\alpha)) = 1 - \max_{1\le k\le 5} Pl(x_k) = 0$ since $Pl(x_1) = Pl(x_2) = 1$ (the first case) and $Pl(x_2) = 1$ (the second case). In the third case, $Pl(x_1) = Pl(x_2) = \alpha$ and $Pl(x_3) = Pl(x_4) = Pl(x_5) = 1-\alpha$. Therefore, $Con_{pl}(F_3(\alpha)) = 1 - \max_{1\le k\le 5} Pl(x_k) = \min\{\alpha, 1-\alpha\}$.

In this example, the measure of auto-conflict is given by $Con_{aut}(F_1(\alpha)) = Con_{aut}(F_2(\alpha)) = 0$ and $Con_{aut}(F_3(\alpha)) = 2\alpha(1-\alpha)$. ♦

The measure of conflict Con_{pl} is easy to calculate and satisfies many desirable properties (particularly axioms B1–B4). As a result, it is popular in applications. At the same time—see Example 2—it becomes insensitive in the presence of disjoint focal elements.

4.3. Metric approach

In this case, the measure of external conflict in a body of evidence $F=(\mathcal{A},m)$ is given by

$$Con_{int}(F)=\inf_{F'\in\mathcal{V}(X)} d(F,F'), \qquad (3)$$

where d indicates some metric between bodies of evidence (see sub-subsection 4.3.1 in [1]), and $\mathcal{V}(X)$ is a set of bodies of evidence with zero internal conflict, i.e., the ones satisfying condition I1. This can be, e.g., the set of categorical or simple bodies of evidence. Such an approach was considered in [24] and was applied to estimate the reliability of expert weather forecasts. In the general case, such a measure may not satisfy all the desirable properties of measures of conflict. The result of calculating an internal conflict significantly depends on the choice of the set $\mathcal{V}(X)$. In addition, the solution procedure of the optimization problem (3) may have high computational complexity.

Example 3. We find the internal conflict in the bodies of evidence $F_i(\alpha)$, $i=1,2,3,$ (Example 1) using formula (3), where the metric $d=d_J$ is given by

$$d_J(F_1,\,F_2)=$$
$$\sqrt{\tfrac{1}{2}\sum\nolimits_{A,B\in 2^X\setminus\{\varnothing\}}\tfrac{|A\cap B|}{|A\cup B|}(m_1(A)-m_2(A))(m_1(B)-m_2(B))},$$
$$F_1=(\mathcal{A}_1,m_1)\,,\; F_2=(\mathcal{A}_2,m_2)\,.$$

(For details, see the paper [25] and sub-subsection 4.3.1 of part I in [1].) Let $\mathcal{V}(X)$ be the set of simple bodies of evidence on the set X of the form $F_{\{x\}}^{\omega}=(1-\omega)F_{\{x\}}+\omega F_X$, where $\omega\in[0,1]$ and $x\in X$. In this case, $Con_{int}(F_i(\alpha))=\min\limits_{1\le i\le 5}\min\limits_{\omega_i\in[0,1]} d_J(F_i(\alpha),F_{\{x_i\}}^{\omega_i})$, $i=1,2,3$.

For the set $B_1=\{x_1,x_2,x_3\}$ ($A\subseteq B_1$), we particularly obtain

$$d_J(F_1(\alpha),\,F_{\{x_i\}}^{\omega_i})=$$
$$\tfrac{1}{\sqrt{2}}\sqrt{h^2(\alpha,\,\omega_i)-\alpha(1-\omega_i)-\tfrac{2}{3}(1-\alpha)(1-\omega_i)},\; i=1,\,2,$$
$$d_J(F_1(\alpha),F_{\{x_3\}}^{\omega_3})=\tfrac{1}{\sqrt{2}}\sqrt{h^2(\alpha,\omega_3)-\tfrac{2}{3}(1-\alpha)(1-\omega_3)}\,,$$
$$d_J(F_1(\alpha),F_{\{x_i\}}^{\omega_i})=\tfrac{1}{\sqrt{2}}h(\alpha,\omega_i)\,,\; i=4,\,5,$$

where

$$h(\alpha,\omega)=\sqrt{\alpha^2+(1-\alpha)^2+\omega^2+(1-\omega)^2-\tfrac{4}{5}\alpha\omega-\tfrac{6}{5}(1-\alpha)\omega}\,.$$

Now,

$$\min_{\omega_i\in[0,1]} d_J(F_1(\alpha),\,F_{\{x_i\}}^{\omega_i})=d_J(F_1(\alpha),\,F_{\{x_i\}}^{\omega_i})\Big|_{\omega_i=\frac{38-11\alpha}{60}}=$$
$$\tfrac{1}{30}\sqrt{\tfrac{3479}{4}\alpha^2-841\alpha+239},\; i=1,\,2,$$
$$\min_{\omega_3\in[0,1]} d_J(F_1(\alpha),\,F_{\{x_3\}}^{\omega_3})=d_J(F_1(\alpha),\,F_{\{x_3\}}^{\omega_3})\Big|_{\omega_3=\frac{38+4\alpha}{60}}=$$
$$\tfrac{1}{30}\sqrt{896\alpha^2-676\alpha+239},$$
$$\min_{\omega_i\in[0,1]} d_J(F_1(\alpha),\,F_{\{x_i\}}^{\omega_i})=d_J(F_1(\alpha),\,F_{\{x_i\}}^{\omega_i})\Big|_{\omega_i=\frac{8-\alpha}{10}}=$$
$$\tfrac{1}{10}\sqrt{99\alpha^2-84\alpha+36},\; i=4,\,5.$$

Consequently,

$$Con_{int}(F_1(\alpha))=\min_{1\le i\le 5}\min_{\omega_i\in[0,1]} d_J(F_1(\alpha),\,F_{\{x_i\}}^{\omega_i})=$$
$$\tfrac{1}{30}\sqrt{\tfrac{3479}{4}\alpha^2-841\alpha+239}.$$

For the set $B_2=\{x_2,\,x_3,\,x_4\}$ ($A\cap B_2\neq\varnothing$, but $A\not\subseteq B_2$ and $B_2\not\subseteq A$), we obtain $Con_{int}(F_2(\alpha))=Con_{int}(F_1(\alpha))$ $\forall\alpha\in[0,\,1]$; for the set $B_3=\{x_3,\,x_4,\,x_5\}$ ($A\cap B_3=\varnothing$),

$$Con_{int}(F_3(\alpha))=\min\Big\{\tfrac{1}{20}\sqrt{351\alpha^2-376\alpha+144},$$
$$\tfrac{1}{30}\sqrt{896\alpha^2-676\alpha+239}\Big\}.$$

As is easily seen, $Con_{int}(F_1(\alpha))=Con_{\mathrm{int}}(F_2(\alpha))\le Con_{\mathrm{int}}(F_3(\alpha))$ $\forall\alpha\in[0,\,1]$.♦

4.4. Decompositional approach

The decompositional approach proceeds from the assumption that an information source formed by a body of evidence with a great internal conflict could be heterogeneous. For example, information about the predictive stock value is obtained using several different techniques. In this case, a body of evidence $F=(\mathcal{A},\,m)$ can be treated as the result of combining several decomposed bodies of evidence $F_i=(\mathcal{A}_i,\,m_i)\in\mathcal{F}(X)$, $i=1,\dots,\,l,$ using some combining rule $\otimes$: $F=F_1\otimes\ldots\otimes F_l$. For a fixed combining rule $\otimes$ and a fixed measure of (external) conflict $Con_{ext}:\underbrace{\mathcal{F}(X)\times\ldots\times\mathcal{F}(X)}_{l}\to[0,1]$ (see part I in [1]), the internal decomposition conflict Con_{dec} in the body of evidence F can be therefore estimated [26, 27] by the formula

$$Con_{dec}(F)=Con_{ext}(F_1,\dots,\,F_l)$$

provided that

$$F=F_1\otimes\ldots\otimes F_l\,.$$

This equation has a set of solutions. Hence, we can formulate optimization problems on finding the greatest $\overline{Con_{dec}^{\otimes}}(F)$ and smallest $\underline{Con_{dec}^{\otimes}}(F)$ conflicts:

$$\overline{Con_{dec}^{\otimes}}(F)=\sup_{F=F_1\otimes\ldots\otimes F_l} Con_{ext}(F_1,\dots,\,F_l)\,,$$
$$\underline{Con_{dec}^{\otimes}}(F)=\inf_{F=F_1\otimes\ldots\otimes F_l} Con_{ext}(F_1,\dots,\,F_l)\,. \qquad (4)$$

Let $S_n=\left\{\mathbf{s}=(s_i)_{i=1}^n : s_i\ge 0\;\forall i=1,\dots,\,n,\;\sum\nolimits_{i=1}^n s_i=1\right\}$ be an n-dimensional simplex.

Consider special cases of the problem.

Decomposition using Dempster rule. Let decomposition be performed using the Dempster rule $\otimes_D$. For $l=2$, problems (4) take the following form:

find

$$K(F_1,F_2)=\sum_{\substack{B\cap C=\varnothing,\\ B\in\mathcal{A}_1,C\in\mathcal{A}_2}} m_1(B)m_2(C)\to\sup\ (\inf) \qquad (5)$$

subject to the conditions

$$\mathbf{m}_1=(m_1(B))_{B\in\mathcal{A}_1}\in S_{|\mathcal{A}_1|},\ \mathbf{m}_2=(m_2(C))_{C\in\mathcal{A}_2}\in S_{|\mathcal{A}_2|}, \qquad (6)$$

$$\left(1-K_0(F_1,F_2)\right)m(A)=\sum_{\substack{B\cap C=A,\\ B\in\mathcal{A}_1,C\in\mathcal{A}_2}} m_1(B)m_2(C),\ A\in\mathcal{A}, \qquad (7)$$

$$K(F_1,F_2)<1. \qquad (8)$$

These are quadratic programming problems under the linear (6) and quadratic (7), (8) constraints. Note that in the general statement (5)–(8), the measure of the decomposition conflict $\underline{Con}_{dec}^{\otimes_D}(F)=0$ is achieved for the bodies of evidence $F_1=F$ and $F_2=F_X$. For two bodies of evidence satisfying conditions (6) and (7) (without condition (8)), the greatest value of the conflict $K(F_1,F_2)=1$ is achieved, e.g., for the bodies of evidence $F_i=(\mathcal{A}_i,m_i)\in\mathcal{F}(X)$, $i=1,2$, in which $B\cap C=\varnothing\quad\forall B\in\mathcal{A}_1,\forall C\in\mathcal{A}_2$. The latter bodies are in no way related to the body F.

Decomposition using disjunctive consensus rule. Let decomposition be performed using the disjunctive consensus rule $\otimes_{\cup}$ (1). Then condition (1) will be applied instead of condition (7) for estimating internal conflict. Therefore, the problem is finding bodies of evidence with the greatest (smallest) canonical conflict (5) that satisfy conditions (1) and (6).

Remark 3. When using the disjunctive consensus rule, sometimes it seems convenient to consider an empty set a focal element in a body of evidence. This can be interpreted as $x\notin X$ and the value $m(\varnothing)$ as the degree of belief to $x\notin X$. The corresponding solutions will be called generalized and denoted by $\mathcal{CON}_{dec}^{\otimes_{\cup}}(F)$. Then the greatest value of the canonical conflict (5) satisfying conditions (1) and (6), $\overline{\mathcal{CON}}_{dec}^{\otimes_{\cup}}(F)=1$, is achieved on the decomposition of the body of evidence F of the form $F_1=F$, $F_2=F_{\varnothing}$.

Clearly, in the general statement, the problem of finding the largest and smallest internal conflicts $\overline{Con}_{dec}^{\otimes}(F)$ and $\underline{Con}_{dec}^{\otimes}(F)$ often leads to trivial solutions.

At the same time, the assumption about the heterogeneous information source of a body of evidence with a great internal conflict implies the following: the bodies of evidence composing the initial body of evidence should be, in some sense, simpler than the latter. In addition, the combining method may impose restrictions on the decomposed set of bodies of evidence. In particular, we identify several constraints on the decomposable set of bodies of evidence:

– structural constraints,
– conflict constraints,
– constraints related to combining rules,
– mixed constraints.

Structural constraints imply that the decomposed set of bodies of evidence belongs to some class of simple-structure bodies of evidence. Examples of such classes are simple bodies of evidence (or their generalizations, see below), consonant bodies of evidence, and others.

For example, the paper [28] defined internal conflict as a conflict between the so-called *generalized simple BBAs* (the bodies of evidence $F_A^{\omega}=(1-\omega)F_A+\omega F_X$, $\omega\in(0,\infty)$) into which an initial non-dogmatic body of evidence ($m(X)>0$) is uniquely decomposed. (Shafer called such a decomposition canonical). If the initial body of evidence is dogmatic ($m(X)=0$), then before the decomposition, the BBA should be discounted with a small parameter $\varepsilon>0$: $m(X)=\varepsilon>0$. The mass functions of the other focal elements are recalculated proportionally to the initial values. According to the report [29], a non-dogmatic body of evidence F can be decomposed into generalized simple BBAs in two stages. The first stage is calculating the *commonality function* $q(A)=\sum_{B\supseteq A}m(B)$. In the second stage, the weights ω_B of the bodies of evidence $F_B^{\omega_B}$ are calculated by the formula $\omega_B=\prod_{A\supseteq B}q(A)^{(-1)^{|A|-|B|+1}}$ for each subset $B\in 2^X\setminus X$. As a result, $F=\otimes_{B\in 2^X\setminus X}F_B^{\omega_B}$, where $\otimes=\otimes_{ND}$ is the unnormalized Dempster rule [29]. Another measure of internal conflict, $Con_{dec\_simple}(F)=\tilde{m}(\varnothing)$, where $\tilde{F}=(\mathcal{A},\tilde{m})=\otimes_{B\in 2^X\setminus\{\varnothing,X\}}F_B^{\omega_B}$, was proposed in [28]. Clearly,

$$Con_{dec\_simple}(F)=\sum_{\substack{B_{i_1}\cap\ldots\cap B_{i_k}=\varnothing,\\ B_{i_1},\ldots,B_{i_k}\in 2^X\setminus\{\varnothing,X\}}}\prod_{s=1}^{k}\left(1-\omega_{B_{i_s}}\right)\times \prod_{B\in\left(2^X\setminus\{\varnothing,X\}\right)\setminus\left\{B_{i_1},\ldots,B_{i_k}\right\}}\omega_B. \qquad (9)$$

Example 4. For $X=\{x_1,x_2\}$ and $F=\alpha F_{\{x_1\}}+\beta F_{\{x_2\}}+(1-\alpha-\beta)F_X$, where $\alpha,\beta\ge 0$ and $\alpha+\beta<1$, we obtain $q(\varnothing)=1$, $q(\{x_1\})=1-\beta$, $q(\{x_2\})=1-\alpha$, and $q(X)=$

$1-\alpha-\beta$. Therefore, $\omega_{\varnothing}=\dfrac{(1-\alpha)(1-\beta)}{1-\alpha-\beta}$, $\omega_{\{x_1\}}=\dfrac{1-\alpha-\beta}{1-\beta}$, and $\omega_{\{x_2\}}=\dfrac{1-\alpha-\beta}{1-\alpha}$. Hence, $Con_{dec\_simple}(F)=\tilde{m}(\varnothing)=(1-\omega_{\{x_1\}})\,(1-\omega_{\{x_2\}})=\dfrac{\alpha\beta}{(1-\alpha)(1-\beta)}$. ♦

For calculating the measure Con_{dec_simple} for the body of evidence from Example 1, we need the following result.

Lemma. *Let* $F=\alpha F_A+\beta F_B+(1-\alpha-\beta)F_X$, *where* $\alpha,\beta\in(0,1)$, $\alpha+\beta<1$, *and* $A,B\in 2^X$. *The following statements are true:*

– *If* $A\subseteq B\subseteq X$, *then* $\omega_A=1-\alpha$, $\omega_B=\dfrac{1-\alpha-\beta}{1-\alpha}$, *and* $\omega_D=1\quad\forall D\in 2^X\setminus\{\varnothing, A, B, X\}$.

– *If* $A\cap B\neq\varnothing$, $A\nsubseteq B$, *and* $B\nsubseteq A$, *then* $\omega_A=\dfrac{1-\alpha-\beta}{1-\beta}$, $\omega_B=\dfrac{1-\alpha-\beta}{1-\alpha}$, $\omega_{A\cap B}=\dfrac{(1-\alpha)(1-\beta)}{1-\alpha-\beta}$, *and* $\omega_D=1\quad\forall D\in 2^X\setminus\{\varnothing, A\cap B, A, B, X\}$.

– *If* $A\cap B=\varnothing$, *then* $\omega_A=\dfrac{1-\alpha-\beta}{1-\beta}$, $\omega_B=\dfrac{1-\alpha-\beta}{1-\alpha}$, *and* $\omega_D=1\quad\forall D\in 2^X\setminus\{\varnothing, A, B, X\}$.

Corollary. Let $F=\alpha F_A+\beta F_B+(1-\alpha-\beta)F_X$, where $\alpha,\beta\in(0,1)$ and $\alpha+\beta<1$. The following statements are true:

– $Con_{dec\_simple}(F)=0$ if $A\cap B\neq\varnothing$.

– $Con_{dec\_simple}(F)=\dfrac{\alpha\beta}{(1-\alpha)(1-\beta)}$ if $A\cap B=\varnothing$.

Example 5. Consider the bodies of evidence $F_i(\alpha)=\alpha F_A+(1-\alpha)F_{B_i}$, $\alpha\in[0,1]$, $A=\{x_1, x_2\}$, $|B_i|=3$, $i=1, 2, 3$, on the set $X=\{x_1,..., x_5\}$ (Example 1). First, we perform discounting with a small parameter $\varepsilon>0$ to obtain the bodies of evidence $F_i(\alpha,\varepsilon)=\alpha(1-\varepsilon)F_A+(1-\alpha)(1-\varepsilon)F_{B_i}+\varepsilon F_X$, $i=1, 2, 3$. According to the corollary, in the first case $B_1=\{x_1, x_2, x_3\}$ (if $A\subseteq B_1$) and the second case $B_2=\{x_2, x_3, x_4\}$ (if $A\cap B_2\neq\varnothing$, but $A\nsubseteq B_2$ and $B_2\nsubseteq A$), we have $Con_{dec\_simple}(F_i(\alpha, \varepsilon))=0$, $i=1,2$. In the third case $B_3=\{x_3,x_4,x_5\}$ (if $A\cap B_3=\varnothing$), we have $Con_{dec\_simple}(F_3(\alpha, \varepsilon)) = (1-\omega_{\{x_1,x_2\}})(1-\omega_{\{x_3,x_4,x_5\}}) = \dfrac{\alpha(1-\alpha)(1-\varepsilon)^2}{(\alpha(1-\varepsilon)+\varepsilon)((1-\alpha)(1-\varepsilon)+\varepsilon)}$. As $\varepsilon\to+0$, $Con_{dec\_simple}(F_3(\alpha))=\begin{cases}1, & \alpha\in(0, 1),\\ 0, & \alpha=0\vee\alpha=1.\end{cases}$ ♦

This example shows that the measure of conflict Con_{dec_simple} is rather rough for dogmatic bodies of evidence. Moreover, decomposition into generalized simple BBAs has other disadvantages. First of all, the bodies of evidence F_A^{ω}, $\omega\notin[0, 1]$, require a certain interpretation. In this case, we cannot say that the initial body of evidence results from combining information from several other sources. In addition, the decomposition can contain up to $2^{|X|}-1$ generalized simple BBAs, different from the meaningless body of evidence F_X. Nevertheless (see the next example), the initial body of evidence may result from combining several more complex bodies of evidence than generalized simple BBAs without internal conflict.

Example 6. Consider two bodies of evidence, $F_1=\alpha F_{\{x_2,x_3\}}+(1-\alpha)F_{\{x_1,x_2,x_3\}}$ with $\alpha\in(0,1)$ and $F_2=\beta F_{\{x_1,x_4\}}+(1-\beta)F_{\{x_1,x_2,x_4\}}$ with $\beta\in(0,1)$, on the set $X=\{x_1, x_2, x_3, x_4\}$. These bodies are consonant: each has no conflict with itself. The canonical measure of conflict is $K=K(F_1, F_2)=\alpha\beta$. Combining these bodies of evidence using the unnormalized Dempster rule yields $F=F_1\otimes_{ND}F_2=(1-\alpha)\beta F_{\{x_1\}}+\alpha(1-\beta)F_{\{x_2\}}+(1-\alpha)(1-\beta)F_{\{x_1,x_2\}}$. Decomposing this body of evidence into generalized simple BBAs and calculating the corresponding measure of conflict, we obtain $Con_{dec\_simple}(F)=\dfrac{\alpha\beta(1-\alpha)(1-\beta)}{(1-\alpha+\alpha\beta)(1-\beta+\alpha\beta)}<K$ (Example 4). In other words, the conflict between the initial consonant bodies of evidence will be greater than the one from decomposing the combined body of evidence into generalized siymple BBAs. ♦

A decompositional approach to estimating the internal conflict in bodies of evidence, close to [28], was considered in the paper [30]. The cited authors studied the conflict function $f_{\varnothing}\left(\left\{B_{i_1},...,B_{i_k}\right\}\right)=\prod_{s=1}^{k}\left(1-\omega_{B_{i_s}}\right)\prod_{B\in\left(2^X\setminus\{\varnothing,X\}\right)\setminus\left\{B_{i_1},...,B_{i_k}\right\}}\omega_B$ on sets of disjoint subsets $\left\{B_{i_1},...,B_{i_k}\right\}$, $B_{i_1}\cap...\cap B_{i_k}=\varnothing$ (see formula (9)) and the *local conflict function* $\overline{f}_{\varnothing}(A)=\sum_{\substack{A\in\left\{B_{i_1},...,B_{i_k}\right\},\\ B_{i_1}\cap...\cap B_{i_k}=\varnothing}}\dfrac{1}{\left|\left\{B_{i_1},..., B_{i_k}\right\}\right|}f_{\varnothing}\left(\left\{B_{i_1},..., B_{i_k}\right\}\right),\qquad A\subsetneq X$. These functions were used in [30] to choose the least-conflict information sources for combining in the robot localization problem.

Conflict constraints mean that the decomposed set of bodies of evidence belongs to the class of ones with a smaller internal conflict than the original body of evidence by another (non-decomposition) measure of conflict.

Example 7. Let $X=\{x_1,x_2,x_3\}$ and $F=\alpha F_{\{x_1\}}+\beta F_{\{x_2\}}+\gamma F_{\{x_3\}}+(1-\alpha-\beta-\gamma)F_{\{x_2,x_3\}}$, where $\alpha,\beta,\gamma>0$, $\alpha+\beta+\gamma<1$. Consider the decomposition of the body of evidence F using the unnormalized Dempster rule: $F=F_1\otimes_{ND}F_2$. Assume that the decomposition belongs to the class of non-conflict bodies of evidence (zero auto-conflict: $Con_{aut}(F_i)=0$, $i=1,\,2$). As is easily shown, the unique decomposition has the form

$$\begin{cases} F_1=\lambda_1 F_{\{x_2,x_3\}}+\lambda_2 F_{\{x_1,x_2\}}+\lambda_3 F_{\{x_2\}}, \\ F_2=\mu_1 F_{\{x_2,x_3\}}+\mu_2 F_{\{x_1,x_3\}}+\mu_3 F_{\{x_3\}}. \end{cases}$$

Due to $F=F_1\otimes_{ND}F_2$, the nonnegative coefficients λ_i,μ_i, $i=1,2,3,$ satisfy the system of equations

$$\begin{cases} \lambda_1+\lambda_2+\lambda_3=1, \quad \mu_1+\mu_2+\mu_3=1, \\ \lambda_1\mu_1=1-\alpha-\beta-\gamma, \quad \lambda_2\mu_2=\alpha, \\ (\lambda_2+\lambda_3)\mu_1=\beta, \quad \lambda_1(\mu_2+\mu_3)=\gamma. \end{cases}$$

This system is solvable if $\alpha(1-\alpha-\beta-\gamma)=\beta\gamma$, and the solution is given by $\lambda_1=1-\alpha-\beta$, $\mu_1=1-\alpha-\gamma$, $\lambda_2=\dfrac{\alpha}{\alpha+\gamma}$, $\mu_2=\alpha+\gamma$, and $\lambda_3=\mu_3=0$. Then $Con_{dec}(F)=K(F_1,F_2)=0$. However, for example, $Con_{aut}(F)=\alpha(1-\alpha)+\beta\gamma$. ♦

Constraints related to combining rules. The choice of a combining rule imposes constraints on the set of admissible bodies of evidence. This is due to the different nature of these rules. For example, the conjunctive rule is optimistic, and the disjunctive rule is pessimistic. Constraints on the set of admissible bodies of evidence, agreed with the nature of combining rules, can be defined, e.g., using imprecision indices [6]. From now on, we will use the normalized generalized Hartley measure $H_0(F)=\sum_{A\in\mathcal{A}}m(A)\log_{|X|}|A|$ as an imprecision index. Nevertheless, all results are valid for a broader class of such indices, particularly for strict linear imprecision indices [26, 27].

Recall that the Dempster rule is optimistic (Proposition 1). When decomposing a body of evidence F into two bodies of evidence $F_i=(\mathcal{A}_i,m_i)\in\mathcal{F}(X)$, $i=1,2$, the problem of estimating its internal conflict can be formulated as follows: find the greatest (smallest) value of the functional $K(F_1,F_2)$ under the constraints (6)–(8) and the conditions

$$H_0(F)\le H_0(F_i)\,,\ i=1,2\,. \tag{10}$$

We denote their solutions by $\underline{Con}^{\otimes_D}_{dec\_gen}(F)$ and $\overline{Con}^{\otimes_D}_{dec\_gen}(F)$, respectively. Note that the bodies of evidence $F_1=F$ and $F_2=F_X$ satisfy conditions (10) because $H_0(F_X)=1$. Therefore, $\underline{Con}^{\otimes_D}_{dec\_gen}(F)=0$. Then the problem is finding bodies of evidence with the greatest canonical conflict (5) that satisfy conditions (6)–(8) and (10).

Besides the lower-bound constraints (10), the upper-bound constraints can be introduced on the amount of ignorance in the information contained in the decomposed bodies of evidence: $H_0(F_i)\le H_{\max}$, $i=1,2$, where $H_{\max}$ is the maximum admissible level of ignorance.

If the body of evidence F is decomposed using the disjunctive consensus rule, conditions (1) are replaced for conditions (7) in the internal conflict problem. Moreover, for the disjunctive consensus rule and any linear imprecision index (particularly H_0), we have the relation

$$H_0(F)\ge H_0(F_i)\,,\ i=1,\,2\,. \tag{11}$$

(For details, see Proposition 1.)

Thus, in this case, the problem is finding bodies of evidence with the greatest (smallest) canonical conflict (5) that satisfy conditions (1), (6), and (11). We denote by $\underline{Con}^{\otimes_\cup}_{dec\_gen}(F)$ and $\overline{Con}^{\otimes_\cup}_{dec\_gen}(F)$ the solutions of the corresponding problems.

Example 8. Let $X=\{x_1,x_2\}$ and $F=\alpha F_{\{x_1\}}+\beta F_{\{x_2\}}+(1-\alpha-\beta)F_X$, where $\alpha,\beta\ge 0$, $\alpha+\beta<1$. According to [26], $\overline{Con}^{\otimes_D}_{dec\_gen}(F)=\dfrac{\alpha\beta}{(1-\alpha)(1-\beta)}$; if $\sqrt{\alpha}+\sqrt{\beta}\le 1$, then $\underline{Con}^{\otimes_\cup}_{dec\_gen}(F)=2\sqrt{\alpha\beta}$. For $\sqrt{\alpha}+\sqrt{\beta}>1$, the corresponding decomposition problem for finding the measure of conflict $\underline{Con}^{\otimes_\cup}_{dec\_gen}$ has no solution. (However, it has a generalized solution; see Remark 3.) Note that $\overline{Con}^{\otimes_D}_{dec\_gen}=Con_{dec\_simple}$ on the set $X=\{x_1,x_2\}$ (Example 4). ♦

The paper [31] established some properties for the measures of conflict obtained by decomposition with constraints related to combining rules. In particular, it was shown therein that $\overline{Con}^{\otimes_D}_{dec\_gen}(F)=1$ in the case of complete conflict of focal elements and $\overline{Con}^{\otimes_D}_{dec\_gen}(F)=0$ for simple bodies of evidence.

The general disadvantage of the decompositional approach is high computational complexity. However, it is compensated by good interpretability in the case of a heterogeneous information source.

CONCLUSIONS

This paper has reviewed current research on the inconsistency (conflict) of information in a single source within belief function theory. In particular, the following aspects can be highlighted:

- There are several requirements to measures of internal conflict: minimum under some degree of non-conflict of focal elements, antimonotonicity with respect to specialization, nondecrease under optimistic combining, and nonincrease under mappings of the basic set.
- These properties underlie the axiomatics of a measure of internal conflict. A general form of such a measure is found; on the set of probability measures, it coincides with some entropy functional (particularly with the Shannon entropy for an appropriately chosen generating function).
- There are several methods for estimating internal conflict: an entropy approach, methods based on auto-conflict calculation and contour function maximization, and metric and decompositional approaches.

The internal conflict estimation methods considered differ in the conditions to satisfy desired properties, their sensitivity and computational complexity, and the estimation model: the average mass distribution of conflicting focal elements, the distance to the set of non-conflicting bodies of evidence, auto-conflict, the measure of logical consistency of focal elements, the heterogeneity of information sources, etc.

Of course, there are still open problems in estimating the internal conflict of bodies of evidence:
– exploring the properties of internal conflict measures based on a particular model;
– finding a general form of a measure of internal conflict for other systems of axioms;
– studying measures of conflict for bodies of evidence defined on a metrical space;
– and others.

Applied problems related to the estimation of internal conflict are topical. Among them, we mention reducing the internal conflict in a body of evidence (including evidence obtained by expert data processing). This problem can be solved by generalizing the initial body of evidence (see condition I2) or by decomposing it into internally non-conflicting bodies of evidence (see subsection 4.4).

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