CONSTRUCTING THE CES PRODUCTION FUNCTION BASED ON THE DISCRETE WEIBULL DISTRIBUTION

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Abstract. This paper considers a probabilistic approach to obtaining the CES production function. It consists in calculating the mean and median of the Leontief function (the quantity of output) as a random variable depending on the capacities of production factors, i.e., the ratios of the factors to their per-unit values. The type of the cumulative distribution function of the minimum from a set of independent random variables is substantiated. Explicit expressions are derived for the mean and median of the quantity of output as CES functions when the factor capacities have (continuous) Weibull distributions. Discretely distributed production factors are considered using the example of a geometric law. An attempt is made to derive the CES function when the factor capacities have discrete Weibull distributions. The difficulties arising in the analytical use of the mean of the Leontief function are described.

Keywords: production function, CES production function, probabilistic approach, Weibull distribution, discrete Weibull distribution, geometric distribution, mean, median.

INTRODUCTION

Traditionally, production functions that establish the relationship between production factors $X_1,...,X_n$ and the quantity Q of output by an enterprise (or a country) are described in terms of the marginal rate of substitution S_{ij} and the elasticity of substitution σ_{ij} of a factor X_i by a factor X_j; by assumption, the factors take deterministic values [1]. In particular, for two factors X_1 and X_2, the property $\sigma_{12} = \text{const}$ is possessed by the CES (constant elasticity of substitution) function.

However, starting from the 1950s, a probabilistic approach to the description of production functions had stood out and was particularly developed in 1990–2015; for example, see [2–5]. The most significant achievement here was the development of a theoretical apparatus based on the following concepts: technology (idea), local production function, technology menu, and global production function [2]. We briefly explain the essence by the following example [3].

Let X_1 and X_2 be two production factors with some technological parameters x_1 and x_2, respectively. Consider the CES production function

$$Y = A\left(\psi\left(\frac{X_1}{x_1}\right)^{\theta} + (1-\psi)\left(\frac{X_2}{x_2}\right)^{\theta}\right)^{\frac{1}{\theta}},$$

where $A > 0$, $\psi \in (0, 1)$, and $\theta \in (-\infty, 0) \cup (0, 1)$ are constants.

A pair of parameter values (x_1, x_2) is called a *technology* or *idea*. The function Y with fixed values x_1 and x_2 is called the *local production function corresponding to the technology* (x_1, x_2).

Let a relation (*technology menu*) be imposed on the parameter values x_1 and x_2:

$$T_1(x_1)T_2(x_2) = N,$$

where $T_1(x_1)$ and $T_2(x_2)$ are some (unknown) functions of one variable and N is a constant.

In this example, the following problem arises naturally: given the values of the factors X_1 and X_2, find functions $T_1(x_1)$ and $T_2(x_2)$ such that the function Y will reach the largest values under the technology menu:

$$\begin{cases} Y = A\left(\psi\left(\frac{X_1}{x_1}\right)^{\theta} + (1-\psi)\left(\frac{X_2}{x_2}\right)^{\theta}\right)^{\frac{1}{\theta}} \to \max \\ T_1(x_1)T_2(x_2) = N. \end{cases}$$

(In this case, Y will be called the *global production function*.)

Using Lagrange's method of multipliers and the variable separation method, we can show that $F_1(x_1)=1-T_1(x_1)$ and $F_2(x_2)=1-T_2(x_2)$ are Weibull distribution functions. Researchers also studied the inverse problem [4]: reconstruct the global production function as a CES function from the parameters x_1 and x_2 distributed according to the Weibull law.

An alternative approach was later proposed by A.V. Mikheev [6] as follows. Denoting by x_i *the per-unit value of a factor* X_i (its quantity required to manufacture one product), he introduced the *capacity* Q_i of X_i as the ratio of the quantity of X_i to the per-unit value x_i:

$$Q_i = \frac{X_i}{x_i}.$$

With the factor capacities treated as random variables, the mean of the two-factor Leontief production function $Q=\min\{Q_1,Q_2\}$ was found through the double integral [6]:

$$EQ = \int_0^{+\infty} q_1 \left(\int_{q_1}^{+\infty} (p_{Q_1Q_2}(q_1,q_2) + p_{Q_1Q_2}(q_2,q_1))dq_2 \right) dq_1, \quad (1)$$

where $p_{Q_1Q_2}(q_1,q_2)$ stands for the joint density of the random variables Q_1 and Q_2. Based on formula (1), Mikheev established the following result: if Q_1 and Q_2 are independent and obey Weibull distributions with the same shape coefficient $\beta>0$, then EQ is expressed through the means EQ_1 and EQ_2 of Q_1 and Q_2 as the CES function

$$EQ = \left((EQ_1)^{-\beta} + (EQ_2)^{-\beta}\right)^{-\frac{1}{\beta}}.$$

Formula (1) leads to quite bulky calculations. However, another solution is possible: find the law (function or density) of distribution of the random variable $Q=\min\{Q_1,Q_2\}$ and derive the mean EQ by definition. If the random variables Q_1 and Q_2 were independent and identically distributed, this problem would turn into a well-known one of mathematical statistics: find the law of distribution of the minimum realization from a random sample with a given universe [7]. The advantage of this problem is the possibility to work with a random sample of any size *n* (i.e., *n* capacities $Q_1,\ldots,Q_n$ can be considered). Some modification of this problem is of interest for further research.

Note that only continuous models were considered in [2–6]; in reality, however, production factors or their capacities may be discrete variables. Here, we are concerned with the capacities of production factors as discretely distributed random variables and attempt to construct production functions on their basis. It is especially important to try reconstructing the CES function based on the discrete analog of the Weibull distribution using analytical methods.

According to the above considerations, we highlight several tasks:

- propose an effective method for finding the distribution law of the Leontief function from the capacities of production factors as independent random variables;
- show the possibility of obtaining the CES function from the means and medians of the capacities of *n* independent random production factors representing independent random variables with continuous Weibull distributions with the same shape coefficient;
- analyze discretely distributed capacities of production factors on the example of a geometric law;
- make an attempt to construct, by analytical methods, the CES function in the case of independent random capacities of production factors with discrete Weibull distributions with the same shape coefficient.

1. THE GENERAL PROPOSITION

Consider production with *n* non-fungible factors of capacities $Q_1,\ldots,Q_n$. For such factors, we can apply Leontief's production principle: the quantity of output is equal to the smallest of the capacities of the production factors used. In addition, we treat the capacities $Q_1,\ldots,Q_n$ as independent random variables.

The analysis below proceeds from the following result.

Proposition 1. *Let* $Q_1,\ldots,Q_n$ *be independent random variables, each having the distribution function*

$$F_{Q_i}(q) = \begin{cases} f_i(q), & q \ge b_i \\ 0, & q < b_i, \end{cases}$$

where b_i *are some numbers. Then the distribution function of the random variable*

$$Q=\min\{Q_1,\ldots,Q_n\}$$

is

$$F_Q(q) = 1 - \prod_{i=1}^{n}(1-F_{Q_i}(q)). \quad (2)$$

P r o o f.

$$F_Q(q) = P\{Q<q\} = 1-P\{Q\ge q\}$$

$$=1-P\{\min\{Q_1,...,Q_n\}\ge q\} = 1-P\{Q_1\ge q,...,Q_n\ge q\}.$$

But $P\{Q_1 \ge q,...,Q_n \ge q\} = P\{Q_1 \ge q\} \cdot ... \cdot P\{Q_n \ge q\}$ since $Q_1,...,Q_n$ are independent. Thus,

$$F_Q(q) = 1 - P\{Q_1 \ge q\} \cdot ... \cdot P\{Q_n \ge q\},$$

which finally gives formula (2). ♦

Remark. In this paper, we also deal with discretely distributed random variables $Q_1,\dots,Q_n$ with integer values 1, 2, 3, ... or 0, 1, 2, ... For such random variables, Proposition 1 remains valid. (For the sake of simplicity, imagine that b_i are integers and integers are selected from the set $q \ge b_i$.)

2. THE CASE OF CONTINUOUS VARIABLES

Let us represent Q_1 and Q_2 as the capacities of some production factors. Suppose that they are two independent continuous random variables obeying the Weibull laws [8] with the same shape coefficient $\beta > 0$ and coefficients $\alpha_1 > 0$ and $\alpha_2 > 0$, respectively:

$$F_{Q_1}(q) = \begin{cases} 1 - e^{-\alpha_1 q^\beta}, & q \ge 0 \\ 0, & q < 0, \end{cases}$$

$$F_{Q_2}(q) = \begin{cases} 1 - e^{-\alpha_2 q^\beta}, & q \ge 0 \\ 0, & q < 0. \end{cases}$$

Assume that Q (the quantity of output) depends on the capacities Q_1 and Q_2 by Leontief's production principle:

$$Q = \min\{Q_1, Q_2\}.$$

Applying Proposition 1 yields

$$F_Q(q) = \begin{cases} 1 - e^{-\alpha q^\beta}, & q \ge 0 \\ 0, & q < 0, \end{cases}$$

where $\alpha = \alpha_1 + \alpha_2$. Thus, the random variable Q has a Weibull distribution. Its mean is given by [8]

$$EQ = \alpha^{-\frac{1}{\beta}} \Gamma\left(\frac{1}{\beta} + 1\right) = \left(\alpha_1 + \alpha_2\right)^{-\frac{1}{\beta}} \Gamma\left(\frac{1}{\beta} + 1\right), \quad (3)$$

where Γ denotes the gamma function.

We calculate the coefficients α_1 and α_2 from

$$EQ_1 = \alpha_1^{-\frac{1}{\beta}} \Gamma\left(\frac{1}{\beta} + 1\right),$$

$$EQ_2 = \alpha_2^{-\frac{1}{\beta}} \Gamma\left(\frac{1}{\beta} + 1\right)$$

(the formulas for the means EQ_1 and EQ_2 of the random variables Q_1 and Q_2) and substitute the results into (3), thereby obtaining

$$EQ = \left((EQ_1)^{-\beta} + (EQ_2)^{-\beta}\right)^{-\frac{1}{\beta}}. \quad (4)$$

This formula defines the CES production function.

In the following, the special case $\beta = 1$ is also of interest. In this case, the Weibull distribution (for Q_1, Q_2, and Q) turns into the exponential one, and the expression (4) takes the form

$$EQ = \frac{EQ_1 EQ_2}{EQ_1 + EQ_2}. \quad (5)$$

These calculations can be generalized to the case of n factors. In addition, other characteristics of random variables (e.g., medians) can be considered instead of means.

Proposition 2. *Let Q_i $(i = 1,...,n)$ be the capacities of production factors represented as independent continuous random variables with Weibull distributions with the same shape coefficient $\beta > 0$ and coefficients $\alpha_1 > 0,\dots,\alpha_n > 0$:*

$$F_{Q_i}(q) = \begin{cases} 1 - e^{-\alpha_i q^\beta}, & q \ge 0 \\ 0, & q < 0. \end{cases}$$

In addition, let the quantity of output Q be determined by Leontief's production principle:

$$Q = \min\{Q_1,...,Q_n\}.$$

Then the mean EQ and median M of Q are expressed through the means EQ_i and medians M_i of Q_i, respectively, as CES production functions:

$$EQ = \left((EQ_1)^{-\beta} + ... + (EQ_n)^{-\beta}\right)^{-\frac{1}{\beta}}, \quad (6)$$

$$M = \left(M_1^{-\beta} + ... + M_n^{-\beta}\right)^{-\frac{1}{\beta}}. \quad (7)$$

P r o o f. Utilizing Proposition 1, we obtain the Weibull distribution function

$$F_Q(q) = \begin{cases} 1 - e^{-\alpha q^\beta}, & q \ge 0 \\ 0, & q < 0, \end{cases}$$

where $\alpha = \alpha_1 + \ldots + \alpha_n$. The mean EQ and median M of the random variable Q are given by

$$EQ = \alpha^{-\frac{1}{\beta}} \Gamma\left(\frac{1}{\beta} + 1\right) = \left(\alpha_1 + ... + \alpha_n\right)^{-\frac{1}{\beta}} \Gamma\left(\frac{1}{\beta} + 1\right),$$

$$M = \alpha^{-\frac{1}{\beta}} (\ln 2)^{\frac{1}{\beta}} = \left(\alpha_1 + ... + \alpha_n\right)^{-\frac{1}{\beta}} (\ln 2)^{\frac{1}{\beta}}.$$

By analogy, the means EQ_i and medians M_i of the random variables Q_i are given by

$$EQ_i = \alpha_i^{-\frac{1}{\beta}} \Gamma\left(\frac{1}{\beta} + 1\right),$$

$$M_i = \alpha_i^{-\frac{1}{\beta}} (\ln 2)^{\frac{1}{\beta}}.$$

Calculating the coefficients α_i from these formulas and substituting the results into the expressions for EQ and M, we finally arrive at (6) and (7). ♦

3. THE CASE OF DISCRETE VARIABLES

Now we attempt to represent the capacities of production factors as random variables with discrete distributions.

3.1. An Example: Geometric Distribution

Let Q_1 and Q_2 be the capacities of production factors described by two independent random variables with geometric distributions with parameters p_1 and p_2, respectively (the probabilities of success in single trials). Here, we understand the distribution in the following sense: the random variable is the trial number with the first success (possible values are $j = 1, 2, ...$).

Then the probabilities of failure in single trials are $q_1 = 1 - p_1$ and $q_2 = 1 - p_2$, respectively.

The distribution functions of the random variables Q_1 and Q_2 have the form

$$F_{Q_1}(j) = P\{Q_1 < j\} = 1 - q_1^{j-1},$$

$$F_{Q_2}(j) = P\{Q_2 < j\} = 1 - q_2^{j-1}.$$

Let the quantity of output Q be determined by Leontief's production principle:

$$Q = \min\{Q_1, Q_2\}.$$

Proposition 1 yields

$$F_Q(j) = 1 - q_0^{j-1},$$

where $q_0 = q_1 q_2$. Obviously, the random variable Q has a geometric distribution with the parameter $p_0 = 1 - q_0$. We express p_0 through the parameters p_1 and p_2:

$$p_0 = 1 - q_1 q_2 = 1 - (1 - p_1)(1 - p_2) = p_1 + p_2 - p_1 p_2.$$

As is well known,

$$EQ_i = \frac{1}{p_i}, \ i = 1, 2, \tag{8}$$

$$EQ = \frac{1}{p_0} = \frac{1}{p_1 + p_2 - p_1 p_2}. \tag{9}$$

Calculating p_1 and p_2 from (8) and substituting the results into formula (9), we obtain

$$EQ = \frac{EQ_1 EQ_2}{EQ_1 + EQ_2 - 1}. \tag{10}$$

Note that, with another definition of the geometric distribution used, the random variable is the number of failures before the first success. In this case,

$$EQ = \frac{EQ_1 EQ_2}{EQ_1 + EQ_2 + 1}. \tag{11}$$

Here is an elementary example of describing the capacities Q_1 and Q_2 of production factors by geometrically distributed random variables. Suppose that a trial to manufacture a single indivisible product requires consuming the unit capacity of either factor 1 or factor 2. A single trial may be successful (the product passes inspection and testing) or not (otherwise). Let p_1 and p_2 denote the probabilities of success when using factors 1 and 2, respectively. By assumption, the trial number does not affect the probability of success.

The number of the first successful trial to manufacture a single product using the selected factor is the realization of its capacity as a random variable, and this variable obeys the geometric distribution by definition. Obviously, it is advantageous to use the factor with the minimum number of the first success (a reference to Leontief's principle).

Of interest is some similarity of formulas (10), (11) with formula (5) obtained for exponentially distributed factors.

As is well known, the geometric distribution is a discrete analog of the exponential distribution. Let some random variable Q have an exponential distribution with the density function

$$p_Q(q) = \begin{cases} \lambda e^{-\lambda q}, & q \geq 0 \\ 0, & q < 0, \end{cases}$$
$$\lambda > 0.$$

Consider the random variable $Y = \lceil Q \rceil$, i.e., the ceiling of the variable Q. For natural numbers $j = 1, 2, ...$, we have

$$P\{Y = j\} = P\{j - 1 < Q \leq j\} = \int_{j-1}^{j} p_Q(q)dq,$$

implying

$$P\{Y = j\} = (1 - e^{-\lambda})e^{-\lambda(j-1)}.$$

With denoting $q_0 = e^{-\lambda} = 1 - p_0$, it follows that

$$P\{Y = j\} = (1 - q_0)q_0^{j-1} = p_0(1 - p_0)^{j-1}.$$

Thus, the variable Y has a geometric distribution with the parameter $p_0 = 1 - e^{-\lambda}$, being interpreted as the number of the first successful trial.

Note that the floor of $\lfloor Q \rfloor$ also obeys a geometric law with the parameter $p_0 = 1 - e^{-\lambda}$, meaning the number of failures before the first success.

3.2. Discretization of the Weibull Distribution and an Attempt to Construct the CES Function

Let us discretize the Weibull distribution by analogy. In this case, the density of the random variable Q has the form

$$p_Q(q) = \begin{cases} \alpha\beta q^{\beta-1} e^{-\alpha q^{\beta}}, & q \geq 0 \\ 0, & q < 0, \end{cases}$$
$$\beta > 0, \ \alpha > 0.$$

Consider the random variable $Y = \lfloor Q \rfloor$. For $j = 0, 1, 2, \dots$, we obtain

$$P\{Y = j\} = P\{j \le Q < j+1\} = \int_j^{j+1} p_Q(q)dq$$

and, after straightforward transformations,

$$P\{Y = j\} = e^{-\alpha j^\beta} - e^{-\alpha(j+1)^\beta}. \quad (12)$$

Then the distribution function of the random variable Y is given by

$$F_Y(j) = P\{Y < j\} = \sum_{k=0}^{j-1} P\{Y = k\} = 1 - e^{-\alpha j^\beta}, \quad (13)$$

representing the desired discrete Weibull distribution of type 1 [9, 10].

Let us calculate the median M of the random variable Y. For this purpose, we introduce the quantile Q_γ of level γ; its unrounded value is the solution of the equation

$$F_Y(Q_\gamma) = \gamma.$$

In view of formula (13), this equation becomes

$$1 - e^{-\alpha Q_\gamma^\beta} = \gamma.$$

After trivial transformations we obtain

$$Q_\gamma = \left(-\frac{\ln(1-\gamma)}{\alpha}\right)^{\frac{1}{\beta}},$$

and the unrounded value of the median is

$$M = Q_{1/2} = \left(\frac{\alpha}{\ln 2}\right)^{-\frac{1}{\beta}}. \quad (14)$$

Consider now the mean of the random variable Y given formula (12):

$$EY = \sum_{j=0}^{\infty} jP\{Y = j\} = \sum_{j=0}^{\infty} j\left(e^{-\alpha j^\beta} - e^{-\alpha(j+1)^\beta}\right). \quad (15)$$

This series is often calculated numerically [9]; in this paper, we endeavor to derive an analytical expression.

Assuming the convergence of the series (15), we open brackets in the expansion and combine the neighboring similar terms to get

$$EY = \sum_{j=1}^{\infty} (e^{-\alpha})^{j^\beta}. \quad (16)$$

Here we study the case $\beta > 1$. Then each term of the series (16) is smaller than the corresponding term of the convergent geometric progression series $\sum_{j=1}^{\infty} e^{-\alpha j}$; therefore, the series (16) will converge as well.

Of theoretical interest is the case $\beta = 2$ (a discrete analog of the Rayleigh distribution). In the remainder of the paper, we focus on this case, denoting the series (16) by $u(\alpha)$:

$$EY = u(\alpha) = \sum_{j=1}^{\infty} e^{-\alpha j^2} = (e^{-\alpha})^{1^2} + (e^{-\alpha})^{2^2} + (e^{-\alpha})^{3^2} + \dots \quad (17)$$

For the analysis, it seems reasonable to introduce the theta function

$$\theta(s) = \sum_{j=-\infty}^{\infty} (e^{-\pi s})^{j^2}$$

and the function

$$w(s) = \sum_{j=1}^{\infty} (e^{-\pi s})^{j^2} = \frac{1}{2}(\theta(s) - 1).$$

According to [11], the following functional equation is valid:

$$w(s) = \frac{1}{\sqrt{s}} w\left(\frac{1}{s}\right) + \frac{1}{2}\left(\frac{1}{\sqrt{s}} - 1\right).$$

Letting $\pi s = \alpha$, we relate the functions $u(\alpha)$ and $w(s)$ by

$$u(\alpha) = w(s) = w\left(\frac{\alpha}{\pi}\right);$$

hence,

$$u(\alpha) = \sqrt{\frac{\pi}{\alpha}} \cdot u\left(\frac{\pi^2}{\alpha}\right) + \frac{1}{2}\left(\sqrt{\frac{\pi}{\alpha}} - 1\right). \quad (18)$$

For the scale coefficients $0 < \alpha \lesssim 2$, the first term on the right-hand side of (18) can be neglected. In this case, we have the approximate equality

$$EY \approx \frac{1}{2}\left(\sqrt{\frac{\pi}{\alpha}} - 1\right), \quad 0 < \alpha \lesssim 2,$$

which can be written as

$$EY + \frac{1}{2} \approx \frac{\sqrt{\pi}}{2} \alpha^{-\frac{1}{2}}, \quad 0 < \alpha \lesssim 2. \quad (19)$$

For $\alpha \gtrsim 2$, the analysis of the expansion (17) can be restricted to the first term and, consequently,

$$EY \approx e^{-\alpha}, \quad \alpha \gtrsim 2.$$

Based on Proposition 1 and the above considerations for the discrete Weibull distribution, we formulate the following result.

Proposition 3. *Let* Q_i $(i=1,...,n)$ *be the capacities of production factors represented as independent random variables with discrete Weibull distributions with the same shape coefficient* $\beta>0$ *and coefficients* $\alpha_1>0,\ldots,\alpha_n>0$:

$$F_{Q_i}(j)=1-e^{-\alpha_i j^{\beta}}, \quad j = 0, 1, 2, ...$$

In addition, let the quantity of output Q *be determined by Leontief's production principle:*

$$Q=\min\{Q_1,...,Q_n\}.$$

Then the unrounded median M (14) *of the variable* Q *is related to the unrounded medians* M_i *of the variables* Q_i *through a CES function:*

$$M=\left(M_1^{-\beta}+...+M_n^{-\beta}\right)^{-\frac{1}{\beta}}.$$

Moreover, if $\beta=2$ *and* $0<\alpha\lesssim 2$, *where* $\alpha=\alpha_1+...+\alpha_n$, *then the mean* EQ *of* Q *can be approximately related to the means* EQ_i *of* Q_i *through a CES function:*

$$EQ+\frac{1}{2}\approx\left((EQ_1+\frac{1}{2})^{-2}+...+(EQ_n+\frac{1}{2})^{-2}\right)^{-\frac{1}{2}}.$$

Note to Proposition 3. Under the constraint $0<\alpha=\alpha_1+\ldots+\alpha_n\lesssim 2$, the inequalities $0<\alpha_i\lesssim 2$, $i=1,...,n$, hold immediately. Hence, the means EQ_i can be expressed in the desired approximate form (19).

CONCLUSIONS

In this paper, we have obtained CES functions for the means and medians of the capacities of *n* production factors in the case where the capacities are represented as independent random variables with continuous Weibull distributions with the same shape coefficient.

We have proposed to consider discretely distributed capacities of production factors on the example of a geometric law. In this case, according to Leontief's production principle, it is advantageous to use the factor with the minimum number of the first successful trial (product manufacturing).

Also, we have endeavored to construct the CES function in the case of independent random factor capacities with discrete Weibull distributions with the same shape coefficient. As a result, the unrounded values (14) of the medians of factor capacities and the median of the quantity of output have been successfully related. However, difficulties arise when establishing a relationship between the means of these variables.

The main challenge in this study has been to derive, in analytical terms, the mean of a random variable distributed according to the discrete Weibull law. In the special case $\beta=2$ (the shape coefficient), it is possible to introduce the theta function and compile a functional equation. With some restrictions on the values of the scale coefficient of the distribution, it is possible to neglect some part of the functional equation, thereby approximating the required mean (see formula (19)) and deriving the CES function.

In the *general case* (no restrictions on the distribution coefficients), a still open issue is the possibility of relating the means of the capacities of production factors represented as random variables with discrete Weibull distributions with the same shape coefficient.

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