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# UPPER BOUNDS ON TRAJECTORY DEVIATIONS FOR AN AFFINE FAMILY OF DISCRETE-TIME SYSTEMS UNDER EXOGENOUS DISTURBANCES

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**Abstract.** We propose a simple upper bound on trajectory deviations for an affine family of discrete-time systems under nonzero initial conditions subjected to bounded exogenous disturbances. It involves the design of a parametric quadratic Lyapunov function for the system. The apparatus of linear matrix inequalities and the method of invariant ellipsoids are used as technical tools. The original problem is reduced to a parametric semidefinite programming problem, which is easily solved numerically. Numerical simulation results demonstrate the relatively low conservatism of the upper bound. This paper continues the series of our previous publications on estimating trajectory deviations for linear continuous- and discrete-time systems with parametric uncertainty and exogenous disturbances. The results presented below can be extended to various robust formulations of the original problem and also the problem of minimizing trajectory deviations for an affine family of discrete-time control systems under exogenous disturbances via linear feedback.

**Keywords**: linear discrete-time system, trajectory deviations, parametric Lyapunov function, bounded exogenous disturbances, linear matrix inequalities, invariant ellipsoids.

## INTRODUCTION

When investigating transients in linear systems, the behavior of the entire system trajectory is of great interest. In this case, the maximum deviation of the trajectory from zero is a crucial characteristic of transients.

There exist different methods for estimating trajectory deviations for a dynamic system; for example, see a survey in the paper [1]. In particular, a regular approach was proposed therein to estimate the maximum deviation for a linear continuous-time system; also, an approach was developed to minimize trajectory deviations via a static linear state feedback law based on linear matrix inequalities (LMIs). The latter approach was extended in [2] to discrete-time systems with structured matrix uncertainty.

Another important and promising line of research in this area concerns the localization method for invariant compact sets. Here, we mention the works of Russian researchers, A.P. Krishchenko, A.N. Kanatnikov, and S.K. Korovin; for example, see the papers [3–6].

The case where uncertain parameters are matrix elements is not common in practice: usually, the matrix coefficients have no direct physical meaning and depend on the parameters in a more sophisticated way. Affine uncertainty is the simplest model of such a *dependent* uncertainty structure; see the monograph [7] for details.

Discrete-time systems with parametric uncertainty were studied in [8–10]. From a technical point of view, the cited works involved the approach from [11]. This approach allows separating the system matrix and the Lyapunov function matrix in a matrix inequality expressing a sufficient condition for the stability of the family under consideration. At the same time, less conservative estimates are obtained by designing a parametric quadratic Lyapunov function.

Also, note the publications [12, 13] devoted to a close topic.

In this paper, we continue the described line of research to derive upper bounds on trajectory deviations for an affine family of discrete-time systems with nonzero initial conditions subjected to bounded exogenous disturbances. The main technical tool is the apparatus of LMIs [14, 15].

The following notations are used below: $\|\cdot\|$ means the spectral norm of a matrix and the Euclidean norm of a vector; $^{\mathrm{T}}$ is the transpose symbol; $I$ denotes an identity matrix of appropriate dimensions. All matrix inequalities are understood in the sense of sign definiteness of corresponding matrices.

## 1. PROBLEM STATEMENT AND SOLUTION APPROACH

Consider a linear discrete-time dynamic system described by

$$x_{k+1} = A(\alpha)x_k + Dw_k \qquad (1)$$

with the state vector $x_k \in \mathbb{R}^n$, a *nonzero* initial condition $x_0$, and an exogenous disturbance $w_k \in \mathbb{R}^m$ satisfying the constraint

$$\|w_k\| \le 1, \quad k = 1, 2, \ldots \qquad (2)$$

Here $D \in \mathbb{R}^{n\times m}$, and Schur matrices $A(\alpha) \in \mathbb{R}^{n\times n}$ belong to the convex family

$$\mathbb{A} = \left\{A(\alpha) : A(\alpha) = \sum\nolimits_{i=1}^{N} \alpha_i A_i, \ \sum\nolimits_{i=1}^{N} \alpha_i = 1, \ \alpha_i \ge 0\right\}. \qquad (3)$$

As is well known, a sufficient condition for the robust quadratic stability of a linear system consists in the existence of a *common* quadratic Lyapunov function

$$V(x) = x^{\mathrm{T}} P^{-1} x, \quad P > 0.$$

According to [8–10], the approach based on designing a *parametric* quadratic Lyapunov function

$$V(x) = x^{\mathrm{T}} P^{-1}(\alpha) x, \quad P(\alpha) > 0,$$

yields significantly less conservative estimates. In addition, the following assertion was established in [8] for system (1)–(3).

**Theorem 1.** *Assume that there exist matrices* $0 < P_i = P_i^{\mathrm{T}} \in \mathbb{R}^{n\times n}$ *and* $G \in \mathbb{R}^{n\times n}$ *such that*

$$\begin{pmatrix} P_i & A_i G & D \\ G^{\mathrm{T}} A_i^{\mathrm{T}} & \mu(G + G^{\mathrm{T}} - P_i) & 0 \\ D^{\mathrm{T}} & 0 & (1-\mu)I \end{pmatrix} \ge 0, \quad i = 1, \ldots, N,$$

*for some* $0 < \mu < 1$.

*Then system* (1)–(3) *has a parametric quadratic Lyapunov function with the matrix*

$$P(\alpha) = \sum\nolimits_{i=1}^{N} \alpha_i P_i.$$

This paper mainly aims at estimating from above trajectory deviations for the family (1) under the exogenous disturbance (2).

For a discrete-time system, the maximum deviation of the trajectory from zero in transients is given by

$$\xi^* = \max_{k=1,2,\ldots} \max_{\|x_0\|=1} \|x_k\|.$$

Estimation of the value $\xi^*$ is very difficult [1], but the method of invariant ellipsoids with the technique of LMIs yields simple upper bounds on this value.

Recall the following well-known result. A matrix $P > 0$ of a quadratic Lyapunov function for some dynamic system defines the so-called *invariant* ellipsoid

$$\mathcal{E} = \left\{x \in \mathbb{R}^n : \quad x^{\mathrm{T}} P^{-1} x \le 1\right\}, \quad P > 0.$$

In other words, a system trajectory starting at any point of the invariant ellipsoid will remain there. Hence, *for any* initial condition from the ball $\mathcal{B} = \{x \in \mathbb{R}^n : \|x\| \le 1\}$ contained in the ellipsoid, we have the upper bound

$$\|x_k\| \le \lambda_{\max}(P) = \sqrt{\|P\|}$$

for any time instant.

In view of this fact, our aim is to find a *minimum* invariant ellipsoid associated with the matrix $P = P(\alpha)$ of the parametric quadratic Lyapunov function for the family under consideration.

Since

$$\|P(\alpha)\| = \left\|\sum\nolimits_{i=1}^{N} \alpha_i P_i\right\| \le \sum\nolimits_{i=1}^{N} \alpha_i \|P_i\| \le$$
$$\sum\nolimits_{i=1}^{N} \alpha_i \max_i \|P_i\| \le \max_i \|P_i\|,$$

within the proposed approach we will minimize the upper bound on the major semiaxis of the invariant ellipsoid with the matrix $P(\alpha)$, i.e., the value

$$\max_i \|P_i\|.$$

Further, the condition $\mathcal{B} \subseteq \mathcal{E}$ is equivalent to the requirement

$$P(\alpha) \ge I$$

and is ensured by $P_i \ge I$, $i = 1, \ldots, N$. Indeed,

$$P(\alpha) = \sum\nolimits_{i=1}^{N} \alpha_i P_i \ge \sum\nolimits_{i=1}^{N} \alpha_i I = I.$$

According to Schur's complement lemma, the matrix inequality

$$\begin{pmatrix} P_i & A_i G & D \\ G^{\mathrm{T}} A_i^{\mathrm{T}} & \mu\left(G+G^{\mathrm{T}}-P_i\right) & 0 \\ D^{\mathrm{T}} & 0 & (1-\mu)I \end{pmatrix} \geq 0$$

can be equivalently written as

$$\begin{pmatrix} P_i - \frac{1}{1-\mu}DD^{\mathrm{T}} & A_i G \\ G^{\mathrm{T}} A_i^{\mathrm{T}} & \mu\left(G+G^{\mathrm{T}}-P_i\right) \end{pmatrix} \geq 0 .$$

Due to the upper bound

$$\|x_k\| \leq \sqrt{\|P(\alpha)\|} \leq \max_{i=1,\ldots,N} \sqrt{\|P_i\|},$$

we therefore arrive at the following assertion.

**Theorem 2.** *Let* $P_i^*$, *i* = 1, ..., *N, be the solution of the convex optimization problem*

$$\min \max_{i=1,\ldots,N} \|P_i\|$$

*subject to the constraints*

$$\begin{pmatrix} P_i - \frac{1}{1-\mu}DD^{\mathrm{T}} & A_i G \\ G^{\mathrm{T}} A_i^{\mathrm{T}} & \mu\left(G+G^{\mathrm{T}}-P_i\right) \end{pmatrix} \geq 0 ,$$

$$P_i \geq I,\ i = 1,\ldots,N,$$

*with respect to the matrix variables* $P_i = P_i^{\mathrm{T}} \in \mathbb{R}^{n\times n}$ and $G \in \mathbb{R}^{n\times n}$ *and the scalar parameter* $0 < \mu < 1$, *where the matrices* $A_i$, $D$ *are given by*(1), (3).

*Then the solutions of system* (1) *under all admissible exogenous disturbances* (2) *have the upper bound*

$$\|x_k\| \leq \max_{i=1,\ldots,N} \sqrt{\|P_i^*\|} .$$

The optimization problem in Theorem 2 is a parametric semidefinite programming problem. It can be easily solved numerically through one-dimensional optimization by varying the parameter μ within the range (0, 1). In particular, the CVX package [16] can be effectively used in MATLAB.

## 2. AN EXAMPLE

Consider the system from [8] in a slightly modified form:

$$A_1 = \begin{pmatrix} 0.0061 & -0.2630 & 0.2748 \\ 0.1266 & 0.1242 & -0.3029 \\ -0.5100 & 0.4678 & -0.9712 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 0.1330 & 0.2009 & 0.1672 \\ 0.1224 & -0.5987 & 0.3100 \\ -0.5235 & 0.0297 & -0.4784 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} -0.2733 & -0.1868 & -0.0077 \\ -0.0253 & -0.2828 & 0.6112 \\ -0.2412 & -0.0844 & -0.8024 \end{pmatrix},$$

$$D = \begin{pmatrix} -0.4 \\ -0.5 \\ 0.2 \end{pmatrix}.$$

Solving the one-dimensional optimization problem in Theorem 2 yields (for μ = 0.873) the matrices

$$P_1^* = \begin{pmatrix} 4.0127 & 0.4418 & -2.1495 \\ 0.4418 & 4.3296 & 0.6922 \\ -2.1495 & 0.6922 & 2.8445 \end{pmatrix},$$

$$P_2^* = \begin{pmatrix} 2.7515 & 0.7640 & -1.2374 \\ 0.7640 & 4.8052 & -0.1782 \\ -1.2374 & -0.1782 & 2.2964 \end{pmatrix},$$

$$P_3^* = \begin{pmatrix} 2.2435 & 1.9108 & -0.7729 \\ 1.9108 & 3.9362 & -1.1877 \\ -0.7729 & -1.1877 & 1.4804 \end{pmatrix}$$

of the parametric Lyapunov function and the matrix

$$G^* = \begin{pmatrix} 2.7991 & -0.0704 & -1.4510 \\ 2.2860 & 5.1487 & -0.6981 \\ -1.0359 & 0.2141 & 1.9738 \end{pmatrix}.$$

Hence,

$$\sqrt{\|P_1^*\|} = 2.3791,\quad \sqrt{\|P_2^*\|} = 2.2774,\quad \sqrt{\|P_3^*\|} = 2.3791,$$

and finally we have the upper bound

$$\|x_k\| \leq \max\left\{\sqrt{\|P_1^*\|}, \sqrt{\|P_2^*\|}, \sqrt{\|P_3^*\|}\right\} = \sqrt{\|P_1^*\|} = 2.3791 .$$

For comparison, the common quadratic Lyapunov function for this system, found according to [5], has the matrix

$$P_{comm}^* = \begin{pmatrix} 27.1113 & -9.3697 & -23.8293 \\ -9.3697 & 76.0285 & 6.4098 \\ -23.8293 & 6.4098 & 47.9982 \end{pmatrix},$$

yielding more than triple the rough estimate:

$$\|x_k\| \leq \sqrt{\|P_{comm}^*\|} = 9.0666 .$$

Figure 1 shows the projections of the ellipsoids with the matrices $P_1^*$, $P_2^*$, and $P_3^*$ (the thin solid lines) and the invariant ellipsoid with the matrix $P_{comm}$ (the thick dashed line) on the plane ($x_2$, $x_3$); the dotted line corresponds to the projection of a unit sphere.

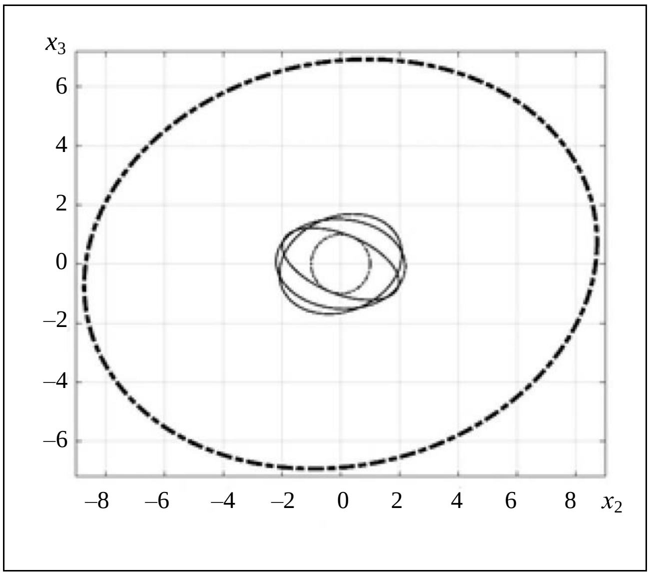


**Fig. 1. The projection of ellipsoids on the plane ($x_2$, $x_3$).**

Figure 2 shows the central part of Fig. 1 and the projection of the system trajectory under the initial condition

$$x_0 = \begin{pmatrix} 0 \\ 0.1391 \\ -0.9903 \end{pmatrix}, \ \|x_0\| = 1,$$

and the admissible exogenous disturbance

$$w_k = \mathrm{sign}(\sin(k/4)\cos(k/7)), \quad k = 1, 2, \ldots$$

(the dotted line).

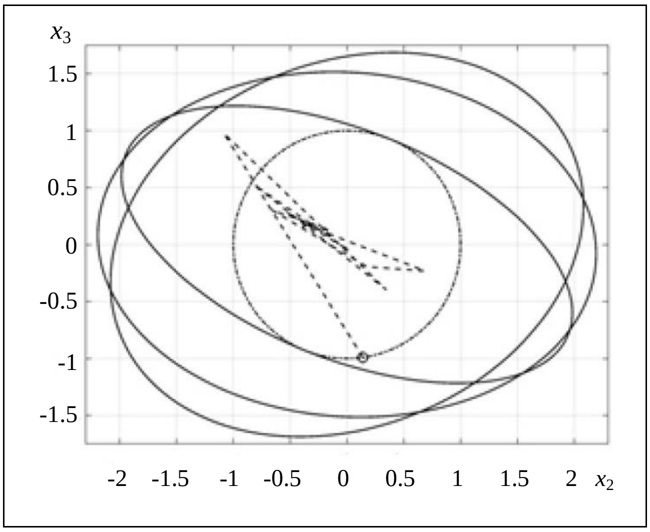


**Fig. 2. The projection of ellipsoids and system trajectory on the plane ($x_2$, $x_3$).**

Fig. 3 shows the dynamics of the value $\|x_k\|$ (the solid line) and its upper bound (the dashed line).

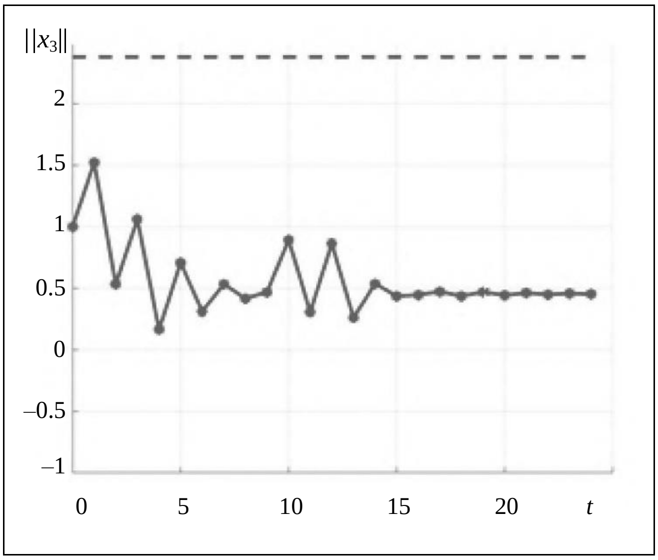


**Fig. 3. Dynamics of $\|x_k\|$ and its upper bound.**

## 3. CONCLUSIONS

This paper has presented a simple upper bound on a crucial characteristic of transients—the maximum trajectory deviation from zero—for an affine family of discrete-time systems with nonzero initial conditions subjected to bounded exogenous disturbances. Developing our previous research works, the estimation approach proposed above involves the design of a parametric quadratic Lyapunov function for the system under consideration. The apparatus of linear matrix inequalities and the method of invariant ellipsoids are used as technical tools. The original problem has been reduced to a parametric semidefinite programming problem, which is easily solved numerically, particularly in MATLAB using the CVX package.

We expect to extend these results to various robust formulations of the original problem and the problem of minimizing trajectory deviations for an affine family of discrete-time control systems under exogenous disturbances via linear feedback.

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