DOI: http://doi.org/10.25728/cs.2021.4.2

A PARAMETRIC LYAPUNOV FUNCTION FOR DISCRETE-TIME CONTROL SYSTEMS WITH BOUNDED EXOGENOUS DISTURBANCES: ANALYSIS

M.V. Khlebnikov[1] and Ya.I. Kvinto[2]

Trapeznikov Institute of Control Sciences, Russian Academy of Sciences, Moscow, Russia

[1] khlebnik@ipu.ru, [2] yanakvinto@mail.ru

Abstract. This paper considers a linear discrete-time dynamic system subjected to arbitrary bounded exogenous disturbances described by a matrix from a convex affine family. A simple approach to designing a parametric quadratic Lyapunov function for this system is proposed. It involves linear matrix inequalities and a fruitful technique to separate the system matrix and the Lyapunov function matrix in the matrix inequality expressing a stability condition of the system. Being well known, this technique, however, has not been previously applied to dynamic systems with nonrandom bounded exogenous disturbances. According to the numerical simulations, the parametric quadratic Lyapunov function-based approach yields appreciably less conservative results for the class of systems under consideration than the common quadratic Lyapunov function-based one.

Keywords: dynamic system, linear discrete-time system, parametric quadratic Lyapunov function, common quadratic Lyapunov function, bounded exogenous disturbances, robustness, linear matrix inequalities, analysis problem, conservatism, structured matrix uncertainty.

INTRODUCTION

The study of dynamic systems with parametric uncertainty and exogenous disturbances is of great theoretical and practical interest. A classical approach to solving this class of problems is based on designing a common quadratic Lyapunov function for the entire family of systems [1–4], and the apparatus of linear matrix inequalities [5] is a convenient technique.

However, as is well known [6], using a common Lyapunov function often yields rather conservative results. In this regard, let us consider the problem of designing a parametric Lyapunov function for continuous- and discrete-time systems with uncertainty. In the papers [6–8], the advantages of a parametric quadratic Lyapunov function were presented; as shown therein, this approach decreases the solution's degree of conservatism compared to the common quadratic Lyapunov function-based one. Among the relatively recent publications on the subject, we mention, e.g., [1, 9–11]. In the paper [8], an efficient method was proposed for designing a parametric quadratic Lyapunov function using linear matrix inequalities when analyzing the stability of an affine family of continuous-time systems. Later on [7], this approach was extended to the case of discrete-time systems with parametric uncertainty. In the paper [12], the result of [7] was generalized to the case of a discrete-time system with parametric and structured matrix uncertainty.

In this paper, we design a parametric quadratic Lyapunov function for an affine family of discrete-time systems subjected to arbitrary bounded exogenous disturbances. Such problems often arise in applications and have a transparent physical motivation; for example, see [9, 13]. As an important technique, we employ and generalize an equivalent representation of the system's stability condition in the form of a matrix inequality, first proposed in [7] (also, see [14]). This representation allows separating the system matrix and the Lyapunov function matrix in the matrix inequality. The technique gave rise to several further generalizations, but it has not been previously applied to dynamic systems with nonrandom bounded exogenous disturbances. The corresponding assertion (see Theorem 1 below) is novel, like Theorem 2 obtained on its basis. According to the numerical simulations, the parametric quadratic Lyapunov function-based approach yields appreciably less conservative results for the class of systems under consideration than the common quadratic Lyapunov function-based one.

A rather close problem was examined in [6], where new sufficient conditions for robust quadratic stability of a discrete-time system with parametric uncertainty were established. The problem statement introduced below is more general, and the apparatus of linear matrix inequalities is used as a convenient technique.

This paper is organized as follows. Section 1 describes the problem statement and the approach to solve it. The main results are formulated in Section 2. The numerical simulation results are presented and discussed in Section 3.

Throughout the paper, $\|\cdot\|$ denotes the Euclidean norm of vectors and the spectral norm of matrices, the symbol $^{\mathrm{T}}$ indicates transposition, I is an identity matrix of compatible dimensions, and all matrix inequalities are understood in the sense of positive or negative (semi-) definiteness of the corresponding matrices.

1. PROBLEM STATEMENT AND SOLUTION APPROACHES

Consider a linear discrete-time dynamic system described by

$$x_{k+1} = A(\alpha)x_k + Dw_k, \tag{1}$$

where $x_k \in \mathbb{R}^n$ denotes the state vector, x_0 is an initial condition, and $w_k \in \mathbb{R}^m$ is an exogenous disturbance satisfying the constraint

$$\| w_k \| \le 1, \quad k = 1, 2, \ldots \tag{2}$$

Let $D \in \mathbb{R}^{n\times m}$ and the matrices $A(\alpha) \in \mathbb{R}^{n\times n}$ belong to the convex family

$$\mathbb{A} = \left\{ A(\alpha):\ A(\alpha) = \sum_{i=1}^{N} \alpha_i A_i,\ \sum_{i=1}^{N} \alpha_i = 1,\ \alpha_i \ge 0 \right\}. \tag{3}$$

Assume that the system (1) is stable: all matrices $A(\alpha) \in \mathbb{A}$ are Schur (their eigenvalues lie inside the unit circle), and the pair (A, D) is controllable.

The main problem is to design a parametric quadratic Lyapunov function for the system (1) and (2).

First of all, we discuss an approach to designing a parametric quadratic Lyapunov function for the dynamic system

$$x_{k+1} = Ax_k + Dw_k,\quad \| w_k \| \le 1,\quad k = 1, 2, \ldots, \tag{4}$$

with matrices $A \in \mathbb{R}^{n\times n}$ and $D \in \mathbb{R}^{n\times m}$, the state vector $x_k \in \mathbb{R}^n$, an initial condition x_0, and an exogenous disturbance $w_k \in \mathbb{R}^m$ (2).

According to the paper [15] (also, see the monograph [16]), a matrix $0 < P \in \mathbb{R}^{n\times n}$ satisfying the linear matrix inequality

$$\frac{1}{\mu} APA^{\mathrm{T}} - P + \frac{1}{1-\mu} DD^{\mathrm{T}} \le 0 \tag{5}$$

for some $0 < \mu < 1$ will define a quadratic Lyapunov function of the form

$$V(x) = x^{\mathrm{T}} P^{-1} x$$

for the system (4) and (2).

For further presentation, we need the following technical result.

Theorem 1. *The assertions below are equivalent:*

I. *There exists a matrix $P > 0$ such that*

$$\frac{1}{\mu} APA^{\mathrm{T}} - P + \frac{1}{1-\mu} DD^{\mathrm{T}} \le 0 \tag{6}$$

for some $0 < \mu < 1$.

II. *There exist matrices $0 < P = P^{\mathrm{T}} \in \mathbb{R}^{n\times n}$ and $G \in \mathbb{R}^{n\times n}$ such that*

$$\begin{pmatrix} P & AG & D \\ G^{\mathrm{T}} A^{\mathrm{T}} & \mu\left(G + G^{\mathrm{T}} - P\right) & 0 \\ D^{\mathrm{T}} & 0 & (1-\mu)I \end{pmatrix} \ge 0 \tag{7}$$

for some $0 < \mu < 1$.

P r o o f. Applying Schur's complement lemma to the matrix inequality (7) two sequential times gives the equivalent relations

$$\begin{pmatrix} P - \frac{1}{1-\mu} DD^{\mathrm{T}} & AG \\ G^{\mathrm{T}} A^{\mathrm{T}} & \mu\left(G + G^{\mathrm{T}} - P\right) \end{pmatrix} \ge 0 \tag{8}$$

and

$$P - \frac{1}{1-\mu} DD^{\mathrm{T}} - \frac{1}{\mu} AG(G + G^{\mathrm{T}} - P)^{-1} G^{\mathrm{T}} A^{\mathrm{T}} \ge 0. \tag{9}$$

Letting $G = G^{\mathrm{T}} = P$ in (9), we arrive at inequality (6). Thus, Assertion I implies Assertion II.

Now, we prove the converse. Multiplying (8) on the left by $\begin{pmatrix} I & -\frac{1}{\mu} A \end{pmatrix}$ and on the right by $\begin{pmatrix} I \\ -\frac{1}{\mu} A^{\mathrm{T}} \end{pmatrix}$ yields

$$\begin{pmatrix} I & -\frac{1}{\mu} A \end{pmatrix} \begin{pmatrix} P - \frac{1}{1-\mu} DD^{\mathrm{T}} & AG \\ G^{\mathrm{T}} A^{\mathrm{T}} & \mu\left(G + G^{\mathrm{T}} - P\right) \end{pmatrix} \begin{pmatrix} I \\ -\frac{1}{\mu} A^{\mathrm{T}} \end{pmatrix} \ge 0$$

or $P - \frac{1}{1-\mu} DD^{\mathrm{T}} - \frac{1}{\mu} AG^{\mathrm{T}}A^{\mathrm{T}} - \frac{1}{\mu} AGA^{\mathrm{T}} + \frac{1}{\mu} A\big(G + G^{1} - P\big)A^{\mathrm{T}} \geq 0.$ This condition is equivalent to (6). The proof of Theorem 1 is complete. ♦

Well, Theorem 1 reduces inequality (5) to the equivalent form (7), linear in the matrix variables P and A.

Let us return to the system (1). Due to the convexity of (3), a solution $P > 0$ of the combined matrix inequalities

$$\frac{1}{\mu} A_i P A_i^{\mathrm{T}} - P + \frac{1}{1-\mu} DD^{\mathrm{T}} \leq 0, \quad 0 < \mu < 1, \quad i = 1, \ldots, N,$$

or, by Theorem 1, of the equivalent ones

$$\begin{pmatrix} P & A_i G & D \\ G^{\mathrm{T}} A_i^{\mathrm{T}} & \mu\big(G + G^{\mathrm{T}} - P\big) & 0 \\ D^{\mathrm{T}} & 0 & (1-\mu)I \end{pmatrix} \geq 0, \quad 0 < \mu < 1, \tag{10}$$

will define the common quadratic Lyapunov function $V(x) = x^{\mathrm{T}} P^{-1} x$ for the affine family (1)–(3).

Note a rather high degree of conservatism for this result: an appropriate matrix $P > 0$ must satisfy all inequalities (10) for the same value μ.

Appreciably less conservative results are obtained using the *parametric* quadratic Lyapunov function $V(x) = x^{\mathrm{T}} P^{-1}(\alpha) x$ with the matrix

$$P(\alpha) = \sum_{i=1}^{N} \alpha_i P_i, \quad 0 < P_i = P_i^{\mathrm{T}} \in \mathbb{R}^{n\times n}. \tag{11}$$

Resting on Theorem 1, we will establish a sufficient condition for the existence of a parametric quadratic Lyapunov function (11) for the affine family (1)–(3); see the next section.

2. MAIN RESULT

Due to the convexity of (3) and the structure of the parametric quadratic Lyapunov function (11), it suffices to require that the component P_i correspond to the vertex A_i in the system (1). The following assertion is the main result of the paper.

Theorem 2. *Assume that there exist matrices* $0 < P_i = P_i^{\mathrm{T}} \in \mathbb{R}^{n\times n}$ *and* $G \in \mathbb{R}^{n\times n}$ *such that*

$$\begin{pmatrix} P_i & A_i G & D \\ G^{\mathrm{T}} A_i^{\mathrm{T}} & \mu\big(G + G^{\mathrm{T}} - P_i\big) & 0 \\ D^{\mathrm{T}} & 0 & (1-\mu)I \end{pmatrix} \geq 0, \quad i = 1, \ldots, N, \tag{12}$$

for some $0 < \mu < 1$.

Then the system (1), (3), and (2) *has a parametric quadratic Lyapunov function with the matrix*

$$P(\alpha) = \sum_{i=1}^{N} \alpha_i P_i. \tag{13}$$

P r o o f. Due to (5), a matrix $P_i > 0$ satisfying the matrix inequality

$$\frac{1}{\mu} A_i P_i A_i^{\mathrm{T}} - P_i + \frac{1}{1-\mu} DD^{\mathrm{T}} \leq 0 \tag{14}$$

for some $0 < \mu < 1$ will define the quadratic Lyapunov function $V(x) = x^{\mathrm{T}} P_i^{-1} x$ for the system (1) at the vertex A_i.

According to Theorem 2, condition (14) is equivalent to the matrix inequality

$$\begin{pmatrix} P_i & A_i G & D \\ G^{\mathrm{T}} A_i^{\mathrm{T}} & \mu\big(G + G^{\mathrm{T}} - P_i\big) & 0 \\ D^{\mathrm{T}} & 0 & (1-\mu)I \end{pmatrix} \geq 0 \tag{15}$$

for some $0 < \mu < 1$.

Multiplying (15) by α_i and summing over $i = 1, \ldots, N$, we obtain

$$\begin{pmatrix} \sum_{i=1}^{N} \alpha_i P_i & \left(\sum_{i=1}^{N} \alpha_i A_i\right) G & \sum_{i=1}^{N} \alpha_i D \\ G^{\mathrm{T}} \sum_{i=1}^{N} \alpha_i A_i^{\mathrm{T}} & \mu\left(\sum_{i=1}^{N} \alpha_i \big(G + G^{\mathrm{T}}\big) - \sum_{i=1}^{N} \alpha_i P_i\right) 0 & \\ \sum_{i=1}^{N} \alpha_i D^{\mathrm{T}} & 0 & \sum_{i=1}^{N} \alpha_i (1-\mu) I \end{pmatrix} \geq 0.$$

Since $\sum_{i=1}^{N} \alpha_i A_i = A(\alpha)$ and $\sum_{i=1}^{N} \alpha_i = 1$,

$$\begin{pmatrix} \sum_{i=1}^{N} \alpha_i P_i & A(\alpha) G & D \\ G^{\mathrm{T}} A^{\mathrm{T}}(\alpha) & \mu\left(G + G^{\mathrm{T}} - \sum_{i=1}^{N} \alpha_i P_i\right) 0 & \\ D^{\mathrm{T}} & 0 & (1-\mu) I \end{pmatrix} \geq 0.$$

Thus, the matrix (13) defines the parametric quadratic Lyapunov function $V(x) = x^{\mathrm{T}} P^{-1}(\alpha) x$ for the system (1)–(3). The proof of Theorem 2 is complete. ♦

Clearly, this approach is conservative primarily because conditions (12) must hold for the same value μ. However, as we will see below, the proposed approach yields less conservative estimates than the common quadratic Lyapunov function-based one.

3. AN EXAMPLE

As an illustrative example, we consider the system from [4] with the vertices

$$A_1 = \begin{pmatrix} 0.0061 & -0.2630 & 0.2748 \\ 0.1266 & 0.1242 & -0.3029 \\ -0.5100 & 0.4678 & -0.9712 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 0.1330 & 0.2009 & 0.1672 \\ 0.1224 & -0.5987 & 0.3100 \\ -0.5235 & 0.0297 & -0.4784 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} -0.2733 & -0.1868 & -0.0077 \\ -0.0253 & -0.2828 & 0.6112 \\ -0.2412 & -0.0844 & -0.8024 \end{pmatrix},$$

and the bounded exogenous disturbances with

$$D = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

Using Theorem 2, we solve the corresponding optimization problem $\min \sum_i \| P_i \|$ and find the matrices

$$\hat{P}_1 = \begin{pmatrix} 1.1381 & 0.8630 & -0.2336 \\ 0.8630 & 1.1608 & 0.2290 \\ -0.2336 & 0.2290 & 0.8764 \end{pmatrix} \cdot 10^4,$$

$$\hat{P}_2 = \begin{pmatrix} 1.1552 & 0.8149 & -0.2110 \\ 0.8149 & 1.3914 & 0.1546 \\ -0.2110 & 0.1546 & 0.3824 \end{pmatrix} \cdot 10^4,$$

and $$\hat{P}_3 = \begin{pmatrix} 1.2724 & 1.1669 & 0.1901 \\ 1.1669 & 1.4113 & -0.2095 \\ 0.1901 & -0.2095 & 0.5410 \end{pmatrix} \cdot 10^4$$

of the parametric quadratic Lyapunov function $V(x) = x^{\mathrm{T}} \left(\sum_{i=1}^{3} \alpha_i \hat{P}_i \right) x.$

For comparison, we calculate the matrix of the common quadratic Lyapunov function for this system by solving the optimization problem $\min \| P \|$ subject to the constraint (10):

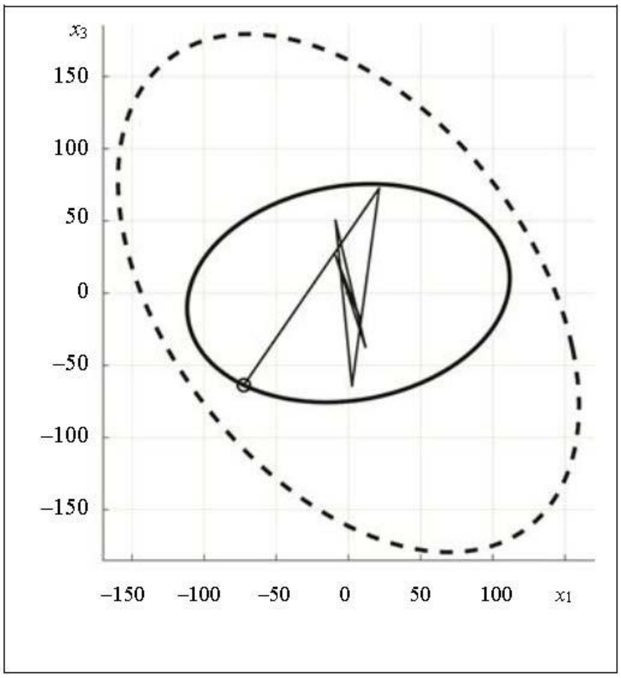
$$\hat{P} = \begin{pmatrix} 2.5519 & -0.3322 & -1.2518 \\ -0.3322 & 4.7636 & 0.5922 \\ -1.2518 & 0.5922 & 3.2176 \end{pmatrix} \cdot 10^4.$$

As is well known, the matrix of a quadratic Lyapunov function is associated with the so-called invariant ellipsoid. (For more details, see the monograph [16].) Recall that the system's trajectory starting at a point inside the invariant ellipsoid will remain there under all admissible exogenous disturbances.

Let us compare the invariant ellipsoids determined by the parametric and common quadratic Lyapunov functions. The figure shows the projections of the corresponding invariant ellipsoids onto the plane (x_1, x_3). The noticeable differences confirm the reduced degree of conservatism of the proposed approach.

Also, this figure shows the projection of the system's trajectory with an initial condition from the invariant ellipsoid with the matrix $\hat{P}(\alpha)$ under the so-called worst exogenous disturbance [16] of the form

$$\tilde{w}_k = \operatorname{sign}\left(D^{\mathrm{T}} \hat{P}^{-1}(\alpha) A(\alpha) x_k \right), \quad k = 1, 2, \ldots$$



Projections of invariant ellipsoids and system's trajectories

The calculations were performed in Matlab using cvx [17].

CONCLUSIONS

This paper has proposed an approach to designing a parametric quadratic Lyapunov function for an affine family of discrete-time systems with arbitrary bounded exogenous disturbances. The approach is remarkable for simplicity and a reduced degree of conservatism, as the numerical simulations show.

Further research will focus on extending these results to the design problem for a family of discrete-time control systems with parametric uncertainty.

REFERENCES

1. Ebihara, Y., Peaucelle, D., and Arzelier, D., *S-Variable Approach to LMI-Based Robust Control*, London: Springer-Verlag, 2015.
2. Mao, W.-J. and Chu, J., Correction to "Quadratic Stability and Stabilization of Dynamic Interval Systems," *IEEE Trans. on Automatic Control*, 2006, vol. 51, no. 8, pp. 1404–1405.
3. Mao, W.-J. and Chu, J., Quadratic Stability and Stabilization of Dynamic Interval Systems, *IEEE Trans. on Automatic Control*, 2003, vol. 48, no. 6, pp. 1007–1012.
4. Pessim, P.S.P., Lacerda, M.J., and Agulhari, C.M., Parameter-Dependent Lyapunov Functions for Robust Performance of Uncertain Systems, *IFAC PapersOnLine*, 2018, vol. 51, no. 25, pp. 293–298.
5. Boyd, S., El Ghaoui, L., Feron, E., and Balakrishnan, V., *Linear Matrix Inequalities in System and Control Theory*, Philadelphia: SIAM, 1994.

6. Ramos, D.C.W. and Peres, P.L.D., A Less Conservative LMI Condition for the Robust Stability of Discrete-Time Uncertain Systems, *Systems & Control Letters*, 2001, vol. 43, pp. 371–378.
7. De Oliveira, M.C., Bernussou, J., and Geromel, J.C., A New Discrete-Time Robust Stability Condition, *Systems & Control Letters*, 1999, vol. 37, pp. 261–265.
8. Geromel, J.C., De Oliveira, M.C., and Hsu, L., LMI Characterization of Structural and Robust Stability, *Linear Algebra and Its Applications*, 1998, vol. 285, pp. 69–80.
9. Cox, P.B., Weiland, S., and Toth, R., Affine Parameter-Dependent Lyapunov Functions for LPV Systems with Affine Dependence, *arXiv:1803.11543v2*. Last revised April 5, 2018.
10. Liu, Z., Theilliol, D., Gu, F., et al., State Feedback Controller Design for Affine Parameter-Dependent LPV Systems, *IFAC PapersOnLine*, 2017, vol. 50, no. 1, pp. 9760–9765.
11. Oliveira, R.C.L.F. and Peres, P.L.D., LMI Conditions for Robust Stability Analysis Based on Polinomially Parameter-Dependent Lyapunov Functions, *Systems & Control Letters*, 2006, vol. 55, pp. 52–61.
12. Khlebnikov, M.V. and Kvinto, Y.I. Robust Stability Conditions for a Family of Linear Discrete-Time Systems Subjected to Uncertainties, *Control Sciences*, 2020, no. 5, pp. 17–21.
13. Gahinet, P., Apkarian, P., and Chilali, M., Affine Parameter-Dependent Lyapunov Functions and Real Parametric Uncertainty, *IEEE Trans. on Automatic Control*, 1996, vol. 41, no. 3, pp. 436–442.
14. Daafouz, J. and Bernussou, J., Parameter Dependent Lyapunov Functions for Discrete Time Systems with Time Varying Parametric Uncertainties, *Systems & Control Letters*, 2001, vol. 43, iss. 5, pp. 355–359.
15. Khlebnikov, M.V., Polyak, B.T., and Kuntsevich, V.M., Optimization of Linear Systems Subject to Bounded Exogenous Disturbances: The Invariant Ellipsoid Technique, *Automation and Remote Control*, 2011, vol. 72, no. 11, pp. 2227–2275.
16. Polyak, B.T., Khlebnikov, M.V., and Shcherbakov, P.S., *Upravlenie lineinymi sistemami pri vneshnikh vozmushcheniyakh: Tekhnika lineinykh matrichnykh neravenstv* (Control of Linear Systems Subjected to Exogenous Distrubances: An LMI Approach), Moscow: LENAND, 2014. (In Russian.)
17. Grant, M. and Boyd, S., CVX: Matlab Software for Disciplined Convex Programming, Version 2.1. URL: http://cvxr.com/cvx/

This paper was recommended for publication by S.A. Krasnova, a member of the Editorial Board.

Received April 30, 2021, and revised July 8, 2021.
Accepted July 14, 2021.

Author information

Khlebnikov, Mikhail Vladimirovich. Dr. Sci. (Phys.-Math.), Trapeznikov Institute of Control Sciences, Russian Academy of Sciences, Moscow, Russia, ✉ khlebnik@ipu.ru

Kvinto, Yana Igorevna. Cand. Sci. (Eng.), Trapeznikov Institute of Control Sciences, Russian Academy of Sciences, Moscow, Russia, ✉ yanakvinto@mail.ru

Cite this article

Khlebnikov, M.V. and Kvinto, Ya.I., A Parametric Lyapunov Function For Discrete-time Control Systems With Bounded Exogenous Disturbances: Analysis. *Control Sciences* **4,** 18–22 (2021). http://doi.org/10.25728/cs.2021.4.2