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INVESTIGATION OF MULTIVARIABLE AUTOMATIC CONTROL SYSTEMS FOR COMPLEX DYNAMIC OBJECTS BASED ON PETROV'S PARADIGM[1]

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Abstract. This paper considers some approaches to studying the properties of multivariable automatic control systems (MACSs), particularly their stability, based on different descriptive models. The theory presented below extends the previously known ideas of Academician B.N. Petrov, which are fundamental in the classical theory of automatic control. Petrov's theory is based on the structural and functional decomposition of MACSs into separate real subsystems and multiple connections between them, described by a new model, and the study of system properties using frequency-domain methods. Therefore, this theory is related to the physical (engineering) approach to dynamic systems analysis. A method for describing MACSs by the individual characteristics of subsystems and multiple connections is suggested. Stability criteria for linear MACSs with identical subsystems and a stability criterion for the system's equilibrium are established. A technology for finding the parameters of periodic motions and assessing their stability for nonlinear MACSs is introduced. Some numerical examples with technical objects illustrate this technology for studying the properties of MACSs.

Keywords: multivariable system, decomposition, frequency-domain methods, linear system, nonlinear system.

Dedicated to the blessed memory of B.N. Petrov, Academician of the USSR Academy of Sciences

INTRODUCTION

In the first half of the twentieth century, complex dynamic objects (CDOs) (aircrafts, power and propulsion systems, electrical installations, complex technological processes in the petrochemical, engineering, and other industries) appeared in operation. Multiple output variables of such objects needed automatic control via appropriate actions applied to some of their control variables. As a result, a new class of controlled systems was created and called multivariable automatic control systems (MACSs) for CDOs. (Note that the terms "multiply connected systems" and "multi-connected systems" are also used in the literature.)

This class of systems has the following peculiarity: when maintaining a given value of its output variable, each subsystem of the system inevitably affects the operation of other subsystems due to the physical processes (aerodynamic, gas-dynamic, electrical, chemical, thermal, etc.) occurring in the controlled object and connecting the subsystems. Practice demanded studying the new class of systems, and new complex problems arose in automatic control theory.

This paper reveals the content of Petrov's paradigm (a model of problems and their solutions) subject to the investigation of MACSs. Petrov's idea is based on the structural and functional decomposition of MACSs for CDOs into physical subsystems and multiple connections between them and analysis of their properties using frequency-domain methods. It extends his earlier idea [1] formulated in 1945, which underlies classical control theory.

We have a modest intention: to show the possibilities of Petrov's paradigm by his students' publications, thus approving it as a new technology for examining the properties of MACSs equally with the technologies

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based on other paradigms. Therefore, when speaking about the advantages of the classical paradigms for studying the stability of MACSs, we would like to emphasize the practical benefits of Petrov's paradigm in solving this problem.

1. MODELS OF MULTIVARIABLE AUTOMATIC CONTROL SYSTEMS FOR COMPLEX DYNAMIC OBJECTS

We separate three types of models of the MACSs for CDOs, writing them in the vector-matrix form:

$$X(s) = W(s)U(s) + Q(s)F(s),$$

$$U(s) = R(s)\left[X^o(s) - X(s)\right],$$

where $X(s), X^o(s), U(s)$, and $F(s)$ are the vectors of controlled variables, reference signals, control variables, and disturbances, respectively; $W(s) = \|W_{ij}(s)\|_{n\times n}$ and $R(s) = \|R_{ij}(s)\|_{n\times n}$ are the matrix transfer functions (MTFs) of the object and controller, respectively (the controller includes the actuator); finally, $Q(s) = \|Q_{ij}(s)\|_{n\times n}$ is the MTF of the disturbance.

Let **the first model** reflect only natural connections between the subsystems through a multivariable controlled object. Then the MTF $R(s)$ is a diagonal matrix, $R(s) = \|R_{ij}(s)\|_{n\times n}$, in which the transfer function $R_i(s)$ of the control device and actuator of the corresponding subsystem stands on the diagonal.

Consider **the second model** of MACSs, in which a multivariable object represents a set of autonomously operating objects (power units, robots, electric motors, etc.). Then its MTF will be a diagonal matrix $W(s) = \|W_i(s)\|_{n\times n}$. In the system, the set of objects is complexly controlled by a multivariable controller with the MTF $R(s) = \|R_{ij}(s)\|_{n\times n}$. The design problems remain the same as in the first model.

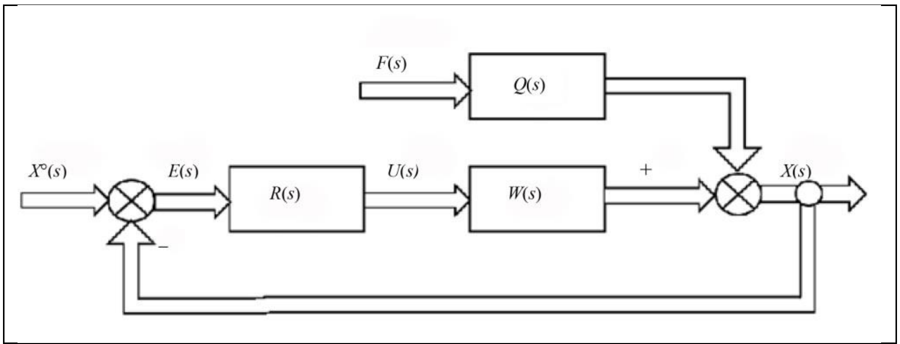

Fig. 1. The structural diagram of MACSs for CDOs.

Finally, in **the third model**, connections between subsystems are through a multivariable controlled object and a multivariable controller. The possibilities of such MACSs are still underinvestigated.

The structural diagram of MACSs is shown in Fig. 1.

For MACSs for CDOs, the first and main problem is stability. Suppose that the stability of each subsystem is established using a well-known classical criterion (Lyapunov, Routh, Hurwitz, Stodola, Nyquist, or Hermite–Mikhailov). In this case, the characteristic equation of the multivariable system is the product of the characteristic equations of the subsystems considering their interconnections. As a result, we obtain a characteristic equation of the form

$$D(s) = a_0 s^n + a_1 s^{n-1} + \ldots + a_{n-1}s + a_n = 0, \qquad (1)$$

but with a very high degree n > (25–40). Nowadays, the system's stability can be assessed by equation (1) without finding the eigenvalues. However, apart from stability analysis, equation (1) yields no constructive conclusions about the reasons of stability of the entire multivariable system using the above criteria. Modern algorithms and programs allow determining the stability of MACSs but not the potential effect on their degree of stability exerted, e.g., by a simultaneous change in the set of physical parameters of the subsystems and their interconnection coefficients: the relationship between the coefficients a_i of equation (1) and the physical parameters of the system is often implicit.

2. VOZNESENSKII'S AUTONOMY PARADIGM

In 1938, I.N. Voznesenskii formulated the autonomy principle of MACSs with respect to free motions; see the paper [2].

To implement this principle, we should consider the third model of MACSs and design artificial connections between the subsystems through a multivariable controller to compensate for the natural connections through the multivariable object. Then the entire MACS will be decomposed into separate stable subsystems. However, the complete compensation of the natural connections is often impossible due to the inertia of the system's elements. In this case, the matter concerns compensating the connections only for a particular operating mode of the system. With a change in the operating modes, the entire MACS needs to be retuned.

Certain success can be achieved in the case of an optimal tuning of MACSs [3].

In the paper [4], the autonomy problem was solved using the structures of the subsystems with infinitely increasing gains without violating the stability of the entire MACS.

When a high performance of the subsystems is achieved, their mutual effect on each other becomes minimal: due to the rapid execution of their control task, the subsystems have no time to respond to the effects of other subsystems fully. As noted by the author of [4], with such properties of the system, it can be made invariant with respect to the load. At the same time, for several important objects, autonomy contradicts technological processes.

As was demonstrated later, multivariable control is vital for complex dynamic objects: it guarantees higher efficiency than the transition to an autonomous structure. Despite this fact, many industrial systems are designed within the autonomy paradigm: metallurgy, power engineering, steam boilers control, and heating turbines control in stable conditions, to name a few.

3. A PARADIGM BASED ON THE STATE-SPACE DESCRIPTION OF MACS

This paradigm implements a purely mathematical approach: an MACS described by an n-order differential equation is reduced to a system of first-order differential equations (the Cauchy form). This approach gave a huge impetus to studying the properties of dynamic systems with feedback control.

Let a dynamic system be described by a system of equations in the vector-matrix form:

$$\begin{aligned} \dot{X} &= AX + BU, \\ Y &= CX, \end{aligned} \tag{2}$$

where $A = \|a_{ij}\|_{n\times n}, B = \|b_{ik}\|_{n\times m}, C = \|c_{\lambda i}\|_{p\times n}$, are numerical matrices with numerical entries; X is the vector of state variables; Y is the vector of output variables; U is the vector of input variables (control).

Note that such MACSs should be designed in the class of fully controllable, fully observable, and structurally stable systems. The MACS structure completely merges with the mathematical structure (2).

By assigning different values to the numerical parameters of the matrices A, B, and C, we determine the system's state X (the motion trajectory, or the solution of system (2)) for any time instant. This process can be repeated. The wide variety of research methods obtained by different modifications of the model (2) does not change the essence of studying dynamic systems in the state space. However, after transforming the characteristic equation of the MACS,

$$D(s) = \det[Is - A] = 0,$$

where I denotes an identity matrix of compatible dimensions, stability analysis reduces to the characteristic equation (1). In other words, the genesis (origin) of the stability of MACSs remains an open problem.

Another drawback of this paradigm, from the engineering viewpoint, is the difficulty of establishing a direct relationship between the state variables and the physical parameters of a real system. For this reason, implementing this paradigm in the design of real MACSs encounters definite difficulties and limitations. Using this paradigm, the design engineer has no clear answer to the question: how will the properties of an MACS change if the characteristics of certain elements of the real system's structure are simultaneously modified? Will the real system remain stable in this case?

Despite the applied problems mentioned above, this paradigm is effective when studying MACSs at the level of their mathematical description (2). Some problems can be eliminated if the model (2) is constructed not top-to-bottom but bottom-to-top: the models of elements are gradually combined into the system (2).

Even if the state matrix A of a multivariable system is the multi-connected composition of the submatrices of the corresponding physical subsystems and their interconnection coefficients, the investigation can be more efficient, e.g., when analyzing the stability margins of the MACS under variations of the individual parameters of the subsystems. However, the problem becomes more complicated if the system characteristics of the subsystems are simultaneously changed. In addition, the stability of an MACS is determined by complexes, i.e., the combinations of two, three, …, n subsystems related by multi-connection characteristics without an explicit matrix form. As we therefore believe, studying the genesis of the stability of MACSs by varying the system characteristics of subsystems will require additional research, causing definite difficulties and limitations in the design of real MACSs.

The state-space paradigm can be successfully applied for assessing the stability of a given MACS by its mathematical model. This paradigm underlies the modern control of dynamic feedback systems and is considered by many researchers [5–15].

4. A PARADIGM BASED ON THE MATRIX TRANSFER FUNCTION DESCRIPTION OF MACS

The matrix description of the MACS dynamics is quite convenient and makes the results visual. However, the MACS description using the MTF (Fig. 1) is somewhat incomplete and reflects the entire system's behavior to some extent only: it provides no information about the behavior of the uncontrolled and unobserved parts of the system. For a complete description of the system, we should pass to the state-space representation (2), whose dimension equals the number of the system's degrees of freedom. The vector of controlled outputs of the MACS has a linear relationship with the vector of state variables.

This paradigm, to some extent, repeats the shortcomings of the state-space paradigm. Performing matrix transformations, the design engineer obtains the final result under the given system parameters and input actions $U(t)$, and the matrix transformations themselves do not reflect physical processes.

The question remains: how to establish a relationship between the physical parameters of a real system and the parameters of matrix transformations? As in the previous paradigm, determining the effect of a change in any physical parameter, e.g., on the system's stability, requires repeating the numerical experiment many times. The problem becomes even more complicated if a group of physical parameters is varied simultaneously.

Unlike classical control theory, e.g., stability criteria cannot be derived in an analytical form through matrix transformations. The stability analysis of MACSs still involves the characteristic equation (1), obtained by transforming the equation

$$D(s) = \det\left[I + W(s)R(s)\right] = 0. \qquad (3)$$

The stability analysis by equations (1) or (3) does not explain how the properties of individual subsystems and the properties of interconnections between the subsystems affect the stability of the entire MACS.

The properties of MACSs were examined using matrix methods by Krasovskii [16], Meerov [17], Chinaev [18], Morozovskii [19], and Sobolev [20]. The applications of the theory of multivariable control systems were described by Bodner [21, 22], Shevyakov [23], Yanushevskii [24], Ray [25], and other researchers.

5. PETROV'S PARADIGM BASED ON THE STRUCTURAL AND FUNCTIONAL DECOMPOSITION OF MACS AND FREQUENCY-DOMAIN METHODS

In the late 1970s, Academician B.N. Petrov posed the following problem: to describe MACSs by larger (than the elements of subsystems) physical blocks and multiple connections between them. Petrov is known in automatic control theory for the paradigm of passing from a system of differential equations to its structural representation by functional blocks with operators and connections between them [1]. The paradigm gave a new and huge impetus to developing and creating the classical automatic control theory. The researchers of MACSs faced a similar problem. The main goal was to preserve the physicality of the structure and all the transformations so that the design engineer knew exactly (without solving the system of differential equations) what changes would contribute to improving the dynamic properties of the MACS.

The solution of this problem—a new description of MACSs through the physical characteristics of the subsystems and multi-connection characteristics—was presented in [26]. This description was used later in [27–29].

According to Fig. 1, an MACS consists of a set of interconnected and closed separate subsystems, each controlling a particular output of the object. We will consider the first model of MACSs, the most common in engineering practice, in which different subsystems are connected through a multivariable controlled object. In this case, as noted above, the controller's MTF $R(s)$ is diagonal, and the transfer functions $R_i(s)$ are located on the diagonal.

The main requirement to describe the dynamic characteristics of MACSs is to make one subsystem (connection) distinguishable from another. In other words, each characteristic must have its "individuality." Therefore, the characteristics of both subsystems and connections are labeled in a systematic way, and their dimensions are indicated.

As a separate ith subsystem, consider a closed loop system with its internal structure, which controls the ith output of the multivariable controlled object. As an individual characteristic (IC) of the ith subsystem, consider the one that fully reflects the investigated properties of the subsystem and expresses these properties and distinctive features. For example, these requirements are satisfied by the individual transfer function $\Phi_i(s)$ in the control mode, when the ith subsystem operates in an isolated (autonomous) mode independently of the other subsystems:

$$\forall i : \Phi_i(s) = \frac{X_i(s)}{X_i^o(s)} = \frac{R_i(s)W_{ii}(s)}{1 + R_i(s)W_{ii}(s)}.$$

Note that its gain-phase response (GPR) $\Phi_i(j)$ and error transfer function $\Phi_\varepsilon(s) = 1 - \Phi_i(s)$ can also be considered as the IC of the ith subsystem. In this case, the ith subsystem corresponds to a real physical system with an independent design value and an individual dynamic characteristic (model), widely used in the classical automatic control theory with its well-developed methods for studying closed loop single-input single-output (SISO) systems.

A multi-characteristic of interconnections (MCI) is introduced to concretize the relationships between the subsystems and express their features. This characteristic (model) reflects the existing relationships between the subsystems, and their mathematical model is built from typical dynamic links of the classical automatic control theory. For the class of MACSs under consideration, cross-connections between the subsystems are determined only by off-diagonal elements $W_{ij}(s)(i \neq j)$ of the MTF $W(s)$ of the multivariable controlled object. They form the connection matrix $\left\| W_{ij}(s)\gamma_{ij} \right\|$, where $\gamma_{ij} = \begin{cases} 1, \; i \neq j, \\ 0, \; i = j, \end{cases} \; i, \; j = \overline{1, n}$. This matrix reflects the individual relationships between the pairs, triples, quads, etc. of the subsystems.

For this class of MACSs, we should identify the absolute effect of cross-connections and, most importantly, their effect relative to direct connections through the controlled object. The latter connections are characterized by the diagonal matrix $\left\| W_{ij}(s)\delta_{ij} \right\|$, where $\delta_{ij} = \begin{cases} 1, \; i = j, \\ 0, \; i \neq j, \end{cases} \; i, \; j = \overline{1, n}$.

This relative connection between the subsystems is considered as the MCI in MACSs. The mathematical model of the MCI between k subsystems is given by

$$H_k(s) = \frac{\det \left\| W_{ij}(s)\gamma_{ij} \right\|_{k \times k}}{\det \left\| W_{ij}(s)\delta_{ij} \right\|_{k \times k}}. \tag{4}$$

The characteristic $H_k(s)$ can be real, complex, or imaginary.

By the nature of its effect, the MCI can be flexible or rigid; stabilizing or destabilizing; forcing, inertial, or lagging. In the general case, it characterizes the sign, magnitude ("strength"), and the character of connections in a group of k subsystems combined into a single whole through this multiple connection.

Thus, the model (4) concretizes the nature of the connections between different subsystems. Changing the sign, parameters, and structure of the model, we can design connections in an MACS, ensuring the required properties of the entire system.

Among the various types of MACSs, a class of homogeneous (identical, single-type) MACSs is often distinguished: the ICs $\Phi_i(s)$ of their subsystems are identical and equal. For this class of MACSs, it is reasonable to introduce the concept of a generalized characteristic of connections (GCC) as the sum of the connection characteristics for the subsystems of one equivalence class. For example, for all interconnected pairs of the subsystems, the GCC will have the form $H_2(s) = \sum_{i,j=1}^{C_n^2} H_{ij}(s)$; for all interconnected triplets in an n-dimensional system, the form $H_3(s) = \sum_{i,j,k=1}^{C_n^3} H_{ijk}(s)$, where $C_n^k = \frac{n!}{k!(n-k)!}$ is the number of k-combinations from n elements; and so on.

The GCC expresses the total connection $H_k(s)$ created by a group of C_n^k identical subsystems. The terms in $H_k(s)$ can be of different signs: the connections between the subsystems within one equivalence class may compensate each other (in this case, $H_k(s) = 0$). The same occurs if there are no connections between the subsystems within the kth equivalence class.

The ICs of subsystems and the MCIs introduced above allow passing from the MACS description at the level of elementary dynamic links to that at the level of subsystems and MCIs formed from these elements.

The new description of MACSs will also have new structural diagrams, e.g., in the form of a loop labeled digraph (Fig. 2).

For the MACSs shown in Fig. 2, the characteristic equation can be expressed through the ICs of the subsystems and their MCIs [27–29]:

$$\begin{gathered} D(\Phi, H) = 1 + \sum_{i,j=1}^{C_n^2} \Phi_i \Phi_j H_{ij} + \\ + \sum_{i,j,k=1}^{C_n^3} \Phi_i \Phi_j \Phi_k H_{ijk} + \ldots + H_{1n} \prod_{i=1}^{n} \Phi_i = 0. \end{gathered} \tag{5}$$

Here the ICs $\Phi_i(s)$ and the MCIs $H_k(s)$ are functions of the complex variable s. Hence, frequency-domain methods can be applied to study the characteristic equation (5).

Consider the characteristic equation (5) for the class of MACSs with homogeneous subsystems. Since

$$\Phi_1(s) = \Phi_2(s) = \cdots = \Phi_n(s) = \Phi(s),$$

we obtain

$$\begin{gathered} D(\Phi, H) = 1 + H_2(s)\Phi^2(s) + H_3(s)\Phi^3(s) + \cdots + \\ = H_k(s)\Phi^k(s) + \cdots + H_n(s)\Phi^n(s) = 0 \end{gathered},$$

where $H_k(s)$ is the GCC of the subsystem of dimension k given by (4).

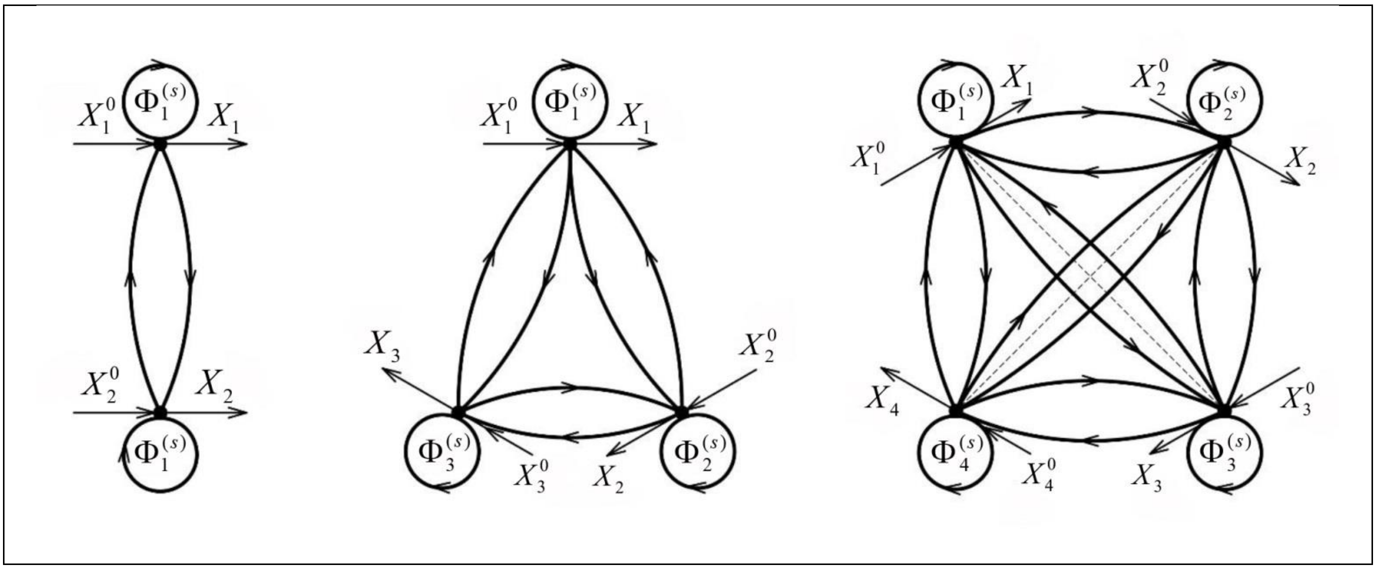

Fig. 2. MACS as a loop labeled digraph

This characteristic equation for one variable $\Phi(s)$ has complex-valued coefficients.

Now let the connection be either through the object or the controller, expressed by numerical coefficients k_{ij}. Then the MCI has the form

$$h_k = \frac{\det\left\|K_{ij}\gamma_{ij}\right\|_{k\times k}}{\det\left\|K_{ij}\delta_{ij}\right\|_{k\times k}}. \tag{6}$$

As a result, the characteristic equation for the homogeneous systems [30] reduces to

$$D(\Phi,h) = 1 + h_2\Phi^2(s) + \\ + h_3\Phi^3(s) + \dots + h_n\Phi^n(s) = 0 \tag{7}$$

The characteristic equation (7) can be written in another form. For $\Phi(s) = \frac{1}{M(s)}$, where $M(s)$ is the characteristic polynomial of the subsystem, the characteristic equation of the MACS [31] becomes

$$D(M,h) = M^n(s) + h_2M^{n-2}(s) + h_3M^{n-3}(s) + \\ + \dots + h_{n-1}M(s) + h_n = 0. \tag{8}$$

Here the Hermite–Mikhailov characteristic polynomial $M(s)$ acts as an IC of the subsystems.

Writing the characteristic equations in the form (7) and (8) opens up new possibilities in the investigation of multivariable systems.

Thus, Petrov's paradigm allows studying separately the individual characteristics $\Phi_i(s)$ and $M_i(s)$ of the subsystems and their GCC $H_k(s)$ and integrating them into a single characteristic of a real MACS to study its system properties.

6. STABILITY ANALYSIS OF LINEAR MACS BASED ON PETROV'S PARADIGM

Since the 1980s, the scientific direction based on Petrov's paradigm was developed at Ufa Aviation Institute; since 1992, at Ufa State Aviation Technical University in the scientific school headed by Prof. B.G. Il'yasov.

In the first stages, the characteristic equation (5) was solved using frequency-domain and numerical methods. They allowed assessing the stability of MACSs for gas turbine engines of supersonic aircrafts in various flight conditions.

The early results were presented in the monographs [27, 28] jointly with researchers from Trapeznikov Institute of Control Sciences and the Central Institute of Aviation Motors.

In that time, the conditions of static stability (the positivity of the free term of the characteristic equation) were derived for the MACS consisting of identical astatic subsystems with $\Phi(0) = 1$,

$$D(h,\Phi) = 1 + h_2 + h_3 + \dots + h_n > 0, \tag{9}$$

and the MACS consisting of identical static subsystems,

$$D(h,\Phi) = 1 + h_2\Phi^2(0) + h_3\Phi^3(0) + \\ + \dots + h_n\Phi^n(0) > 0, \tag{10}$$

where $\Phi(0) = \frac{k}{1+k}$, and k denotes the gain of the open loop subsystem.

In contrast to the matrix form, this form allows easily analyzing the effect of connections between subsystems on the static stability of MACS, i.e., easily assessing the violation of structural stability due to numerical changes in the connections between the subsystems.

For an MACS with different static subsystems, the static stability condition has the form

$$D(h,\Phi)=1+h_2\Phi_1(0)\Phi_2(0)+h_3\prod_{i=1}^{3}\Phi_i(0)+$$
$$+\ldots+h_n\prod_{i=1}^{n}\Phi_i(0)>0$$

For assessing dynamic stability, the stability problem was analytically solved [28, 29] by the *D*-partition method for a three-variable system consisting of three identical second-order subsystems interconnected through the outputs by numerical coefficients. Its characteristic equation

$$D(h,\Phi)=1+h_2\Phi^2(s)+h_3\Phi^3(s)=0 \qquad (11)$$

was represented on the plane of the interconnection coefficients (h_2, h_3); see Fig. 3. The function

$$\Phi(s)=1/(\tau^2s^2+2\xi\tau s+1),$$

where $\tau=0.5$ s and $\xi\in(0,1\ldots1)$, was taken as an individual characteristic of the subsystem.

According to Fig. 3, the less the subsystems are damped, the smaller the stability domain of the entire MACS will be.

Using numerical and frequency-domain methods, we can solve equation (11) for a more complex form and a higher order of the function $\Phi(s)$. Note that this approach was applied for assessing the stability of the designed three-variable ACSs for gas turbine engines of supersonic aircrafts based on their mathematical models [27, 28], for the first time in practice.

The results were used to formulate the fundamental postulates (regularities) for the MACSs consisting of stable identical subsystems interconnected through output variables.

Postulate 1. For this class of MACSs, the static stability conditions (9) and (10) are simultaneously a structural stability condition: in the case of their violation, the stability of MACSs cannot be achieved due to changes in the subsystems' parameters. This conclusion also applies to the MACSs with stable subsystems having different structures and individual characteristics.

Postulate 2. If a structurally unstable subsystem appears in a stable MACS, this will be a sufficient condition for the structural instability of the entire MACS in which all subsystems are interconnected through the output variables by numerical interconnection coefficients: a change in the interconnection coefficients h_2 or h_2 prevents from restoring the structural stability of the entire MACS.

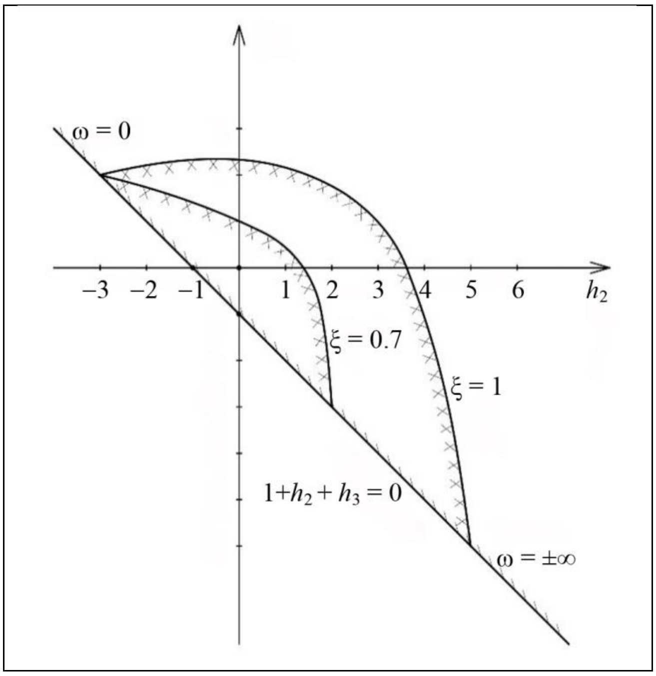

Fig. 3. Stability domains of three-variable ACS under different values of ξ.

Postulate 3. Consider an MACS in which the numerical connections between *n* subsystems are implemented either through a multivariable controlled object or a multivariable controller. For this MACS, there additionally exist *n* critical points located on its stability boundary and determined by the multi-connection equations.

Postulate 4. The critical points are determined by the roots of the mult-connection equation that are obtained either from the characteristic equation (7) by replacing $\Phi(s)$ with *x*,

$$D(h,x)=1+h_2x^2+h_3x^3+\cdots+h_nx^n=0, \qquad (12)$$

or from the characteristic equation (8) by replacing $M(s)$ with *z*,

$$D(h,z)=z^n+h_2z^{n-2}+h_3z^{n-3}+\cdots+h_n=0, \qquad (13)$$

where the equation order *n* is given by the number of interconnected subsystems.

The equation of multiple connections through the interconnection coefficients h_i in Petrov's paradigm was introduced for the first time.

Postulate 5. Consider an MACS in which identical subsystems are rigidly interconnected through the output variables. This MACS is dynamically stable if and only if the corresponding subsystem's GPR $\Phi(j\omega)$ neither hits nor encircles any critical point of the multi-connection equation (12) as the frequency ω varies from 0 to + ∞, and the Hermite–Mikhailov characteristic polynomial $M(j\omega)$ does not hit but encircles all critical points of the multi-connection equation (13).

Postulate 6. For an MACS with identical subsystems, stability margins (gain and phase margins) are defined as the distance of the corresponding frequency response of a subsystem on the complex plane to the nearest critical point of the multi-connection equations (12) or (13); see [32]. This postulate also holds if identical subsystems have elements with pure delay [33].

Postulate 7. A linear MACS with *n* identical subsystems interconnected through the output variables lies on the boundary of oscillatory stability: steady oscillations (periodic motions with frequency ω_n and amplitude α_n) occur in the system if one of the characteristics of an identical subsystem passes through the nearest critical point of the multi-connection equations (12) or (13). In this case, the amplitude and frequency of oscillations are determined from the corresponding individual characteristics of the subsystem and the multi-connection characteristics [30, 31].

Postulate 3 was used to formulate two frequency-domain stability criteria.

Criterion 1. A linear MACS with identical subsystems and numerical interconnection coefficients is stable if and only if the subsystem's GPR $\Phi(j\omega)$ neither hits nor encircles any critical point given by the roots of the multi-connection equation (12) as the frequency ω varies from 0 to + ∞ [30].

This criterion was confirmed by numerical examples in [31, 32]; also, see Fig. 4.

Example 1. The transfer function of the closed loop stable and separate subsystem of a three-variable system has the form $\Phi(s)=1/(s^3+3s^2+2s+1)$.

The subsystems have the multiple connection given by

$$h=\begin{pmatrix}0 & 0.2 & 0.4\\ 0.2 & 0 & 0.5\\ 0.2 & 0.5 & 0\end{pmatrix}.$$

The characteristic equation of the system with numerical coefficients has the form

$$D(\Phi)=1+h_2\Phi^2+h_3\Phi^3=0, \qquad (14)$$

where h_3 = –0.6 and h_2 = –0.37 according to formula (6). Replacing the function Φ in (14) with the complex variable *x*, we obtain the multi-connection equation

$$D(x)=1+h_2x^2+h_3x^3=0. \qquad (15)$$

Its roots (the critical points) are x_1 = –5.64, x_2 = –2.00, and x_3 = 1.48.

We construct on the complex plane the hodograph of the function $\Phi(j\omega)$ for $\omega\in(0,+\infty)$. On the same plane, we arrange the critical points x_i, $i=\overline{1,3}$. By Criterion 1, the multivariable system will be stable since the GPR $\Phi(j\omega)$ of the autonomous subsystem does not encircle any critical point of equation (15) as the frequency ω varies from 0 to + ∞; see Fig. 4. ♦

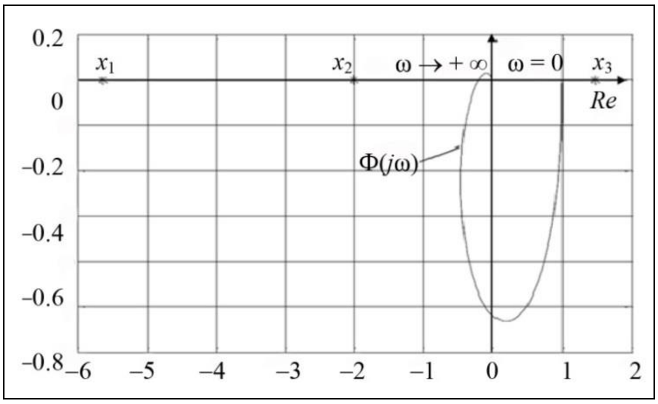

Fig. 4. The hodograph of Φ(jω) and critical points x_i, $i=\overline{1,3}$.

Criterion 2. A linear MACS with identical subsystems and numerical interconnection coefficients is stable if and only if the subsystem's characteristic hodograph (the Hermite–Mikhailov curve) neither hits nor encircles any critical point given by the roots of the multi-connection equation (13) as the frequency ω varies from 0 to + ∞ [31].

A numerical example of calculating a three-variable system confirms this criterion; see Fig. 5.

Example 2. We write the characteristic equation of Example 1 according to formula (8):

$$D(M)=M^3+h_2M+h_3=0, \qquad (16)$$

where the coefficients h_3 = –0.06 and h_2 = –0.37 are given by (6).

The critical points satisfy the equation $z^3+h_2z+h_3=0$. The closed separate subsystems have the characteristic equation corresponding to a stable subsystem: $M(s)=s^3+3s^2+2s+1=0$. The roots of the critical points equation are z_1 = 0.68, z_2 = –0.5, and z_3 = –0.18.

We construct on the complex plane the Hermite–Mikhailov curve $M(j\omega)$ for $\omega\in(0,+\infty)$. On the same plane, we arrange the roots z_i. According to Fig. 5, the Hermite–Mikhailov curve $M(j\omega)$ encircles all the roots. By Criterion 2, the multivariable system is stable. This conclusion is confirmed by the transient processes of the system. ♦

Postulate 8. According to the studies presented above, the postulates also hold for an MACS containing identical subsystems with a digital or discrete-time control part [31].

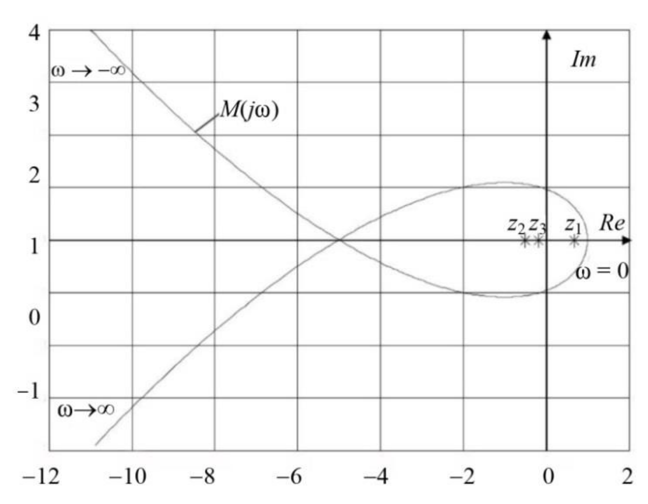

Fig. 5. The Hermite–Mikhailov curve $M(j\omega)$ and critical points z_i, $i=\overline{1,\,3}$, in Example 2.

7. STUDYING THE PROPERTIES OF NONLINEAR MACS BASED ON PETROV'S PARADIGM

This approach can also be used to study the properties of nonlinear MACSs. For example, consider a class of nonlinear MACSs with identical subsystems containing elements with nonlinear static characteristics. The subsystems are connected through a multivariable controlled object.

Let the harmonic linearization method be applied to this class of MACSs. An additional strict requirement is that the characteristics of all the subsystems and resulting closed loops satisfy the filtering condition.

We represent the nonlinear MACS as the interconnection of a nonlinear element (NE) and the linear part W_{lin} of the system; see Fig. 6.

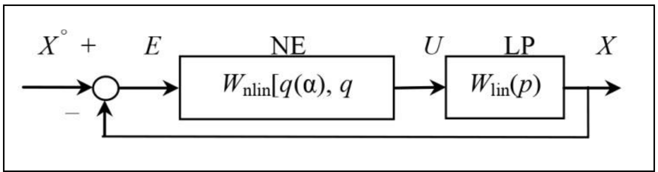

Fig. 6. The structural diagram of nonlinear MACS.

Here p denotes the differentiation operator. The functions $W_{\text{lin}}(p)$ and $W_{\text{nlin}}\left[q(\alpha),\, q'(\alpha)\right]$ form the operators of the system's linear part (LP) and nonlinear element (NE), respectively. The latter element is subjected to harmonic linearization.

The individual characteristic of the harmonically linearized identical subsystem has the form

$$\Phi(p,a)=\frac{W_{\text{nlin}}\left[q(\alpha),q'(\alpha)\right]W_{\text{lin}}(p)}{1+W_{\text{nlin}}\left[q(\alpha),q'(\alpha)\right]W_{\text{lin}}(p)},$$

where $q(\alpha)$ and $q'(\alpha)$ are the harmonic linearization coefficients; α is the input signal amplitude.

We write the characteristic equation for the entire MACS with identical nonlinear subsystems:

$$D(p,\alpha)=1+h_2\Phi^2(p,\alpha)+h_3\Phi^3(p,\alpha)+ \\ +\cdots+h_n\Phi^n(p,\alpha)=0 .$$

Passing from the function $\Phi(p,\alpha)$ to $M(p,\alpha)=1/\Phi(p,\alpha)$, we obtain

$$D(p,\alpha)=M^n(p,\alpha)+h_2M^{n-2}(p,\alpha)+ \\ +h_3M^{n-3}(p,\alpha)+\cdots+h_n=0 .$$

For these two equations, the multi-connection equations (12) and (13), respectively, hold as well.

We introduce the frequency-domain characteristics with the change $p=j\omega$. Then each characteristic equation is a function of ω and α: $D(\omega,\ \alpha)=0$. Postulates 1–8 are true for the harmonically linearized MACS. Hence, we may formulate another postulate for it.

Postulate 9. Consider a nonlinear MACS with harmonically linearized identical subsystems. This MACS is stable if and only if the characteristics $\Phi(j\omega,\alpha)$ do not encircle any critical point as the frequency ω varies from 0 to $\pm\infty$, and the curves $M(j\omega,\alpha)$, where the amplitude α belongs to some range, encircle all critical points of the multi-connection equations without hitting them.

Postulate 10. Periodic motions occur in the nonlinear MACS if either the characteristic $\Phi(j\omega,\alpha)$ or the Hermite–Mikhailov curve (the subsystem's characteristic polynomial) $M(j\omega,\alpha)$ hit a critical point of the corresponding multi-connection equation (12) or (13). The frequency ω_{per} and amplitude α_{per} of periodic motions are determined using classical control theory methods. The amplitude α_{per} is calculated by the corresponding characteristic; the frequency ω_{per}, by the multi-connection equation.

Postulate 11. Like in the classical control theory, the stability of periodic motions is assessed by the direction of deformation of the curves $\Phi(j\omega,\alpha)$ or $M(j\omega,\alpha)$ under increasing the amplitude α.

The periodic motions in a linear homogeneous MACS with fuzzy controllers in separate subsystems were analyzed using the same technique.

Example 3. It is required to investigate a nonlinear three-variable system for the presence of self-oscillations. The multivariable system consists of identical nonlinear subsystems interconnected through the output variables Y by numerical coefficients. The nonlinear subsystem is a standard structure consisting of a nonlinear element (NE) and a linear part (LP); see Fig. 7.

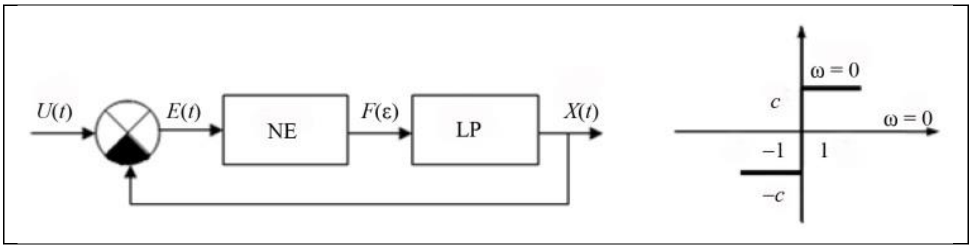

Fig. 7. The structural diagram of nonlinear MACS.

The nonlinear element is a relay (Fig. 7) with the harmonic linearization coefficients $q(\alpha) = 4c/\pi\alpha$, where $c = \pi$, and $q'(\alpha) = 0$.

Let the linear part have the transfer function $W_{lp} = \dfrac{2}{p(p^2+p+1)}$.

Then the characteristic polynomial $M(p)$ of the subsystem's harmonically linearized equation is written as

$$M(p) = \alpha(p^3+p^2+p)+8. \qquad (17)$$

The interconnection coefficients of the subsystems are given by the matrix

$$h = \begin{bmatrix} 0 & 0.75 & -1.45 \\ 0.18 & 0 & 0.75 \\ 0.75 & 0.18 & 0 \end{bmatrix}.$$

The characteristic equation of the three-variable system has the form

$$D(M) = M^3 + h_2 M + h_3 = 0. \qquad (18)$$

For the autonomous subsystems, the self-oscillation parameters are $\omega_{per} = 1$ and $\alpha_{per} = 8$. Replacing M with z, from formula (17) and equation (13) we obtain the critical point equation

$$D(z) = z^3 + h_2 z + h_3 = 0. \qquad (19)$$

We calculate the coefficients h_2 and h_3 of the characteristic equation (18) by formulas (6). For the given numerical values of the interconnection coefficients, we obtain $h_2 = 0.8175$ and $h_3 = 1.728$. Then the roots of (19) are $z_1 = -0.976$ and $z_{2,3} = 0.488 \pm 1.238\,j$.

This problem can be solved graphically. We construct the hodograph of the function $M(j\omega)$ (17) for $\omega \in (0, +\infty)$ and $\alpha \in (0, +\infty)$. On the same plane, we arrange the eigenvalues of the critical point equation, z_i (see Fig. 8). Of all the roots, the nearest critical one is $z_1 = -0.976$: the curve $M(j\omega, \alpha)$ hits the remaining roots, and the nonlinear MACS is unstable according to the above criterion.

Hence, there are stable periodic motions with the parameters $\omega_{per} = 1$ and $\alpha_{per} = 8.976$ in this multivariable system. Note that under numerical connections in multivariable systems, the frequency ω of the subsystems remains the same, whereas the amplitude α_{per} of oscillations changes compared to the autonomous subsystem. ♦

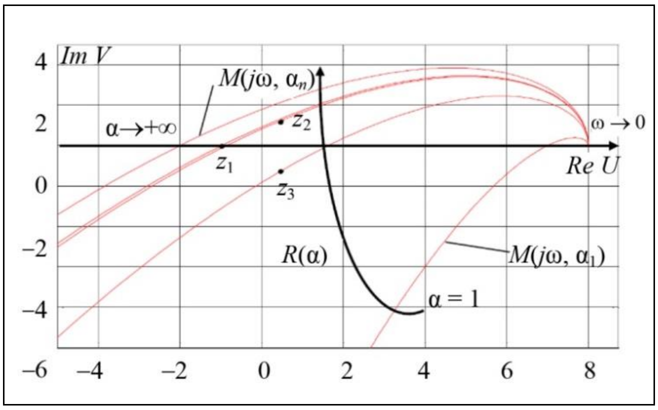

Fig. 8. The Hermite–Mikhailov curves $M(j\omega, \alpha)$ and critical points z_i, $i = \overline{1, 3}$, in Example 3.

Thus, Petrov's paradigm allows extending traditional control theory methods to the class of linear and nonlinear MACSs, including those with logical elements [32], artificial intelligence elements [35], adaptive systems [36], and variable structure systems [37]. What is important, the physical meaning of the effect of system elements on the properties of the entire MACS is preserved.

CONCLUSIONS

Academician B.N. Petrov and his students proposed a new paradigm for studying MACSs based on the description of the characteristic equation of a linear MACS through the individual characteristics of its subsystems and multiple connections between them. Within this paradigm, the system properties are investigated in the frequency domain. Such an approach was pioneering in the theory of multivariable systems.

Forming the multi-connection equation from the interconnection coefficients and finding new critical points for the subsystems to assess the stability of the entire MACS was novel in the theory of multivariable systems. As a result, new criteria for the stability of MACSs were established.

Petrov's paradigm involves the structural and functional decomposition of MACSs and frequency-domain methods to investigate the properties of MACSs. This paradigm fundamentally differs from the existing approaches: it preserves the physical meaning of each element of the subsystem and each connection and their role in the properties of MACSs.

Over the past 40 years, the theory of MACSs based on Petrov's paradigm was developed in the works of his students: new theoretical results were obtained and used in applications. As was shown, the theoretical and practical results established for the linear and nonlinear MACSs based on Petrov's paradigm keep the "spirit" of the classical control theory and the physical (engineering) sense of the ongoing research of complex systems.

Petrov's paradigm is a significant contribution of Russian scientists to global research as a new approach to studying various classes of MACSs for CDOs and revealing their unique properties.

Investigations of MACSs based on Petrov's paradigm open up new opportunities for studying various classes of MACSs for complex dynamic objects.

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