DOI: http://doi.org/10.25728/cs.2021.6.3

THE TECHNOLOGICAL CORE MODEL OF A LARGE-SCALE ECONOMIC SYSTEM: OPTIMAL CHARACTERISTICS

V.B. Gusev

Trapeznikov Institute of Control Sciences, Russian Academy of Sciences, Moscow, Russia

✉ gusvbr@mail.ru

Abstract. This paper considers the technological core model of an economic system and mathematical methods of its analysis. As a formalized criterion for the effectiveness of structural innovations, an indicator of productivity is proposed. The problem of finding an equilibrium state that optimizes the productivity of the technological core of the economy is formally stated. The method of equivalent transformation of the model considering the achieved value of indicators is developed. Several propositions on the properties of the equilibrium state are proved. A multi-stage process for calculating the trajectory that brings the economic system closer to the equilibrium state is constructed. The developed model uses the intersectoral balance of national accounts of the economy. The model is analyzed by determining the preferred structure of outputs at the development stages of the economic system's technological core. The phased process of changing the structure of outputs that asymptotically brings the technological core to the productivity maximum is calculated on an example of Russia's data. The results allow assessing the potential growth of economic productivity within the existing technological order by eliminating structural disproportions.

Keywords: structural disproportions, technological core of the economy, productivity optimum, plans for phased development, equilibrium state.

INTRODUCTION

Despite the existing high potential for development, the modern economy of Russia faces crisis phenomena. At the macrolevel, they include low GDP growth rate, the economy's critical dependence on oil and gas exports, unstable and undervalued exchange rates, a small share of the manufacturing sector, the economy's dependence on external sanctions, and ineffective management mechanisms. These factors cause the incomplete realization of the economy's potential. The results below demonstrate the opportunities for improving the economy's effectiveness based on the planned structural modification of its technological core.

This paper analyzes the effective use of the existing technological potential of an economic system and ways of its development [1, 2]. Comparing such indicators as labor productivity and economic growth rates for developed countries shows that similar technological processes can yield different results. To a certain extent, this difference can be explained by the structural features of the economies of these countries. As noted in [2], there is an increasing awareness that the economy's structure causes the main limitations of economic growth in Russia; the matter concerns an ineffective structure of production, an unproductive structure of incomes, an outdated structure of exports, and an irrational regional structure of the distribution of productive forces. A possible approach to accelerate economic growth is finding a preferred structure of economic activities and ways to implement this structure. According to numerical calculations, such a possibility is justified.

The analysis method under consideration uses the technological core model of a multisectoral economic system. The technological core of an economic system will be understood as a set of economic activities available for observation and measurement and the costs incurred to achieve the results of these activities, sufficient to represent the state of this system adequately. An example of the technological core is the set of factors used by the Federal State Statistics Service (Rosstat) to form the intersectoral balance in the national accounting system [3, 4]. The technological core model describes a static picture of the impact of economic activities on the volume of goods and services supplied. The parameters of this impact characterize the achieved level of technological development of the economy and the possible limits of its use. Technological core subset models can have several types, including goods and services, final consumption expenditures, accumulation, and net exports. The interpretation of the results will differ, depending on the model type. The main data source, the intersectoral balance, reflects the volume of goods and services consumed by different sectors and activities [5].

The important point is that the analyzed technological links form a stable Schur matrix of specific costs of full rank [6]. This feature allows determining the productivity potential (the excess of output over costs in the autonomous mode) and suggesting a way to improve the effectiveness of the economy's technological core by varying the output and prices. It is also possible to find "bottlenecks" in the technological interaction system: to identify the services, outputs, or sectors currently limiting GDP growth (there will be no growth in other sectors within their growth) and determine the potential effect of increasing production in these critical sectors. These questions are settled by optimizing an objective function of technological core productivity subject to different-type constraints [7]. In the resulting solutions, the potential demand is balanced by the supply [8], and the optimal outputs therefore form an equilibrium.

The development of new technologies is a problem with a high degree of uncertainty. Only a small share of innovations seems effective and is embedded in the economy's technological structure. New technologies are included in the economy's structure after assessing their potential. Moreover, the model allows determining the admissible output increase for sectors significant by non-economic criteria.

The reproduction model [9] of a multiproduct system is adopted for defining the multiplier of output (and an indicator of economic system productivity) as a function of structural proportions of outputs and prices for the produced goods and services of sectors. This indicator reflects the ratio of output and costs. Maximizing it, we find the potential of the technological core and the balanced structure of outputs and prices in the reproduction mode.

The inertial character of economic processes should be considered when implementing the calculated parameters of the output structure in practice. For this purpose, we describe a procedure for calculating the indicative forecast of output indicators [9]. Several propositions on the properties of the applied calculation procedures are formulated and proved. The results calculated for the structure of Russia's intersectoral balance are presented to demonstrate the prospects of the proposed approach.

1. INDICATORS OF STABLE DEVELOPMENT

We consider models and methods of management describing stable self-sufficient development of the economy. For this purpose, we use the closed input-output model of the Leontief type [3].

The *productivity* of a single-product (scalar) model of an economic system is defined as $\pi = Y/Z$, where Z denotes the total intermediate costs (input), and Y is the gross value added (GDP). We denote by V the gross output and by $a = Z/V$ the materials consumption. If the gross output is represented as the sum $V = Y + Z$, then the productivity equals

$$\pi = (V - Z)/Z = 1/a - 1. \tag{1}$$

Throughout the paper, all indicators are written in value form at comparable prices of the base year. In the scalar model, economic productivity depends only on the materials consumption parameter: the lower it is, the higher the productivity will be.

For the multiproduct (vector) model of the economy, different configurations of the input $\mathbf{Z}$ and output $\mathbf{V}$ vectors yield different productivity. Note that the relative values of their components are important. Therefore, we can pose the following problem: choose the model parameters with the highest productivity by varying the structure of these vectors.

The productivity potential of a multiproduct economic system is defined as the maximum productivity under the natural constraints imposed on the output $\mathbf{V}$ and input $\mathbf{Z}$ vectors:

$$\pi^* = \max_{\mathbf{V},\mathbf{Z}} \pi.$$

When passing to the multiproduct economy model, we assume that the direct costs Z_{ij} of sector j for the output of products or services of type i and the outputs V_j of products and services of type j are given. These data are used to calculate the specific cost coefficients

$$a_{ij} = Z_{ij} / V_j, \; i = 1, \ldots, n \;\; j = 1, \ldots, n.$$

which form the technological matrix $\mathbf{A}$. Here n is the number of sectors. The total costs of sector i have the form

$$Z_i = \sum_{j=1}^{n} a_{ij} V_j, \; i = 1, \ldots, n.$$

By analogy with formula (1), we define the productivity of sector i as the value-added share in the intermediate consumption value. We define the productivity of the economic system's core as the minimum of all sectoral productivities:

$$\pi = \min_i Y_i / Z_i = \min_i \{(V_i - Z_i) / Z_i\}.$$

The input-output model can be described by an equality expressing the balance between the outputs (supply) and total costs (demand) of all sectors:

$$V_i(t) = \gamma_i \sum_{j=1}^{n} a_{ij} V_j(t),\ i = 1, ..., n, \tag{2}$$

where γ_i denotes the output multiplier of sector i, $\gamma_i \geq 1$.

We formulate an optimization problem for the output structure V_i: maximize the lower bound of the output multipliers

$$\gamma = \min_i \gamma_i,$$

i.e., find

$$\gamma^* = \max_{\gamma, \mathbf{V}} \gamma, \tag{3}$$

subject to the technological output balance constraint corresponding to condition (2). According to this constraint, direct costs include the costs of all activities and cannot be less than the volume of a certain regulated share of outputs:

$$V_i(t) \geq \gamma \sum_{j=1}^{n} a_{ij} V_j(t). \tag{4}$$

The relations (3) and (4) represent a bilinear programming problem. The outputs satisfying condition (4) are called balanced. Thus, the problem reduces to determining a balanced output vector $\mathbf{V}$ with the maximum lower bound of the multiplier γ.

If problem (3), (4) has a solution ($\mathbf{V}$, γ), the maximum of the productivity indicator can be calculated as $\pi^* = \gamma - 1$. It represents the value-added share in the intermediate consumption value in the optimal balanced mode of technological development of the economic system. Since $\pi \leq \pi^*$, the inequality $\gamma \geq 1/a$ always holds. In the optimal balanced mode, we have

$$a = 1/\gamma.$$

Now we give an illustrative example of the two-dimensional case of this optimal output structure problem. In this case, the balance condition (4) takes the form

$$\begin{aligned} V_1 &\geq \gamma(a_{11}V_1 + a_{12}V_2), \\ V_2 &\geq \gamma(a_{21}V_1 + a_{22}V_2). \end{aligned} \tag{4`}$$

Let ($\mathbf{V}$, γ) be the solution of problem (3), (4`) under the natural assumptions $\gamma a_{11} < 1$ and $\gamma a_{22} < 1$. (They are necessary for the technological core to be productive.) Suppose that the specific costs $a_{ij}$ and the components $V_i$ of the output vector are positive. We solve the system of inequalities (4`), which is satisfied under the condition

$$(1 - \gamma a_{11})(1 - \gamma a_{22}) \leq \gamma^2 a_{21} a_{12}.$$

An admissible solution of this system consists of an eigenvector $\mathbf{V}^*$ and the multiplier $\gamma^* = 1/a^*$ corresponding to an eigenvalue a^* of the matrix $\mathbf{A} = \left[a_{ij}\right]$, $i, j = 1, 2$. It will be called the eigenvalue multiplier. This multiplier is given by

$$\gamma^* = \left((a_{11} + a_{22})/2 \pm \sqrt{(a_{11} - a_{22})^2/4 + a_{12}a_{21}}\right) / $$

$$\left(a_{11}a_{22} - a_{21}a_{12}\right) =$$

$$\frac{2}{\left(a_{11} + a_{22}\right) \mp \sqrt{(a_{11} + a_{22})^2 - 4\left(a_{11}a_{22} - a_{12}a_{21}\right)}}.$$

The model parameters make economic sense if

$$1 < \gamma < \min(1/a_{11}, 1/a_{22}).$$

We denote by $\gamma_{\min}$ and $\gamma_{\max}$ the minimum and maximum values, respectively, of the eigenmultiplier γ^*.

In the case of strong intersectoral links,

$$a_{11}a_{22} < a_{21}a_{12},$$

the eigenmultipliers are real and have different signs, whereas the constraints (4`) hold if $\gamma_{\min} \leq \gamma \leq \gamma_{\max}$. Then we obtain

$$\gamma_{\max} = \frac{2}{\left(a_{11} + a_{22}\right) + \sqrt{(a_{11} + a_{22})^2 - 4\left(a_{11}a_{22} - a_{12}a_{21}\right)}} <$$

$$\frac{2}{\left(a_{11} + a_{22}\right) + \sqrt{(a_{11} + a_{22})^2 - 4\left(a_{11}a_{22}\right)}} < \frac{1}{\max(a_{11}, a_{22})},$$

$$\gamma_{\min} < 0,$$

$$\gamma_{\max} > 0.$$

Since $\gamma_{\max} < 1/\max(a_{11}, a_{22})$, condition (3) is satisfied, the relations (4`) hold on the strict equality, and $\gamma^* = \gamma_{\max}$.

If

$$a_{11}a_{22} = a_{21}a_{12},$$

we have

$$\gamma^* = 1/(a_{11} + a_{22}),$$

and the constraints (4`) are satisfied for $\gamma \leq \gamma^*$. Problem (3), (4`) has the solution $\gamma = \gamma^*$, and the relations (4`) hold on the strict equality.

If

$$a_{11}a_{22} > a_{21}a_{12}$$

(a small impact of intersectoral links), the values $\gamma_{\min}$ and $\gamma_{\max}$ are positive,

$$\gamma_{\max} = \frac{2}{(a_{11}+a_{22}) - \sqrt{(a_{11}+a_{22})^2 - 4(a_{11}a_{22} - a_{12}a_{21})}} > \frac{1}{\max(a_{11},\, a_{22})}.$$

In this case,

$$\gamma_{\min} = \frac{2}{(a_{11}+a_{22}) + \sqrt{(a_{11}+a_{22})^2 - 4(a_{11}a_{22} - a_{12}a_{21})}} < \frac{1}{\max(a_{11},\, a_{22})},$$

and the constraints (4`) are satisfied under the conditions $\gamma \le \gamma_{\min}$ and $\gamma \ge \gamma_{\max}$. Since $\gamma_{\max} \ge 1/\max(a_{11},\, a_{22})$, which is characteristic for weak intersectoral links, the solution is $\gamma^* = \gamma_{\min}$.

Thus, in each case, the solution (whenever exists) coincides with the eigenmultiplier; moreover, when solving the technological core productivity problem (3), (4`), we can use the equality constraints

$$V_1 = \gamma(a_{11}V_1 + a_{12}V_2),$$
$$V_2 = \gamma(a_{21}V_1 + a_{22}V_2),$$

instead of condition (4`).

If the multiplier values γ^* are real and $\gamma_{\min} > 1$, then the matrix $\mathbf{A} = \left[a_{ij}\right],\ i,\ j = 1,\, 2,$ is stable (Schur matrix) [6].

To find the eigenvector $\mathbf{V}^*$, we solve the system of inequalities (4`) with an additional normalization condition, e.g., in the form of bounds:

$$\mathbf{V}_i' \le \mathbf{V}_i \le \mathbf{V}_i'',\ i = 1,\, 2\,.$$

It is convenient to construct models and analyze calculation results using dimensionless relative prices. In this form, the models and interpretation of the results become more compact and clear.

2. EQUILIBRIUM OUTPUT INDICES

Consider the operations of transforming the technological matrix in current prices to that in relative prices and conversely.

We denote by $\mathbf{D}(\mathbf{X})$ a diagonal matrix with diagonal elements $X_1, X_2, ..., X_n$: $\mathbf{D} = \mathrm{diag}(\mathbf{X})$. Also, we denote by $\mathbf{C}(\mathbf{X})$ a diagonal matrix with diagonal elements $1/X_1, 1/X_2, ..., 1/X_n$.

As outputs change, specific cost estimates $\boldsymbol{a_{ij}}$ also change. To fix the changes in the outputs $\boldsymbol{V_i}$ at the previous step, we recalculate the direct cost coefficients:

$$\bar{a}_{ij} = a_{ij}\ V_j\ / V_i\ \ i = 1, \ldots, n,\ \ j = 1, \ldots, n,$$

or

$$\bar{\mathbf{A}} = \mathbf{D}(\mathbf{V})\mathbf{A}\mathbf{C}(\mathbf{V})\,. \tag{5}$$

Proposition 1. *If λ is some eigenvalue, $\mathbf{V}$ is the corresponding eigenvector of the matrix $\mathbf{A}$, and all $X_i \ne 0,\ i = 1,..,n$, then the transformed matrix $\bar{\mathbf{A}}$ has the same eigenvalues, and the eigenvector $\mathbf{v}$ equals the original one up to the expansion transformation $\mathbf{D}(\mathbf{X})$:*

$$\mathbf{v} = \mathbf{D}(\mathbf{X})\mathbf{V}\,.$$

All propositions are proved in the Appendix.

Obviously, the inverse to the transformation $\mathbf{D}(\mathbf{X})$ has the form

$$\mathbf{V} = \mathbf{D}(\mathbf{X})^{-1}\mathbf{v} = \mathbf{C}(\mathbf{X})\mathbf{v}\,.$$

The values v_i are interpreted as the proportions (indices) of the outputs (products and services). Thus, the transformation (5) does not change the eigenvalue of the specific cost matrix (technological matrix $\mathbf{A}$) when passing to the scale of output proportions $\bar{\mathbf{A}}$, and the corresponding eigenvector is subjected to the expansion $\mathbf{D}\,(\mathbf{X})$. This transformation of the technological matrix will be called a *deformation*.

Assume that the eigenvector of outputs $\mathbf{V}^0$ in the absolute scale is used for passing to the relative scale (price proportions) by deforming the technology matrix:

$$\mathbf{A}^1 = \mathbf{C}(\mathbf{V}^0)\mathbf{A}\mathbf{D}(\mathbf{V}^0)\,. \tag{6}$$

In this case, the eigenvector of the technological matrix becomes a unit vector after the deformation. In other words, the vectors $\mathbf{V}^0$ and $\mathbf{v}^0$ describe an equilibrium state of the outputs in different scales.

We formulate the following problem: find the output structure maximizing the multipliers under balanced outputs and costs. In the relative scale of output proportions, this optimization problem for the structure of outputs v_i is written as

$$\max_{v_i} \gamma, \tag{7}$$

subject to the technological output constraint

$$v_i(t) \ge \gamma \sum_{j=1}^{n} \bar{a}_{ij} v_j(t),\ i = 1, \ldots, n, \tag{8}$$

where t is the current time instant. Condition (8) is equivalent to condition (4): multiplying the former on the left by the positive matrix $\mathbf{D} = \mathbf{D}(\mathbf{V})$, we obtain

$$\mathbf{v} = \mathbf{D}\mathbf{V} \ge \mathbf{D}\gamma\mathbf{A}\mathbf{V} = \gamma\mathbf{D}\mathbf{A}\mathbf{C}\mathbf{D}\mathbf{V} = \gamma\bar{\mathbf{A}}\mathbf{v}\,.$$

For calculations, we choose a lower bound μ on the output indices. Then the corresponding constraint has the form

$$v_i(t) \ge \mu > 0,\ i = 1, ..., n. \tag{9}$$

This constraint is normalizing: it sets the scale of the indices without affecting their relations. Its economic meaning is as follows: for an equilibrium output vector, there should be no excessive drop in the outputs of sectors that are little involved in technological chains but have noneconomic importance (social sphere, security, ecology, etc.)

The solution under consideration has an economic interpretation if $\bar{\mathbf{A}}$ is a Schur matrix (its maximum eigenvalue is less than 1 by magnitude, i.e., the multiplier value $\gamma > 1$), and its elements and all components of its eigenvector are nonnegative.

Proposition 2. *Let* $\bar{\mathbf{A}}$ *be a positive Schur matrix with real eigenvalues. Then the positive solution of problem* (7), (8) *is implemented on the equality*

$$v_i(t) = \gamma^* \sum_{j=1}^{n} \bar{a}_{ij} v_j(t),\ i = 1, \ldots, n,$$

representing the eigenvector of this matrix and the multiplier γ^* *corresponding to the eigenvalue*

$$a^* = 1/\gamma^* > \max_i \{a_{ii}\}.$$

The solution implemented on the equality will be called an equilibrium; the corresponding state of the system, a technological equilibrium. Only the equilibrium states are economically reasonable: if the strict equality does not hold, then

$$\exists i : \overline{\Delta V}_i = V_i - \gamma \sum_{j=1}^{n} a_{ij} V_j > 0$$

(surplus output of the products).

The eigenvector of the matrix $\bar{\mathbf{A}}$ does not necessarily satisfy the optimality condition (7), or additional constraints can be imposed on the outputs besides (8) such that the equilibrium condition is violated. In this case, the productivity of the technological core turns out below the potential value $\pi^* = \gamma^* - 1$.

The indicator of technological effectiveness of the economic system, $u = \pi/\pi^*$, shows the degree of its closeness to the state of technological equilibrium. Obviously,

$$0 \le u \le 1,\ \max u = 1.$$

3. INDICATIVE PLAN-FORECAST FOR JOINT DEVELOPMENT OF SECTORS

If the optimal outputs differ significantly from the current ones, it is impossible to implement a jump or change the structure of outputs according to conditions (7)–(9) in a short period. We define a framework (directive) optimal output trajectory corresponding to additional implementability conditions by imposing extra constraints on changes in the outputs at planning stages. The feasibility of such constraints should be ensured by organizational capabilities and the availability of resources to increase the output of the corresponding sectors. The calendar duration of each planning stage also depends on these capabilities.

To determine a more rational plan for the development of sectors, we use the local problem by supplementing the expressions (6)–(9) with a constraint on the admissible change in the output indices at a rate $0 > \theta > 1$ per plan stage:

$$v_i(t) \le \theta v_i(t-1),\ \ i = 1, \ldots, n.$$

Repeating the procedures of finding the optimal solution and recalculating the direct costs matrix from stage to stage, we obtain an indicative multistage plan-forecast for the joint development of the sectors in the economy’s technological core. The absolute and relative outputs are used in the indicative plan calculation procedure. Let $\mathbf{V}^1$ be the vector of current outputs. At the first stage, the deformation of the technological matrix is applied for passing to the relative outputs $\mathbf{v}^1$:

$$\mathbf{A}^2 = \mathrm{diag}(\mathbf{V}^1)^{-1} \mathbf{A}\, \mathrm{diag}(\mathbf{V}^1).$$

The corresponding transformation implements the inverse transition for the vectors of intermediate calculations $\mathbf{v}$:

$$\mathbf{V} = \mathrm{diag}(\mathbf{V}^1)\, \mathbf{v}.$$

Then the vector of relative outputs $\mathbf{v}^k$ are found from the optimization problems

$$\max_{\gamma^k, \mathbf{v}^i} \gamma^k, \tag{10}$$

where $k = 1, 2, \ldots$ denotes the stage number, with the technological output constraint

$$\mathbf{v}^k \le \gamma^k \bar{\mathbf{A}}^k \mathbf{v}^k \tag{11}$$

and the relative output growth condition with a rate $\theta > 1$ per plan stage,

$$\mu \mathbf{I} \le \mathbf{v}^k \le \theta \mathbf{I}, \tag{12}$$

where $\mathbf{I}$ denotes a unit vector of compatible dimension. Then the indicative plan-forecast for the joint development of sectors can be calculated using the following propositions.

Proposition 3. *The sequence* $\mathbf{V}^k$ *and the estimate* γ^k *tend to the solution of problem* (7)–(9) *in a finite number of steps*.

Remark. Let $\mathbf{V}^*$ be the solution of problem (7)–(9). When deforming the technological matrix $\mathbf{A}$ by the matrix $\mathbf{D} = \mathrm{diag}(\mathbf{v}^*)$, the solution of the planning problem becomes trivial: $\mathbf{v} = \gamma \mathbf{I}$, where $\mathbf{v} = \mathbf{D}^{-1}\mathbf{V} = \mathbf{C}\mathbf{V}$. In other words, when the technological optimum is achieved, the structure of outputs becomes equilibrium, and there are no further changes.

Proposition 4. *Assume that there exists a solution of the local problems for* $k \ge 1$. *Then the indicative outputs can be obtained at stage* i *in absolute units of the form*

$$\mathbf{V}^k = \prod_{j=k}^{1} \mathrm{diag}(\mathbf{v}^j) \cdot \mathbf{V}^0 . \qquad (13)$$

Proposition 5. *If the coefficient* $\theta > 1$ *is constant for all stages, then the relative growth of outputs becomes the same starting from some stage. This property is similar to the highway property of optimization models of economic dynamics* [9].

Consider the optimum vector of outputs for problem (10)–(12). Note that the output structure $\mathbf{v}^k$ obtained by solving this problem differs at the initial stages from the equilibrium and is not stable for a fixed technological matrix **A.** However, changes in the output structure should lead to changes in this matrix in accordance with formula (5):

$$\bar{\mathbf{A}}^k = \mathbf{D}(\mathbf{v}^k)\mathbf{A}\mathbf{C}(\mathbf{v}^k) .$$

The resulting output structure tends to a stable equilibrium structure between planning stages; see the remark to Proposition 3.

The transformation (13) is applied to pass from the obtained indicators to the value-form outputs.

4. CALCULATION OF INDICATIVE PLAN-FORECAST

Consider the numerical technological core model (10)–(12). As a data source, let us use Russia's intersectoral balance for 2016; see [5]. It reflects the volume of goods and services consumed by different sectors and activities. (The base input-output tables are developed quinquennially for the years ending with 1 and 6.) An Excel library was used for numerically solving the corresponding mathematical programming problems. Its analog was described in [10].

Figure 1 shows the graph of the coefficient of productivity for the technological core when balancing the outputs at successive indicative planning stages. The upper limit of change in the output proportions was increased with a rate of $\theta = 1.5$ per stage. The productivity estimate can be written as $\pi = (\gamma - 1) \cdot 100\%$.

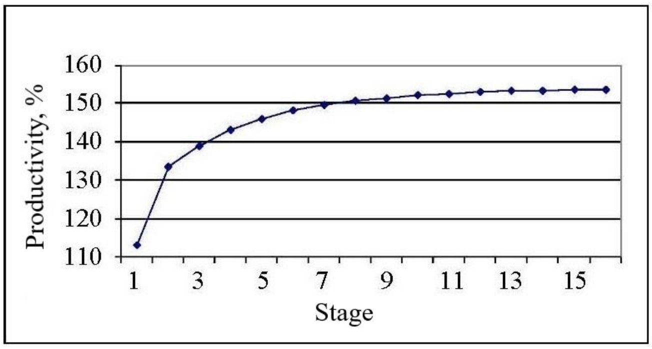

Fig. 1. The coefficient of technological core productivity when optimizing output proportions at successive indicative planning stages.

Next, Fig. 2 presents the indicative dynamics of output proportions for some sectors when solving the problem at successive indicative planning stages based on Russia's economic data. For a considerable part of economic activities, the output indices immediately reach the level $\theta = 1.5$ and remain there at all subsequent stages. The graphs correspond to the first eight rows of 98 types of economic activities in the intersectoral balance table for which the estimated outputs differ from $\theta = 1.5$ for $k > 1$.

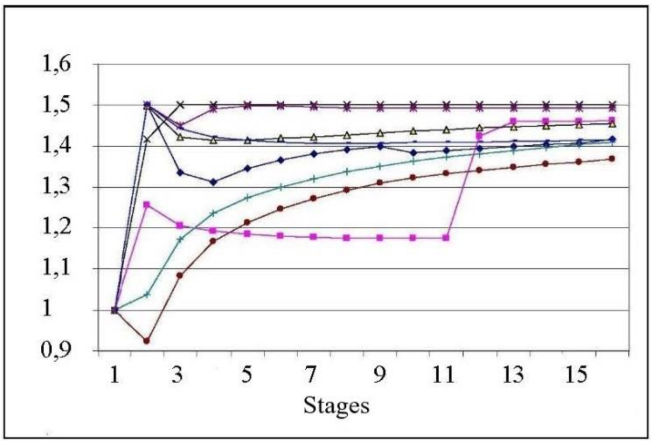

Fig. 2. Indicative dynamics of output proportions for some sectors at successive indicative planning stages for Russia's economic data:
agricultural output;
services related to hunting, trapping, and breeding of wild animals;
fish and other fishery and fish-farming products; services related to fishery and fish-farming;
oil, including oil derived from bituminous minerals; oil shale (bituminous) and bituminous sandstone;
other mining products;
meat, meat products, and other processed animal products;
fish and fish products, processed and canned;
fruit, vegetables, and potatoes, processed and canned.

Figures 1 and 2 illustrate the highway property [11] of the development model of the Russian economy's technological core: on a sufficient horizon, the output indices reach a constant level. According to formula (13), the outputs in value terms grow at rates v_i, $i = 1, \ldots, n$.

The above calculations allow estimating the growth potential of the technological core of the economic system. They demonstrate that changes in the output structure may significantly increase the indicator of productivity. To estimate the realizability of the existing potential of the technological core, we should supplement the balancing conditions with constraints on labor and raw materials, final consumption costs, accumulation, stock formation, and export-import flows. Thus, the obtained dynamics of output indices can be treated as their upper bound.

CONCLUSIONS

According to the Rosstat data and the calculation results, the state of the Russian economy is not equilibrium because the real productivity of the technological core (112%, see the first point in Fig. 1) is noticeably lower than the potential value (above 152%, see the asymptotic value of productivity). This observation gives hope in increasing the indicator of productivity in real conditions significantly. Such a possibility should be implemented by elaborating strategic plans for the development of the economy. Along with the choice of priority areas of technological core development, it requires adequate methods for forecasting multisectoral dynamics considering the main aspects of economic activity: stock formation, accumulation, final consumption of the state and households, and export-import flows [12–14]. New-level planning also involves appropriate organizational mechanisms [15].

In addition to the applied aspect, the results of this paper illustrate the specifics of the proposed methodology with relevant calculation and analysis tools. The class of problems discussed above has several specific features. Real interest in macroeconomic studies of this kind is connected with high-dimensional models. (The list of economic activities analyzed can be counted in hundreds.) Besides, the practical application of technological core models requires effective algorithms for solving the mathematical programming problems of the considered type [16–18] and linguistic calculation management tools, which should be integrated into a working environment [10, 19]. Free access to actual verified data and modern information technology, including the corresponding computational environment and interface devices, is necessary as well. For example, the paper [20] described a similar open-access toolkit (Thread Pool Executor of Akka) to process high-dimensional problems.

APPENDIX

P r o o f of Proposition 1. Let λ and $\mathbf{x}$ be an eigenvalue and the corresponding eigenvector of the matrix **A**, respectively:

$$\mathbf{Ax} = \lambda\mathbf{x}.$$

Multiplying both sides of this equation on the left by the matrix $\mathbf{D}$, we obtain

$$\mathbf{DAx} = \mathbf{D}\lambda\mathbf{x}.$$

Since $\mathbf{CD} = \mathbf{E}$ is an identity matrix, it follows that $\mathbf{x} = (\mathbf{CD})\mathbf{x}$ and

$$\mathbf{DA(CD)x} = \lambda(\mathbf{Dx}), \text{ or}$$

$$\mathbf{DAC(Dx)} = \lambda(\mathbf{Dx}).$$

In other words, $\mathbf{Dx}$ is an eigenvector of the matrix $\mathbf{DAC}$, and λ is its eigenvalue. ♦

P r o o f of Proposition 2. Let $\mathbf{V}^*$ be an eigenvector of matrix **A**. Since the number of inequalities in the constraint coincides with the dimension of the output vector, the solution of the bilinear programming problem

$$\max_{\gamma, Vi} \gamma, \tag{A.1}$$

$$V_i \geq \gamma \sum_{j=1}^{n} a_{ij} V_j, \tag{A.2}$$

is achieved on the equality

$$V_i = \gamma^* \sum_{j=1}^{n} a_{ij} V_j, \; i = 1, \ldots, n, \tag{A.3}$$

where $\gamma^* = 1/a^*$, and a^* is the eigenvalue of the matrix **A**.

Indeed, considering the condition $\mathbf{V} > 0$ and its corollary $\gamma < 1/\max_i \{a_{ii}\}$, we eliminate the variables V_i from (A.2), arriving at the following inequality for the characteristic polynomial $L(\gamma)$ of degree n:

$$L(\gamma) = (1-\gamma a_{11})(1-\gamma a_{22})\ldots(1-\gamma a_{nn}) - a_{12}a_{21}(1-\gamma a_{33})\ldots$$
$$- a_{13}a_{31}(1-\gamma a_{22})\ldots \geq 0.$$

As a result, we obtain the optimization problem

$$\max \gamma,$$
$$L(\gamma) \geq 0,$$
$$1 \leq \gamma < 1/\max_i \{a_{ii}\}.$$

This problem has a unique finite solution γ^* coinciding with a root of the polynomial $L(\gamma)$ if the value γ^* satisfies (A.3). The converse is true as well (otherwise, the matrix **A** would have more than n eigenvalues).

Assume on the contrary that Proposition 2 is false: the maximum γ^* is reached on the half-interval $1 \leq \gamma^* < 1/\max_i \{a_{ii}\}$ on the strict inequality $L(\gamma) > 0$. In the neighborhood of γ^*, the analytic function $L(\gamma)$ can be approximated by a segment of the Taylor–Lagrange series:

$$L(\gamma) = L(\gamma^*) + L'(\gamma^*)\left(\gamma - \gamma^*\right) + L''\left(\gamma^* + \theta\left(\gamma - \gamma^*\right)\right)\left(\gamma - \gamma^*\right)^2$$
$$0 < \theta < 1.$$

Hence, in this neighborhood, there exist an admissible point $(\gamma^* + \delta)$ and a constant $\varepsilon > 0$ such that $L(\gamma^* + \delta) > 0$ and

$$0 < \delta < \min\Big\{L(\gamma^*)/\left(\left|L'(\gamma^*)\right| + \varepsilon\right),$$
$$\sqrt{L(\gamma^*)/\left(\left|L''(\gamma^*)\right| + \varepsilon\right)}, 1/\max_i \{a_{ii}\} - \gamma^*\Big\}.$$

In other words, if $L(\gamma^*) > 0$, the value γ^* cannot deliver maximum to the polynomial $L(\gamma)$ under the condition $L(\gamma) \geq 0$. Therefore, the maximum value γ is achieved at the root of the polynomial $L(\gamma)$, and the solution of problem (A.1), (A.2) is reached on equality (A.3). ♦

P r o o f of Proposition 3. Consider an auxiliary bilinear programming problem of the form

$$\max_{\mathbf{v}} \gamma$$

subject to the output constraint

$$\mathbf{v} \geq \gamma \mathbf{A}^0 \mathbf{v}$$

and the relative output growth condition with an unbounded rate per one planning stage:

$$\mathbf{I} \leq \mathbf{v}.$$

Let $\mathbf{v}^*$ be the solution of this problem. Assume that $\theta^* = \max_i v_i^*$. Then the proposition holds for $\theta = \theta^*$, and

$$\mathbf{V}^1 \leq \theta^* \mathbf{V}^0.$$

For $1 < \theta = \theta_1 < \theta^*$, the planning problem will be solved in two stages: for $\theta = \theta_1$ and $\theta = \theta_2 = \theta^* / \theta_1$. Solving the planning problem in the two stages, we also obtain the solution $\mathbf{v}^*$. At the last stage, we have the conditions of the previous problem for one stage:

$$\mathbf{I} \leq \mathbf{v}^1 \leq \theta_1 \mathbf{I},\ \mathbf{v}^2 \leq \theta^* / \theta_1 \mathbf{I},$$

$$\mathbf{V}^2 \leq \theta^* \mathbf{V}^0.$$

Following similar considerations, we can divide the interval $\left[1, \theta^*\right]$ into any finite number of segments and the solution of the planning problem into the corresponding number of stages. ♦

P r o o f of Proposition 4. For $i = 1$, we have $\mathbf{D} = \mathrm{diag}(\mathbf{v}^1)$, $\mathbf{V}^0$ is the initial output vector, and $\mathbf{V}^1$ is the output vector after the first stage. Then $\mathbf{V}^1 = \mathbf{D}\mathbf{V}^0$. For $i > 1$, we obtain $\mathbf{V}^i = \mathrm{diag}(\mathbf{v}^i)\mathbf{V}^{i-1}$ by induction. Therefore, $\mathbf{V}^i = \mathrm{diag}(\mathbf{v}^i) \prod_{j=i-1}^{1} \mathrm{diag}(\mathbf{v}^j)\ \mathbf{V}^0$. ♦

P r o o f of Proposition 5. Since the value $\gamma^* = \max_{\mathbf{v}} \gamma$ is bounded, the output multiplier will reach the constant level γ^* starting from some step k^*. Moreover, under the hypotheses of Proposition 3, the inequality constraints

$$\mathrm{v}^k(t) \leq \theta \cdot \mathrm{I}$$

where $1 < \theta \leq \theta^*$ and $\theta^* = \max_i v_i^*$, will be satisfied on the equality

$$\mathrm{v}^k(t) = \theta \cdot \mathrm{I},\ k \geq k^*.\ ♦$$

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This paper was recommended for publication by V.V. Klochkov, a member of the Editorial Board.

Received February 19, 2021, and revised September 21, 2021. Accepted October 7, 2021

Author information

Gusev, Vladislav Borisovich. Cand. Sci. (Phys.-Math.), Trapeznikov Institute of Control Sciences, Russian Academy of Sciences, Moscow, Russia
✉ gusvbr@mail.ru

Cite this article

Gusev, V.B. The Technological Core Model of a Large-Scale Economic System: Optimal Characteristics, *Control Sciences* **6,** 25–33 (2021). http://doi.org/10.25728/cs.2021.6.3

Translated into English by *Alexander Yu. Mazurov,*
Cand. Sci. (Phys.-Math.),
Trapeznikov Institute of Control Sciences,
Russian Academy of Sciences, Moscow, Russia
✉ alexander.mazurov08@gmail.com