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UNCONSTRAINED OPTIMIZATION OF A TIME-VARYING OBJECTIVE FUNCTION ON A DISCRETE TIME SCALE

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Abstract. This paper develops an approximate method to optimize a time-varying objective function on a discrete time scale. The method should provide an admissible (controllable) error value. The conditions to be satisfied by the time scale, the objective function, and the environment's parameters are established. The unconstrained optimization of a time-varying objective function that depends on the control vector components is considered on a discrete set of time instants. To find a solution, a discrete gradient constrained optimization method is proposed. Efficiency conditions for the gradient method are formulated. A lower bound on the solution error is obtained in terms of the time step, the rate of change of the objective function, and its first- and second-order derivatives with respect to the control vector components. The method is illustrated on a numerical example of an optimal controller design for a time-varying plant with a nonlinear objective function. According to the numerical experiments, the wide-range variations of the controller's parameters have no significant effect on the qualitative behavior of the resulting trajectory. The method can be used to calculate an optimal control function for a system with a discrete-valued objective function.

Keywords: time-varying system, optimal controller, unconstrained optimization, lower bound on error.

INTRODUCTION

Purposeful developing systems, such as the national and regional economy or large multiple product farms, use optimal management mechanisms to maximize a target indicator [1]. This indicator can be total output, added value, profit, etc. Under crisis conditions, in an unpredictable environment (economic sanctions, financial catastrophes, force majeure), the management methods based on knowledge of normal business processes do not provide the desired result. For example, the international division of labor, which usually plays a positive role, becomes useless under sanctions. In such a situation, successful economic management should primarily focus on its internal capabilities and closed technological cycles within the economic system, thereby being autonomous in some sense [2].

Mathematical models for managing developing systems under crisis conditions may have little or even no accuracy. In such cases, management has to be limited only to a set of target indicators. In addition, the available statistical reports usually provide economic indicators only for certain periods (month, quarter, or year). All these factors restrict the applicability of any methods involving a smooth objective function and should be considered when developing appropriate management and decision-making methods.

Under uncertain behavioral rules of an object, the most appropriate method is optimal control. It consists in determining and maintaining a mode of operation in which the optimal (minimum or maximum) value of some criterion characterizing the object's performance is achieved. The construction of management mechanisms for autonomous system models has much in common with the design of an optimal controller that automatically finds and maintains the optimal value of the controlled variable. It ensures some stability of the controlled object (often called plant). The optimal controller's applicability is restricted since we cannot manage the long-term consequences of its operation.

In addition, in the case of a limited amount of information about the object, its inertial properties may be neglected.

In the 1960s, optimal (extremal) control formed an independent branch in the theory of nonlinear automatic control systems [3], and optimal controllers became widespread. For example, they were used in optimal relay systems [4] and pulse self-adjusting (adaptive) and optimal automatic control systems [5]; when tuning resonance loops and automatic measuring devices; when finding the optimal parameters of tunable models; when controlling chemical reactors and heating units for flotation and crushing [6].

Depending on the available information about the plant, the control laws in optimal controllers involve various approaches, differing in their validity and convergence of the result to the optimum. For example, in the paper [7], a heuristic extreme regulation algorithm was proposed to simulate the metabolic process. At each iteration, this algorithm performs a random search for the best response. The convergence of the process was demonstrated using an example for a particular object. In the paper [8], the air supply u in the furnace was regulated using the optimal control of the inertial object's static characteristic f as follows:

$$\Delta u(i+1) = h \operatorname{sign}[\Delta f(i+1)\,\Delta u(i)],$$

where u and f are the scalar control parameter and the measurable response parameter (control criterion), respectively; i denotes the time instant. The convergence of the control process in the cited paper was also demonstrated experimentally. A particular case of an optimal controller with a nonlinear objective function was considered in [9] for a linear dynamic system described by an ordinary first-order differential equation. The control step was chosen constant, and its sign was inverse to that of the derivative with respect to the control variable. The convergence of the control process was proved under the exactly known system's dynamics and derivatives of the objective function with respect to the time and control variables. The algorithm proposed below does not require this knowledge.

Consider a time-varying autonomous system model. For this system, let a control u be designed by

$$F(y, u) \to \operatorname*{opt}_{u}$$

at each point t, where $u(t)$ denotes the control vector; $F(y, u)$ is an objective function that satisfies smoothness and convexity in u; $y(t, p, x, u)$ denotes the system's state vector; p is the parameters vector; finally, $x(t)$ is the environment's state vector. If such an optimal control exists, it depends on the current state of the system and can be determined, e.g., using the gradient-based unconstrained optimization method from the first-order optimality condition

$$\frac{dF}{du} = 0\,.$$

If at the corresponding time instants only the values of the objective function are known, and the current state of the system is considered implicitly, we will find the control by the value of this function, denoting

$$f(t, u) = F(y(t, p, x(t), u), u)\,.$$

For the nonstationary problems of this type, the convergence of the gradient-based unconstrained optimization method was considered in the paper [10]. Assuming the exactly known gradient of the objective function, the convergence of the discrete-time iterative process

$$u(t_{k+1}) = u(t_k) - \gamma_k \nabla_u f(t_k, u(t_k))$$

was established under the requirement

$$\exists a > 0 : \|u^*(t_{k+1}) - u^*(t_k)\| \le a\,,$$

where $u^*(t)$ is the optimum of the function $f(t, u)$ at the time instant t; a specifies the deviation of the limit value from the optimum $u^*(t_k)$ as $k \to \infty$; the step γ_k is determined by the properties of the matrix $\nabla_{uu} f$. The results of [10] were further developed in the paper [11] by weakening the convergence condition of the iterative process:

$$\|\nabla_u f(t_{k+1}, u) - \nabla_u f(t_k, u)\| \le a,\ a > 0\,.$$

When considering nonstationary unconstrained optimization problems in [10, 11], both the deviation of the solution from the exact value at the current time instant t and the limit deviation as $t \to \infty$ were controlled. As a disadvantage of purely gradient methods, note the relatively slow convergence to the exact solution, which can be explained by the following fact: when approaching the optimum of the objective function with smooth derivatives, the gradient norm $\|\nabla_u f(t_k, u)\|$ may tend to 0 faster than the growth of the step γ_k. Therefore, the approximate solution "lags" the exact counterpart at every step. The methods proposed in [8, 9] allow the advance of the exact solution, which does not reduce their errors but accelerates their convergence.

If the objective function values are measured at discrete instants, the derivative $\nabla_u f(t, u)$ can be estimated only approximately, for example, using the spline representation of the function $f(t, u)$ (ambiguously). The expected consequence of such assumptions is an increase in the solution error compared to the methods based on the accurate estimation of the objective function's derivatives. Moreover, the solution error will not vanish over time, i.e., it cannot be eliminated. Naturally, its value should increase with an increase in the discretization step of the time interval and the rate of change of the objective function. Accordingly, "the deviation of the solution from the exact value" becomes an incorrect concept due to the latter's ambiguity. It can be replaced by "the deviation of the solution from an exact value" or "the deviation of the solution from the set of exact values."

This paper develops an approximate method to optimize a time-varying objective function on a discrete time scale. The method should provide an admissible (controllable) error value. The conditions to be satisfied by the time scale, the objective function, and the environment's parameters are established. This method can be used to design an optimal controller for a system defined at discrete time instants.

1. PROBLEM STATEMENT

Let the objective function $f(t, u)$, where t is the scalar time and $u \in R^n$, be continuously differentiable with respect to both variables and convex in u. Also, let this function together with the vector $u(t)$ be given at discrete time instants $t_1 < t_2 < \ldots < t_i$.

Consider the unconstrained optimization of the objective function at the time instant t_{t+1}:

$$f(t_{i+1}, u) \to \underset{u}{\mathrm{opt}}\,.$$

More precisely, the problem is to find the control vector value $u = u(t_{i+1})$ approximating the objective function value $f(t_{i+1}, u)$ to the optimum using the values $f(t_j, u(t_j))$, $j \le i$, and estimate the resulting error.

2. BASIC RESULTS

Let the function $f(t,u)$ have continuous first-order partial derivatives with respect to both variables. We introduce the following notations for the kth components of the vectors: $u^k{}_i = u^k(t_i)$, $k = 1,\ldots, n$, $f_i = f(t_i, u_i)$, where $i = 1, 2,\ldots$ The first-order partial derivatives on the two-point data have the approximations

$$\left.\frac{\partial f}{\partial u^k}\right|_i \cong \frac{f_{i+1} - f_i}{u^k{}_{i+1} - u^k{}_i} = \frac{\Delta f_i}{\Delta u^k{}_i},$$

written in the vector form as

$$\nabla_u f_i \cong \overline{\nabla}_u f_i\,.$$

Here the gradient and its approximations apply to the values of the variables t_i and u_i. Also, we denote by $\overline{\nabla}_f u_i$ the vector composed of $\dfrac{\Delta u^k{}_i}{\Delta f_i}$.

Proposition 1. Assume that:

1. The function $f(t,u)$ is continuously differentiable with respect to both variables.
2. The values of this function are given on the discrete set $\{t_i\}$ with the step Δt.
3. For each t_i, $\left.\dfrac{\partial f}{\partial t}\right|_{t_i} \neq 0$.
4. There exists a stationary point of this function in the variable u.

Then for some value h_i, the iterative method

$$u_{i+1} = u_i + h_i \overline{\nabla}_f u_i \Big/ \left(\left\|\overline{\nabla}_u f_i\right\| + \alpha_i\right),$$

$$0 \le \alpha_i \le h \Big/ \left(\frac{\partial f_i}{\partial t}\Delta t_i\right) - \left\|\overline{\nabla}_u f_i\right\|,$$

yields a sequence of values u_i deviating from the stationary point $\tilde{u}$ so that the differences $u^k{}_i - \hat{u}^k$ and $u^k{}_{i+1} - \hat{u}^k$, $k = 1,\ldots, n$, are of opposite sign. Thus, the result u_i fluctuates around the stationary points $\nabla_u f\big|_{\hat{u}, t_i} = 0$.

Proposition 2. In addition to the conditions of Proposition 1, assume that:

1. The function (t, u) has second-order partial derivatives with respect to the variable u that form the matrix $\nabla_{uu} f$.
2. At each point (t_i, u_i), this matrix satisfies the strong convexity in u: $\left\|\nabla_{uu} f_i\right\| > 0$.

Then for $|h_i| \ge \left|\dfrac{\partial f}{\partial t}\left\|\overline{\nabla}_u f\right\|\Delta t\right|_i$ and $0 \le \alpha_i \le h \,/\, / \left(\dfrac{\partial f_i}{\partial t}\Delta t_i\right) - \left\|\overline{\nabla}_u f_i\right\|$, where the gradient approximation applies to the values of the variables t_i and u_i, the iterative method

$$u_{i+1} = u_i + h_i \overline{\nabla}_f u_i \Big/ \left(\left\|\overline{\nabla}_u f_i\right\| + \alpha_i\right)$$

yields a sequence of values u_i deviating from the stationary points alternately by each coordinate in the opposite directions by the value Δu_i. Moreover, the lower bound $\inf\left\|\Delta u_i\right\| = \left|\dfrac{h_i \Delta t \left\|\overline{\nabla}_f u_i\right\|}{\left\|\nabla_{uu} f_i\right\|}\right|^{\frac{1}{2}}$ of its norm is achieved for $|h_i| = \left|\dfrac{\partial f}{\partial t}\left\|\overline{\nabla}_u f\right\|\Delta t\right|_i$.

Proposition 3. In the maximization problem, the sign of the step h_i obeys the following rule: h_i has the same sign as $\left(\Delta f(t_i, u_i) - \Delta t_i \left.\dfrac{\partial f}{\partial t}\right|_{\hat{t} \in [t_i,\, t_{i+1}]}\right)$ if

$\Delta f(t_i, u_i) \ge 0$ and the opposite sign otherwise $(\Delta f(t_i, u_i) < 0)$.

The proofs of Propositions 1–3 are postponed to the Appendix.

Remark 1. Given an admissible solution error δ, the admissible class of all functions $f(t, u)$ for which $\|\Delta u_i\| \le \delta$ must satisfy the inequality $\left| \frac{\left.\frac{\partial f}{\partial t}\right|_i \Delta t \|\nabla_f u_i\| \|\nabla_u f_i\|}{\|\nabla_{uu} f_i\|} \right|^{\frac{1}{2}} \le \delta$ on the time interval under consideration (when needed, on the entire definitional domain). Therefore, greater values $\left|\left.\frac{\partial f}{\partial t}\right|_i\right|$ and Δt lead to greater errors.

Remark 2. According to Proposition 1, a fluctuating process approximates the optimum at the first iteration of the method. According to Remark 1, the process will not leave the tube $\|\Delta u_i\| \le \delta$.

Remark 3. Proposition 3 is applicable if the derivative $\frac{\partial f}{\partial t}$ varies in the period Δt_i so that the sign of $\left(\Delta f(t_i, u_i) - \Delta t_i \left.\frac{\partial f}{\partial t}\right|_t \right)$ is fixed for $t \in [t_i, t_{i+1}]$. Under a fixed step Δt, the value $\Delta t \left.\frac{\partial f}{\partial t}\right|_{\hat t}$ can be estimated using the three-point approximation

$$\Delta t \left.\frac{\partial f}{\partial t}\right|_{\hat t} \cong$$

$$\cong \frac{(f_i - f_{i-1})(u_{i+1} - u_i) - (f_{i+1} - f_i)(u_i - u_{i-1})}{u_{i+1} - 2u_i - u_{i-1}}.$$

This estimate can be obtained from the system of equations

$$f_{i+1} - f_i \cong \frac{\partial f}{\partial u}(u_{i+1} - u_i) + \Delta t \left.\frac{\partial f}{\partial t}\right|_{\hat t},$$

$$f_i - f_{i-1} \cong \frac{\partial f}{\partial u}(u_i - u_{i-1}) + \Delta t \left.\frac{\partial f}{\partial t}\right|_{\hat t},$$

assuming that the derivative $\frac{\partial f}{\partial t}$ has a small variation in the period Δt.

Remark 4. Choosing the value α in the iterative method so that $\alpha_i \ge \alpha = \text{const} \ge 0$, we obtain a greater range of the increment

$$\Delta u_{i+1}(\alpha) = h \bar\nabla_f u_i / \left(\|\bar\nabla_u f_i\| + \alpha \right) \ge \Delta u(\alpha_i),$$

for which the differences $u_i^k - \hat u_i^k$ and $u_{i+1}^k - \hat u_i^k$ have opposite signs. The greater the difference $\alpha_i - \alpha$ is, the greater the range of fluctuations $u_i - \hat u_i$ will be.

The admissible values of the parameters α_i and h_i can be chosen within a rather wide range without any accurate estimates of the values $\left.\frac{\partial f}{\partial t}\right|_i$ and $\|\nabla_{uu} f_i\|$. The closer the parameter α_i to 0 is, the greater the range of Δu_i will be. Decreasing the parameter h_i reduces the range of Δu_i; however, for very small h_i, the algorithm diverges: the total error increases between iterations.

Remark 5. Since the external factors affect the objective function through its derivative $\left.\frac{\partial f}{\partial t}\right|_{t_i}$, the range of control values directly depends on the value of this effect.

Remark 6. Since the values of the target function $f(t, u)$ are calculated (or measured) only at the nodes of the discrete time grid and for the corresponding values of the control vectors, it is possible to construct a spline approximation of this function of the required smoothness and apply exact methods of gradient descent for it [10, 11]. However, the accuracy of the obtained result will remain finite, since such a spline approximation is not unique. In addition, the computational complexity of this method will significantly exceed the complexity of the proposed approach.

3. NUMERICAL EXAMPLE

The approximate method for optimizing a discrete time-varying system is illustrated below by a numerical example of optimal controller design for a simplified model of production. This model is described by the following discrete-time finite difference relations. However, according to the problem statement, only the values of the objective function and control at the previous and current time instants are used for solution.

The control (and simultaneously the state parameter) is the production output $u(t)$. The objective function—profit––has the form

$$r(t) = f(t, u(t)) = p(t)u(t) - C_0 - cu^2(t) \to \max_u,$$

where $t = 0, 1, 2, \ldots$; $p(t)u(t)$ gives the income; C_0 are fixed costs; $C_0 + cu^2$ is an estimate of the total costs including production assets, remuneration of labor and direct expenditures; finally, $p(t)$ specifies the unit price of the products (the environment's parameter)

$$p(t) = d\, p(t-1), \ \ p(0) = p_0,$$

where d is the growth coefficient, and p_0 is an initial price.

The marginal profit (the gradient of the objective function with respect to the control variable) is estimated as

$$e(t) = (r(t) - r(t-1))/(u(t) - u(t-1)).$$

At the next step, the control is calculated using the proposed approximate optimization method:

$$u(t+1) = u(t) + h/((e(t)(|e(t)| + \alpha)).$$

Here α is the stabilization parameter, and h is the coupling coefficient.

The figure shows the simulation results for the production output model with the parameters $C_0 = 1$ and $c = 1$ and the optimal controller with the parameters $\alpha = 0.1$ and $h = 1$. The product price with the initial value $p_0 = 3$ varies with the constant rate $d = 1.03$. The exact optimal solution has the form

$$u(t) = \frac{p(t)}{2c}.$$

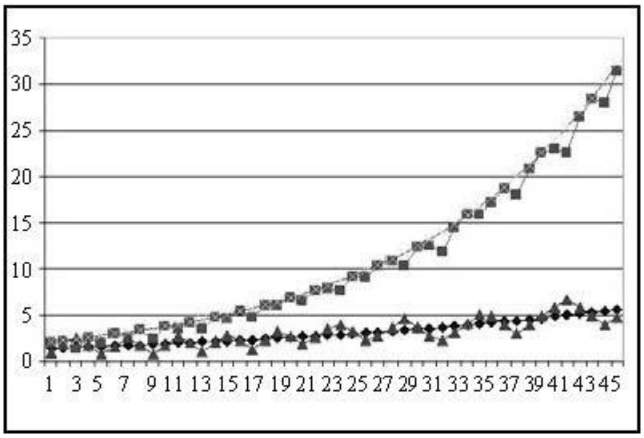

Fig. Simulation results for optimal controller: **optimal solution,** **calculated solution,** **optimal profit, and** **calculated profit.**

The optimal controller quickly (in one step) approximates the production output the optimal one and, over time, tracks the optimal output within the method's error.

The range of fluctuations around the optimal output is conditioned by the discrete nature of the model, the nonstationary behavior of the product price, and an inaccurate choice of the controller's parameters. With an increase in the time derivative of the price, the solution error grows, which agrees with the approximation estimate presented above. At the initial steps, the value h_i has the estimate $|h_i| = \left|\frac{\partial f}{\partial t} \|\bar{\nabla}_u f\| \Delta t\right|_i \cong 1$ and the sign +1. An appropriate estimate of the coefficient α_i was selected from the stability considerations to satisfy $\|\bar{\nabla}_u f_i\| >> \alpha_i > 0$. They can be experimentally refined along the trajectory by maximizing the value achieved by the objective function $f(t, u(t))$ during several steps of the discrete algorithm. In the example, the values of α_i and h_i were constant along the entire trajectory.

CONCLUSIONS

The approximate optimization method proposed in this paper is not very critical to the choice of the parameters α and h. They can be determined in a particular way depending on the applied problem under consideration. In the numerical example, the value of the parameters corresponds to the rate of increase in the unit price of products. In addition, the parameter h can be estimated using finite-difference approximations for the derivatives of the function $f(t, u)$. In this case, the parameter α can be estimated as $\alpha \cong \|\bar{\nabla}_u f_0\|^2 \times h / \max_t \left|\frac{\partial f}{\partial t}\right|$. According to numerical simulations, varying the controller's parameters in a rather wide range has an insignificant effect on the qualitative behavior of the calculated trajectory.

APPENDIX

Proof of Proposition 1. Using the linear part of the Taylor–Lagrange series for the function $f(t, u)$ at the point (t_{i+1}, u_{i+1}), where the remainder is given at the intermediate point $(t_i, \hat{u}^k), t_i \le \hat{t} \le t_{i+1}, \hat{u}^k \in [u^k_i, u^k_{i+1}]$, we obtain

$$f(t_{i+1}, u_{i+1}) - f(t_i, u_i) = .$$
$$= \sum_k \left.\frac{\partial f}{\partial u^k}\right|_{\hat{t},\hat{u}} (u^k_{i+1} - u^k_i) + \left.\frac{\partial f}{\partial t}\right|_{\hat{t},\hat{u}} (t_{i+1} - t_i).$$

According to the first-order optimality condition, let $\left.\frac{\partial f}{\partial u^k}\right|_{\hat{t},\hat{u}} = 0$. Then

$$\Delta u^k_i = u^k_{i+1} - u^k_i =$$
$$= (t_{i+1} - t_i) \left.\frac{\partial f}{\partial t}\right|_{\hat{t},\hat{u}} / \frac{f(t_{i+1}, u_{i+1}) - f(t_i, u_i)}{u^k_{i+1} - u^k_i},$$

which can be written in the vector form as

$$\Delta u_i = \Delta t_i \left.\frac{\partial f}{\partial t}\right|_{\hat{t},\hat{u}} \bar{\nabla}_f u_i. \qquad (1)$$

Now we present a calculation method suitable for numerical implementation. Let α_i and h_i be determined from the relation

$$\frac{\partial f_i}{\partial t} = h_i / \left(\Delta t_i \left(\|\bar{\nabla}_u f_i\| + \alpha_i\right)\right). \qquad (2)$$

Then, near the stationary point $\hat{u}_i \in [u_i, u_{i+1}]$, we have

$$\Delta u_i = h_i \bar{\nabla}_f u / \left(\|\bar{\nabla}_u f_i\| + \alpha_i\right).$$

Since $\hat{u}_i$ is an inner point of the interval $[u_i, u_{i+1}]$, the differences $u^k_i - \hat{u}^k_i$ and $u^k_{i+1} - \hat{u}^k_i$ have opposite signs. Thus, u_i will coordinate-wise fluctuate around the stationary points $\nabla_u f|^{\hat{u}}_i = 0$.

Proof of Proposition 2. We denote by $\overline{\Delta u}$ the method error at the current step due to the discrete time scale. Near the stationary point, the gradient can be estimated as

$$\bar{\nabla}_u f_i = \nabla_{uu} f_i \overline{\Delta u} .$$

If the value of the derivative $\frac{\partial f}{\partial t}$ is exactly known, then for the values h and α satisfying (2), we use condition (1) to obtain

$$\left\|\overline{\Delta u}\right\| = |h|\,\Delta t \left\|\bar{\nabla}_f u_i\right\| / \left(\left\|\nabla_{uu} f_i\right\| \left\|\overline{\Delta u}\right\| + \alpha\right).$$

Solving the quadratic equation for $\left\|\overline{\Delta u}\right\|$ yields

$$\left\|\overline{\Delta u}\right\| = -\frac{\alpha}{2\left\|\nabla_{uu} f_i\right\|} + \left(\frac{\alpha^2 + 4|h|\,\Delta t \left\|\bar{\nabla}_f u_i\right\| \left\|\nabla_{uu} f_i\right\|}{4\left\|\nabla_{uu} f_i\right\|^2}\right)^{\frac{1}{2}} .$$

Obviously, the function $\left\|\overline{\Delta u}\right\|(\alpha)$ is monotonically increasing.

From the relation (2) it follows that

$$h_i = \frac{\partial f_i}{\partial t}\left(\left\|\bar{\nabla}_u f_i\right\| + \alpha_i\right)\Delta t_i ,$$

$$|h_i| \ge \left|\frac{\partial f}{\partial t}\left\|\bar{\nabla}_u f\right\|\Delta t\right|_i . \quad (3)$$

For α = 0, the expressions (2) and (3) give

$$\left\|\overline{\Delta u}\right\|(\alpha = 0) = \left(\frac{|h|\,\Delta t \left\|\bar{\nabla}_f u_i\right\|}{\left\|\nabla_{uu} f_i\right\|}\right)^{\frac{1}{2}} =$$

$$= \left|\frac{\left.\frac{\partial f}{\partial t}\right|_i \Delta t \left\|\bar{\nabla}_f u_i\right\| \left\|\bar{\nabla}_u f_i\right\|}{\left\|\nabla_{uu} f_i\right\|}\right|^{\frac{1}{2}} .$$

Due to this equality, the estimate

$$\left\|\overline{\Delta u}\right\| = \left|\frac{\left.\frac{\partial f}{\partial t}\right|_i \Delta t \left\|\bar{\nabla}_f u_i\right\| \left\|\bar{\nabla}_u f_i\right\|}{\left\|\nabla_{uu} f_i\right\|}\right|^{\frac{1}{2}}$$

is a lower bound for the solution error.

Proof of Proposition 3. According to Proposition 1, the interval $\left[u_i, u_{i+1}\right]$ does not contain the stationary point if for some (or all) coordinates, the differences $u_{i+1} - u_i$ have the same signs.

In this case, assuming $\left.\frac{\partial f}{\partial u^k}\right|_{\hat{t} \in [t_i,\, t_{i+1}]} \ne 0$, we obtain

$$\Delta u^k{}_i = u^k{}_{i+1} - u^k{}_i =$$

$$= \frac{f(t_{i+1}, u_{i+1}) - f(t_i, u_i) - (t_{i+1} - t_i)\left.\frac{\partial f}{\partial t}\right|_{\hat{t}}}{\left.\frac{\partial f}{\partial u^k}\right|_{\hat{t}}},$$

which can be written in the vector form

$$\Delta u_i = \left(\Delta f(t_i, u_i) - \Delta t_i \left.\frac{\partial f}{\partial t}\right|_{\hat{t}}\right) \nabla_f u\Big|_{\hat{t}} .$$

In the maximization problem, for a fixed absolute value of the step h, the descent direction is chosen from the condition

$$\operatorname{sign}(h) = \operatorname{sign}\left(\Delta f(t_i, u_i) - \Delta t_i \left.\frac{\partial f}{\partial t}\right|_{\hat{t}}\right) \text{ if } \Delta f(t_i, u_i) > 0 ,$$

$$\operatorname{sign}(h) = -\operatorname{sign}\left(\Delta f(t_i, u_i) - \Delta t_i \left.\frac{\partial f}{\partial t}\right|_{\hat{t}}\right) \text{ if } \Delta f(t_i, u_i) < 0 .$$

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