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INFORMATION COMMUNITIES IN SOCIAL NETWORKS. PART II: NETWORKED MODELS OF FORMATION[1]

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Abstract. This survey deals with mathematical models for the formation of information communities under uncertainty. The models of opinion dynamics are considered in detail. Within these models, individuals change their opinions under the influence of other individuals in a social network of a nontrivial structure. Two classes of such models are presented: the models with rational (Bayesian) individuals and the models with naive (heuristic) individuals. For each of the classes, conditions for the formation of information communities in social networks are described. For various information communities to emerge in a society with rational agents, the rationality of individuals is often limited, and some assumptions on different awareness of individuals are introduced considering the network structure. For a society with naive individuals, different modifications of the opinion dynamics mechanism are often adopted.

Keywords: social networks, information community, formation of information communities, belief dynamics, naive individuals, rational individuals.

INTRODUCTION

As noted in part I of the survey (see [1]), identifying and studying information communities in social networks – the sets of individuals with similar and stable beliefs about a given issue – is an important problem in many subject areas. To solve this problem, we should understand the patterns of belief dynamics in a social network. Features of information processing by an individual are considered in cognitive science, psychology, and social psychology; for example, see [2, 3]. Formal microlevel models are developed to describe the belief dynamics in networks with these features; for example, see [4–8]. Models of belief dynamics and the formation of information communities in social networks based on microeconomic, cognitive, and socio-psychological foundations were discussed in part I of the survey [1]. Particularly, the concept of an information community was outlined, and a general conceptual model was introduced to describe information processing and decision-making by an individual in a social network. Within this model, agents seek to eliminate uncertainty about the environment's parameter, observing external signals and the actions of their neighbors in the social network. The factors affecting belief dynamics and the formation of information communities in social networks were considered. According to the analysis of the existing models, rational agents in a society of a degenerate structure often reach a true belief about the issue. For various information communities to emerge in such a society, the rationality of individuals and their awareness should be modified somehow; for example, see [9–11]. However, part I of the survey did not touch upon two key factors affecting the formation of information communities: the structure of a social network and agents with heuristic belief updating rules. These issues will be considered below.

Part II of the survey is organized as follows. Section 1 considers the formation of information communities in models with Bayesian agents interacting in the network. Section 2 considers the formation of information communities in a network of agents with heuristic belief updating rules.

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1. FORMATION OF INFORMATION COMMUNITIES IN SOCIAL NETWORKS WITH BAYESIAN AGENTS

In models with a network structure, a finite or countable set of individuals is specified. The main elements of networked models of belief dynamics are the awareness structure of agents, the set of their actions and payoff functions, and the observability of the actions of other agents. Let us describe them in detail.

The awareness structure of agents. The state of the world – the value of the parameter $\theta \in \Theta$ – is a realization of a random variable unobservable by individuals. Each individual i has private information: a private signal s_i representing a random variable whose distribution depends on the value θ. The signal value provides information about the true value of θ. Private signals are conditionally independent of the state of the world θ. The private belief is set initially and does not change over time. The agent's belief in some period t will depend on his observations in previous periods.

The actions and payoffs of agents. In a given period, each agent i can perform an action $x_i \in X$ once, gaining a payoff $u(x_i, \theta)$. When choosing his action, the agent is guided by the subjective probability θ and the expected payoff from performing the action, $U = E_i[u(x_i, \theta)]$, considering all available information. The informative value of the agent's action for observers depends on the set X.

In the case of binary actions $X = \{0, 1\}$ and the state space $\Theta = \{0, 1\}$, the payoff function is defined as $u(x, \theta) = \theta - c$, where $0 < c < 1$. Under uncertainty, the payoff is given by

$$u(x) = (E[\theta] - c)x.$$

The standard way to define continual choice is to assume that the agent chooses an action $x \in R^1$ by maximizing the expected value of a quadratic payoff function:

$$u(x, \theta) = -E\left[(x-\theta)^2\right].$$

The optimal action is $x = E[\theta]$, yielding the agent's expected payoff $U = -\mathrm{Var}(\theta)$.

Public information and action history. The order of agents' actions (interaction protocol) is defined in advance. Agent t ($t \geq 1$) chooses an action in period t. The action history by this period has the form

$$h_t = \{x_1, \ldots, x_{t-1}\}.$$

Agent t knows the action history h_t when choosing his action. At the beginning of period t (before making their decisions), the agents have the following common knowledge:

- the prior probability distribution of the state of the world θ,
- the distribution of private signals and the payoff functions of all agents,
- the action history h_t.

The payoffs of different agents are unobservable.

With the elements described above, *the belief updating process* of individuals is as follows. In a period $t \geq 1$, the probability distribution of the state of the world θ, which is based on the public information h_t only, is called *the public or social belief* $F(\theta \mid h_t)$. Agent t uses the public belief and private information (signal s_t) to form his belief about the state of the world, which has the distribution $F(\theta \mid h_t, s_t)$. Then he chooses an action maximizing the payoff $E[u(x_t, \theta)]$ depending on his belief. The other agents know the payoff function of agent t and his decision model. The observed action x_t is treated by them as the information available to agent t (the private signal s_t). According to this information, the agents update the public belief $F(\theta \mid h_{t+1})$.

Remark. Social learning is *effective* if the individual's action fully reveals his private information. This is possible if the set of admissible actions is large enough.

Let us consider basic models – the models of belief updating with continual and discrete actions – in which individuals observe the actions of *all* predecessors.

In *the model of belief dynamics with continual actions*, the state of the world is the realization of a random variable or vector with the Gaussian distribution $N(\bar{\theta}, 1/\rho_\theta)$ in the initial period. A countable number of individuals $i = 1, 2, ...$, is given. Each individual i receives a private signal s_i representing the sum of the true value and some noise $\epsilon_i \sim N(0, 1/\rho_\epsilon)$:

$$s_i = \theta + \epsilon_i.$$

The agent's payoff is $u(x, \theta) = -E\left[(x-\theta)^2\right]$. Individual t chooses an action $x_t \in R$. The public information at the beginning of period t consists of the prior distribution $N(\bar{\theta}, 1/\rho_\theta)$ and the action history $h_t = \{x_1, \ldots, x_{t-1}\}$.

Suppose that the public opinion about the value θ in period t obeys the Gaussian distribution $N(\mu_t, 1/\rho_t)$. Then the same assumption is valid in the period $t = 1$ for the parameters $\mu_1 = \theta$ and $\rho_1 = \rho_\theta$. As is easily demonstrated, it will hold for each subsequent period. In any period, the public belief is updated in three stages as follows.

- Calculating the belief of agent t. The public belief $N(\mu_t, 1/\rho_t)$ is updated according to Bayes' rule based on the private information $s_t = \theta + \epsilon$. The public belief is the distribution $N(\tilde{\mu}_t, 1/\tilde{\rho}_t)$ with the parameters

$$\tilde{\rho}_t = \rho_t + \rho_\epsilon,$$

$$\tilde{\mu}_t = \alpha_t s_t + (1-\alpha_t)\mu_t, \text{ where } \alpha_t = \rho_\epsilon / \tilde{\rho}_t.$$

• Choosing the action x_t of agent t. The agent seeks to maximize the payoff $-E\left[(x-\theta)^2\right]$. He chooses an action equal to the expectation of θ:

$$x_t = \alpha_t s_t + (1-\alpha_t)\mu_t.$$

• Social learning. Network agents observe the action x_t and update the public belief about θ during the next period. Recall that the decision rule of agent t and the values α_t and μ_t are known to all agents. Therefore, *the observed action x_t fully reveals the private signal s_t*. The public information at the end of period t is identical to the information of agent t: $\mu_{t+1} = \tilde{\mu}_t$ and $\rho_{t+1} = \tilde{\rho}_t$. Hence, in period $(t + 1)$, the belief still has the Gaussian distribution $N(\mu_{t+1}, 1/\rho_{t+1})$, and the learning process can be continued. Note that the action history $h_t = \{x_1, \ldots, x_{t-1}\}$ is equivalent by information content to the sequence of signals ($s_1, \ldots, s_{t-1}$).

The accuracy of public persuasion increases according to the law $\rho_t = \rho_\theta + (t-1)\rho_\epsilon$, i.e., the variance will converge to zero. Also, the significance of private signals tends to zero, and the agents will accordingly imitate each other's actions. Under "noisy" observations of the actions of other agents, the rate of social learning decreases [12]. There are modifications of the basic model of social learning [13] in which the agent *pays for the required accuracy p* of his private signal and then performs an action. Under minimum assumptions about the cost function $c(p)$, it can be proved that the agents will stop "buying" the signal after some time, and social learning will stop.

In *the model of belief dynamics with discrete actions*, the state of the world $\theta \in \Theta = \{0,1\}$ is specified randomly in the initial period, $\mu_1 = P(\theta = 1)$. A finite (N) or countable number of agents is indexed by integer t. Each agent receives a symmetric private signal q> ½: $P(s_t = \theta|\theta) = q$. Agent t chooses an action $x_t \in \{0,1\}$ in period t (and only in this period). The agent's payoff is given by the state of the world:

$$u(x,\theta) = \begin{cases} 0, & x = 0, \\ \theta - c, & x = 1, \end{cases}$$

where $0 < c < 1$. (In the classical BHW-model, proposed by Bikhchandani, Hirshleifer, and Welch [144], the state of the world θ is the agent's payoff from action 1 (accept), and the parameter c describes the costs incurred by action 1.) Since $x \in \{0,1\}$, the payoff can be written as $u(x,\theta) = (\theta - c)x$. Under uncertainty, the agent considers the payoff to be the expectation of $u(x, \theta)$ given the available information.

As before, the information available to agent t is his private signal and the action history h_t. The public belief at the beginning of period t is the probability of state 1 given the public history h_t:

$$\mu_t = P(\theta = 1|h_t).$$

As was shown in [14], *an information cascade* can quickly arise in such models: the agents in sequence ignore their private signals and act in the same way as their predecessors, thereby not providing their followers with new information. In other words, the entire society ineffectively aggregates the available information and may come to wrong beliefs. In a possible modification [15] of this model, the agents acquire information to change their actions. If the information helps break the current consensus and is reasonably "inexpensive," the agents will reach the right beliefs and actions.

Thus, in the models of belief updating with a canonical structure of a social network (in which each agent observes the actions of all predecessors), social interactions will yield one information community with a true or false belief about the issue of interest. (The conditions of true or false belief have been discussed above.) Let us now consider the models of belief updating for Bayesian agents with a more complex topology of social networks.

Formation of information communities in networks with a nontrivial structure

Let us start with the exemplary model of sequential social learning with a nontrivial structure. Consider a countable set of agents (individuals) indexed by $n \in \mathbb{N}$ [16]. Agents make decisions sequentially and once. The payoff of agent n depends on his action and the initial state of the world θ. For simplicity, the state of the world and the actions of agents are assumed binary: for agent n, the action is $x_n \in \{0,1\}$, and the state of the world is $\theta \in \{0,1\}$. The payoff of agent n is given by

$$u_n(x_n, \theta) = \begin{cases} 1, & x_n = \theta, \\ 0, & x_n \neq \theta. \end{cases}$$

Also, the values of the state of the world are supposed equally probable, i.e., $P(\theta = 0) = P(\theta = 1) = 1/2$. The agents do not know the state θ. Each agent forms a belief about the state of the world by observing a private signal $s_n \in \bar{S}$ ($\bar{S}$ is a metric space) and the actions of other agents. The signals are conditionally independently generated according to the probability measure F_θ. The pair (F_0, F_1) is called *the signal structure*. The measures F_0 and F_1 are absolutely continuous relative to each other: a signal that would completely reveal the state of the world is impossible.

Each agent n observes the actions of his neighbors in the social network only, i.e., the agents from the set $B(n) \subseteq \{1, 2, \ldots, n-1\}$. Each network neighborhood $B(n)$ is generated according to some probability distribution Q_n defined on the set of all subsets $\{1, 2, \ldots, n - 1\}$. Each distribution Q_n in the sequence $\{Q_n\}_{n \in N}$ does not depend on other distributions and the realizations of private signals. The sequence $\{Q_n\}_{n \in N}$ forms the social network topology, which is common knowledge, unlike the realized neighborhood $B(n)$ and the private signal s_n. The canonical topology considered in the literature, when each agent observes all previous actions, is realized if for any $n \in \mathbb{N}$ the probability of the neighborhood $\{1, 2, \ldots, n - 1\}$ is equal to 1 in the distribution Q_n. Other options are also possible, for example, the realization of a random graph model.

The information set of agent n is defined as

$$I_n = \{s_n, B(n), x_k \text{ for all } k \in B(n)\}.$$

Let $\boldsymbol{I}_n$ denote the set of all admissible information sets of agent n. The strategy of agent n, σ_n: $\boldsymbol{I}_n \to \{0,1\}$, is a mapping of the set of all admissible information sets into the action set. The strategy profile is a sequence of strategies $\sigma = \{\sigma_n\}_{n \in N}$. For a given strategy profile σ, the sequence of actions of network agents is a random process $\{x_n\}_{n \in N}$. This process generates a measure P_σ. A strategy profile σ^* is a *perfect Bayesian equilibrium* in the class of pure strategies in the social learning game if, for any $n \in N$, the strategy σ^*_n maximizes the expected payoff of agent n under the opponents' strategy profile σ^*_{-n}.

For a given strategy profile σ, the expected payoff of agent n from the action $x_n = \sigma_n(I_n)$ is $P_\sigma(x_n = \theta \mid I_n)$. Hence, for any equilibrium σ^*,

$$\sigma^*_n(I_n) \in \operatorname*{Argmax}_{y \in \{0,1\}} P_{(y, \sigma^*_{-n})}(y = \theta | I_n).$$

This social learning game has a perfect Bayesian equilibrium in the class of pure strategies. The set of all such equilibria is denoted by Σ^*.

The following question is of interest: Will the equilibrium behavior guarantee asymptotic learning? Formally speaking, *asymptotic learning* arises in an equilibrium σ if the value x_n converges in probability to θ:

$$\lim_{n \to \infty} P_\sigma(x_n = \theta) = 1.$$

The agents' actions can be characterized as a function of the sum of two posterior beliefs: the agent's private belief and the *social belief*. In an equilibrium $\sigma \in \Sigma^*$, the decision of agent n, $x_n = \sigma_n(I_n)$, has the form

$$x_n = \begin{cases} 1, & p_n + q_n > 1, \\ 0, & p_n + q_n < 1, \end{cases}$$

and $x_n \in \{0,1\}$ otherwise. Here $p_n = P_\sigma(\theta = 1 \mid s_n)$ is *the private belief*, and $q_n = P_\sigma(\theta = 1 \mid B(n), x_k, k \in B(n))$ is *the social belief*.

The private belief of agent n does not depend on the strategy profile. Using Bayes' rule, it can be written as

$$p_n = \left(1 + \frac{dF_0}{dF_1}(s_n)\right)^{-1}.$$

The support of the private beliefs is the range $[\underline{\beta}, \overline{\beta}]$, where $\underline{\beta} = \inf\{r \in [0, 1] \| P(p_1 \le r) \rangle 0\}$ and $\overline{\beta} = \sup\{r \in [0,1] \mid P(p_1 \le r) < 1\}$. The signal structure has *bounded private beliefs* if $\underline{\beta} > 0$ and $\overline{\beta} < 1$, and *unbounded* if $\underline{\beta} = 1 - \overline{\beta} = 1$. In the latter case, the agents can receive an arbitrarily strong signal in favor a certain state.

Consider some properties of network topologies and signal structures to present further results on asymptotic social learning. A network topology has *expanding observations* if, for all $K \in N$,

$$\lim_{n \to \infty} Q_n\left(\max_{b \in B(n)} b < K\right) = 0.$$

The following theorem was proved: if the network topology $\{Q_n\}_{n \in N}$ does not have expanding observations, then *there exists no equilibrium σ in which asymptotic learning will be achieved*. If the network topology does not have expanding observations, there is a finite set of agents whose actions will be observed with a positive probability by an infinite number of agents. As a result, they will be unable to aggregate information dispersed through the network. (Such a finite set of agents is called *excessively influential* [16].)

If the network topology $\{Q_n\}_{n \in N}$ has expanding observations, and the signal structure (F_0, F_1) implies the unboundedness of private beliefs, then the theorem on asymptotic learning in each equilibrium $\sigma \in \Sigma^*$ holds. In particular, this theorem guarantees learning in the case of moderately influential agents (i.e., their actions are visible to the entire society): they are not the only sources of information in the network. As an example, consider a network in which all other agents observe the actions of the first K agents, but each agent also observes its immediate neighbor, i.e., $B(n) = \{1, 2, ..., K, n - 1\}$. This network topology has expanding observations and, therefore, leads to learning under unbounded private beliefs. This conclusion contradicts the results for non-Bayesian learning models (see [17, 18]), in which the new beliefs of agents are the weighted average of the private beliefs and the beliefs of the agents they observe: if the first K agents are influential (other agents observe their actions), there will be no asymptotic learning.

Now consider the signal structure (F_0, F_1) in which the private beliefs are bounded, and the network topology $\{Q_n\}_{n \in N}$ satisfies one of the following conditions:

– $B(n)=\{1,\ldots,n-1\}$ for all n (see the paper [199]).

– $|B(n)| \le 1$ for all n.

– There exists a constant M such that $|B(n)| \le M$ for all n and

$$\lim_{n\to\infty} \max_{b\in B(n)} b = \infty \text{ a.s.}$$

Then asymptotic learning will not be achieved in any equilibrium $\sigma \in \Sigma^*$. Particularly, there is no asymptotic learning in a network where each agent n chooses $M \ge 1$ neighbors from the set $\{1, \ldots, n–1\}$ uniformly and independently.

Note that in this model, the agents perform their actions once, subsequently gaining their payoffs. In some situations, *action can be postponed:* agents may exchange messages—their information—without considerable costs (except for time) to obtain additional information. An example of the agent's payoff function [20] is

$$u_i(x_i, \theta) = \begin{cases} \delta^\tau \pi \text{ if } x_{i,\tau} = \theta \text{ and } x_{i,t} = \text{wait for } t < \tau, \\ \qquad\qquad x \text{ otherwise,} \end{cases}$$

where $x_i = [x_{i,t}]_{t=0,1\ldots}$ is the action sequence of agent i ($x_i \in \{\text{wait}, 0, 1\}$); $\pi > 0$ specifies the agent's payoff; $\delta \in (0, 1)$ denotes the discount factor. At the qualitative level, an analog of this two-stage model in the case of agents with heuristic belief updating rules is the model [21], in which agents first form their opinions and then simultaneously perform their actions in accordance with the payoff functions.

The conditions for achieving learning in a social network [16] are rather mild. The typical outcome of Bayesian social learning models is a long-term consensus. In order to obtain information communities with different beliefs, it is necessary to relax the rationality requirement for social network individuals.

In particular, the concept of *quasi-Bayesian updating* of agents is widespread [22–24]: each agent in the network believes that the actions of other agents are caused exclusively by their private signals. (This concept is associated with *cognitive constraints*—the limited depth of agent's inference.) The paper [24] considered sequential social learning in a social network (observation network) under the following assumptions. The world can be in one of two equiprobable states $w \in \{0, 1\}$. There is a countable set of agents indexed by $i = 1, 2, 3,\ldots$, which act once and in turn (sequentially). At his move, agent i receives a private signal about the state of the world, $s_i \sim N(1, \sigma^2)$ if $w = 1$, or $s_i \sim N(-1, \sigma^2)$ if $w = 0$. In addition, agent i observes the actions of his predecessors in a directed observation network $N_i \subseteq \{1, 2, ..., i - 1\}$. Based on this information, the agent forms his belief about the state of the world w and chooses an action $a_i \in [0, 1]$ maximizing his utility function $u_i(a_i, w) = -(a_i - w)^2$, more precisely, the expected utility $\mathbb{E}\left[-(a_i - w)^2\right]$ (Thus, the action chosen by him corresponds to his belief about the probability of the event $\{w = 1\}$.) The agents in the model are Bayesian, but the cited authors made *a rather strong assumption about the naivety of network participants*: agent i mistakenly believes that the action of his predecessor j in the observation network is conditioned by the private signal of agent j only (he has no predecessors). In other words, agent i supposes that $a_j = \mathrm{P}[w = 1 \mid s_j]$, underestimating the correlation of the actions of his predecessors. Interestingly, the agent's optimal action can be derived by a rule similar to the updating rule in DeGroot's opinion dynamics model. As was established in [24], the society (all agents) of denser observation networks more often comes in the long run to a false estimate of the state of the world compared to that of sparse networks. *Erroneous learning* takes place: either $\lim_{n\to\infty} a_n = 0$ if $w = 1$, or $\lim_{n\to\infty} a_n = 1$ if $w = 0$. This effect can be explained as follows: in sparse networks, "early" agents do not strongly affect each other, and the consensus reached "includes" more independent sources of information and is likely to be correct. Also, it was demonstrated that the agents will almost surely come to a consensus in the case of continuous actions: their disagreements will disappear. However, if the agents' actions are binary (for any agent i, $a_i \in \{0, 1\}$), and the set of agents is divided into two groups with even and odd numbers so that, in accordance with the stochastic block model, the probability of an observation connection from agent i to agent j is equal to q_s if they belong to the same group, and equal to q_d otherwise ($q_s > q_d > 0$), then there is a positive probability that all odd (even) agents will choose action 0 (action 1, respectively). Thus, *information communities with opposite beliefs about the state of the world* will be formed in the connected network.

In the paper [25], agents were assumed *locally Bayesian*: they process information as Bayesian agents, but each considers his ego (local) network to be the entire initial global network, undirected and connected. (In contrast to the models discussed above, the agent does not suppose that his neighbors are guided only by private signals.) In each period, agents form their beliefs about the state of the world based on the private signal received in the previous period and messages from their neighbors (the complete history of messages from neighbors since agents have perfect memory) and then exchange their beliefs. A simple belief updating rule was proposed: agents attribute unexpected changes in the beliefs of their neighbors to the new private signals they receive. (Agents suppose that there are no other agents outside their local network.) As was shown, agents' beliefs fluctuate without stabilizing in some networks.

Constraints on the rationality of network agents are also imposed in models with repeated actions, in which agents repeatedly revise their beliefs and actions (repeated Bayesian updating). This class includes models of repeated actions (1) with locally optimal agents, (2) with heuristic inclusion of information from neighbors, and (3) with rational expectations of agents.

Models of repeated actions with locally optimal agents. In each period, agents choose the best response based on their current beliefs (formed rationally), neglecting the influence of their actions on other agents and the possibility of obtaining additional information in the future. If the agents' actions are continuous and their prior beliefs coincide, a consensus is reached in any connected network with discrete states of the world [23] and Gaussian states of the world [26]. In the case of discrete actions of agents ($x_n(t) \in \{0, 1\}$), a consensus can also be reached in a connected network; see [27, 28]. In the model [28], each agent performs a locally optimal action in each period, taking into account his current beliefs.

Models of repeated actions with heuristic inclusion of information from neighbors. A striking example is the model [29, 30], partly resembling DeGroot's approach. Agents have prior beliefs about the state of the world $\theta \in \{0, 1\}$. At the beginning of each period, each agent receives a private signal and observes the beliefs of his neighbors. In period t, agent n has the belief $p_n(t) = P(\theta = 1)$. First, he updates the belief according to Bayes' rule, taking into account the received signal $s_n(t)$:

$$p_n'(t) \equiv P(\theta = 1 | s_n(t)) =$$

$$= \frac{P(s_n(t)|\theta = 1)\, p_n(t)}{P(s_n(t)|\theta = 1)\, p_n(t) + P(s_n(t)|\theta = 0)(1 - p_n(t))}.$$

Then he averages the resulting belief based on the beliefs of his neighbors using DeGroot's rule:

$$p_n(t+1) = a_{nn} p_n'(t) + \sum_m a_{nm} p_m(t),$$

where the matrix A specifies the weights of his neighbors. If the signals received by the agents are not informative, then their beliefs are formed according to DeGroot's rule; see Section 2. If the signals are informative, the network graph is strongly connected, and each agent "trusts" himself, then the agents' beliefs will almost surely converge to the true estimate of the state of the world.

Models of repeated actions with rational expectations of agents. In the paper [31], the states of the world are from the set $\Theta = \{0, 1\}$. In the initial period, each agent n receives an informative signal s_n. In each period, agent n observes the action of each neighbor $m \in B(n)$ and chooses the action $x_n(t)$, obtaining the payoff

$$u(x_n(t), h_n(t), s_n) = P(\theta = x_n(t) | h_n(t), s_n),$$

where $h_n(t)$ is the history of neighbors' actions by the beginning of period t. Agents discount their future payoffs with a factor $\lambda \in (0, 1)$ and play a repeated game with incomplete information. If the network graph is L-locally connected and there is an upper bound d on the number of observed neighbors, all agents in an infinite (large) network will almost surely (with a high probability) reach the true estimate of the state of the world. A graph G is L-locally connected if, for each edge (n, m), the length of a path from m to n does not exceed L. The property of L-connectedness and the existence of the bound d can be interpreted as the absence of excessively influential agents in the network.

2. FORMATION OF INFORMATION COMMUNITIES IN SOCIAL NETWORKS WITH HEURISTIC AGENTS

Bayesian models initially do not consider the psychological components of personality. As is known from psychology and social psychology, individuals have cognitive limitations and are subject to various socio-psychological factors (including a predisposition to their point of view, the social impact of some individuals on others, conformism, etc.). Various theories and models are being developed to explain the emerging effects. Since the 1950s, mathematical models with simple empirical rules for updating agents' beliefs have been developed and improved to demonstrate the effects observed in practice. The fundamental works in the field of *opinion dynamics modeling* investigate and describe, first of all, the phenomenon of coordinating the opinions of agents (reaching a consensus) when the interaction between network members gradually decreases the disagreement of their opinions. This phenomenon is explained in social psychology by several reasons, particularly by conformism, the acceptance of evidence (persuasion), incomplete information, uncertainty in one's own decisions, etc.

In classical formal models of opinion dynamics (see [4, 7, 32–35]), the sequential averaging of continuous (*continual*) opinions of agents in discrete time was considered. There are some variations of this kind of models with continuous-time averaging [36, 37].

Here is a slightly modified example of the classical DeGroot's model of reaching a consensus in a network structure. In this structure, each agent from the set $N = \{1, \ldots, n\}$ forms his current opinion at each step as the weighted sum of the opinions of all other network agents and his opinion at the previous step:

$$x_i^{(t+1)} = \sum_{j \in N} a_{ij} x_j^{(t)},\ t \geq 0,$$

where $x_i^{(0)}$ denotes the initial opinion of agent i. The parameter $a_{ij} \in [0,1]$ reflects the degree of influence of agent j on agent i ($\sum_j a_{ij} = 1$).

In matrix form, the opinion dynamics can be written as

$$x^{(t+1)} = Ax^{(t)},$$

where A is a row-stochastic influence matrix.

Note that DeGroot's model has microeconomic foundations and relation to Bayesian models. In particular, if the initial beliefs of individuals are noisy, then the DeGroot updating rule is optimal at the first step [17]: the new opinion of an individual is the weighted sum of the opinions of his neighbors, and the weight of the neighbor's opinion is the accuracy of his information. In subsequent periods, the individual must tune the weights of his neighbors since the incoming information can be repeated. This procedure is not easy, so DeGroot's rule with constant weights can be treated as a behavioral heuristic. Another microeconomic foundation is the representation of agents as players participating in a simple coordination game [38]. In this game, the locally optimal best response dynamics (coinciding with the dynamics in DeGroot's model) yield a Nash equilibrium.

The opinion dynamics in DeGroot's model allow reaching a consensus in a strongly connected social network. The agents' opinions gradually coincide since each agent has a direct or indirect impact on any other agent in the network, and the deviations in their opinions finally vanish.

The structure of the interaction network restricts the possibility of reaching a consensus. For example, in a disconnected network, consensus can only be reached in special cases. The disagreement of opinions can also be observed in strongly connected networks if, e.g., agents have initial beliefs somewhat "insensitive" to any influence [39]. In such models, the agent's opinion at each step is the weighted sum of the opinions at the previous step and his initial opinion:

$$x^{(t+1)} = \Lambda A x^{(t)} + \left[I_n - \Lambda \right] x^0,$$

where $\Lambda = I_n - \mathrm{diag}(A)$.

The initial opinions of agents can be interpreted as individual preferences or ingrained beliefs remaining in force during opinions exchange.

The opinion dynamics similar to the considered ones can be obtained using the model with compound nodes [40] in which each node consists of two agents––external and internal—interacting with each other. Each node exchanges information with other nodes through its external agent, and the internal agent (a trusted person of the external one, his friend or consultant) interacts only with the corresponding external agent.

A multidimensional generalization of the model with "insensitive" agents is the model [41] where each agent has an opinion on several interrelated issues (m different topics). The opinion of agent i ($i \in N$) on m different topics is given by the vector $x_i^{(t)} = \left(x_i^{(t)}(1), \ldots, x_i^{(t)}(m) \right)$. The opinion dynamics of agent i in period t are described by

$$x_i^{(t)} = \lambda_{ii} \sum_{j \in N} a_{ij} y_j^{(t-1)} + \left(1 - \lambda_{ii}\right) x_i^{(0)},.$$

$$y_j^{(t-1)} = C x_j^{(t-1)},$$

where C denotes the mutual influence matrix of the topics under discussion, and $y_j^{(t-1)}$ are convex combinations of agent j on several topics. The dynamics can be written in matrix form:

$$x^{(t)} = \left[(\Lambda A) \otimes C \right] x^{(t-1)} + \left[(I_n - \Lambda) \otimes I_m \right] x^{(0)},$$

where $\otimes$ indicates the Kronecker product, and $\Lambda = I_n$ or $\Lambda = I_n - \mathrm{diag} A$ (depending on the model).

Despite the additional factors of these models (the presence of biases and mutually influencing topics) that preserve some mismatch of opinions, the mutual influence of agents gradually decrease their disagreements. In particular, the averaging assumption implies that opinions will never go beyond the range of initial opinions.

Numerous theoretical results were obtained for opinion dynamics models, often associated with reaching a consensus in networks. Such models are studied using the theory of stochastic matrices and the theory of homogeneous and inhomogeous Markov chains. As is known, opinion dynamics can be modeled by Markov chains. In a homogeneous Markov chain, reaching a consensus is determined by the convergence of the powers of its stochastic matrix. Some sufficient conditions for the convergence of the powers of a stochastic matrix were given in [4, 42]. For the class of stochastic matrices without guaranteed consensus, the necessary conditions for reaching a consensus were presented in [42]. The minimum changes in the initial beliefs of agents leading to a consensus were found in [43]. Also, some particular results were established. For example, in [38], random (stochastically block) networks were considered, and the dependence of the rate of convergence of agents' beliefs (updated by the simple averaging rule) on the value of homophily was determined.

The analysis above deals with long-term consensus. In practice, it is often interesting to know the possibility of reaching disagreements in a finite time. How does the network structure affect *medium-term disagreements and the formation of medium-term information communities*? If the initial beliefs of the agents are the same, a consensus is reached immediately and does not depend on the network structure. In general, consensus depends on the initial beliefs and the network structure. The worst case [44] can be examined: What is the rate of convergence of beliefs (the number of steps required for making the disagreements sufficiently small) under any initial beliefs? For simplicity, irreducible and primitive influence matrices A are considered. For a typical matrix of this kind, the following relation holds almost everywhere:

$$A^t = \sum_{l=1}^{n} \lambda_l^t P_l.$$

Also, this matrix has the following properties:

– The values $\lambda_1 = 1$, λ_2, λ_3, …, and λ_n are different eigenvalues of the matrix A sorted in the non-ascending order of the magnitude.

– The matrix P_l is a projection operator corresponding to a nontrivial one-dimensional subspace associated with the eigenvalue λ_l.

– $P_1 = A^\infty$ and $P_1 x^{(0)} = x^{(\infty)}$.

– $P_l 1 = 0$ for all $l > 0$.

The matrix P_1 corresponds to the resulting influence matrix A^∞ and determines a stable state of the system (the resulting beliefs of the agents). The other matrices $P_{l>1}$ reflect the deviation from this resulting matrix in period t. The domination of the matrix P_1 depends on λ_2: the smaller this value is, the faster the stable state will arise in the network [455]. As it turned out, the agents' beliefs in period t satisfy the inequality

$$\frac{1}{2}\left|\lambda_2\right|^t - (n-2)\left|\lambda_3\right|^t \le \sup_{x^{(0)}\in[0,1]^n} \left\| x^{(t)} - x^{(\infty)} \right\|_\infty \le (n-1)\left|\lambda_2\right|^t.$$

Thus, the value $\left|\lambda_2\right|^t$ determines the maximum deviation of the agent's belief from the resulting belief in the network. The matrix P_2 mainly determines the deviation of beliefs from the resulting belief (consensus) and corresponds to *the metastable state of the network in which most of the disagreements disappear, but some stable part remains valid*.

Since $P_2 = \sigma\rho^{\mathrm{T}}$, where ρ^{T} and σ are the left and right eigenvectors, respectively, of the matrix A corresponding to the eigenvalue λ_2, for sufficiently large t the deviation $x^{(t)} - x^{(\infty)}$ will be equal to $\lambda_2^t \sigma(\rho^{\mathrm{T}} x^{(0)})$. If the eigenvalue λ_2 is a positive real number, then the deviation of agent i from the consensus is proportional to the component σ_i, irrespective of the initial beliefs $x^{(0)}$. The order of the medium-term beliefs of the agents is determined by one network-dependent number. An interesting interpretation was given in [17]: the opinions of individuals about different issues can be approximated well by a line; the individual's position on this line (in the left-right spectrum) determines his opinion about all issues. (For example, the opinions of many people about a wide range of fundamentally unrelated issues can be characterized by a measure of their conservatism/liberality.)

The following question arises immediately: *How does the network structure (influence matrix) affect the preservation of different information communities in the network?* The effect of splitting (dividing) the network into groups can be estimated, e.g., using the Cheeger isoperimetric constant (an indicator of graph bottlenecks) [45]:

$$\Phi_*(A) = \min_{S:S\subseteq N, \sum_{i\in S}\pi_i \le \frac{1}{2}} \frac{\sum_{i\in S, j\notin S} \pi_i A_{ij}}{\sum_{i\in S}\pi_i},$$

where π denotes the left eigenvector of the matrix A corresponding to the eigenvalue λ_1 (the influence vector of agents). This constant is small if there exists some set of agents with at most 50% of social influence and a relatively small external influence. If the influence matrix corresponds to a reversible Markov chain, then $\Phi_*^2(A) \le 1-\lambda_2 \le 2\Phi_*(A)$ [45]. In other words, the Cheeger isoperimetric constant provides lower and upper bounds on λ_2: the smaller this constant is, the greater the eigenvalue λ_2 will be, and the longer the disagreements will hold in the network.

The influence (trust) matrix in the belief dynamics models considered above remains invariable. There are studies in which the influence matrix changes over time. The belief dynamics model was generalized in [46] as follows: the influence matrix evolves at each step, and the iterative process is given by the product of matrices:

$$x^{(t)} = A^{(t)} x^{(t-1)}.$$

In this problem statement, the opinions are coordinated by studying the convergence of inhomogeneous Markov chains. Note that the class of opinion formation models is closely related to extensive research into consensus in multi-agent systems (see the surveys [47, 48]). The theoretical results obtained in this area can be transferred to social networks. (Of course, in the case of simple belief updating rules, which often neglect the specifics of the subject area.)

Consensus formation is given much attention in the publications generalizing the classical DeGroot's model, unlike the issues of social learning (the formation of a society with true beliefs).

Forming true beliefs in DeGroot's model. If the agents' beliefs in the network reflect their estimates of the state of the world $\mu \in [0, 1]$, the agents come to a true or false consensus. Is *social learning* possible in a network where the agents update their beliefs by DeGroot's rule? In [18], the initial beliefs (private signals) $x_i^{(0)}$ were supposed independent random variables on [0, 1] with a mean μ and a positive variance. As is known, under sufficiently weak conditions, the agents' beliefs converge to the same final opinion. To assess convergence to the true resulting estimate in a growing society, the sequence of influence matrices $\left(A(n)\right)_{n=1}^{\infty}$ indexed by n (the number of agents in each network) was considered. Each network (the corresponding influence matrix) was assumed *convergent*: any initial beliefs of agents have some limit with respect to it. A sequence $\left(A(n)\right)_{n=1}^{\infty}$ is said to be *wise* if $\operatorname*{plim}_{n\to\infty} \max_{i\le n} \left| x_i^{(\infty)}(n) - \mu \right| = 0$.

One of the society's "wisdom" conditions is associated with the influence of agents. Without loss of generality, agents are rearranged in the descending order of their influence so that $s_i(n) \ge s_{i+1}(n) \ge 0$ for any i and n, where $s_i(n)$ is the weight of the initial opinion of agent i in the final opinion of network n. As was established, the sequence of converging stochastic matrices . is wise if and only if the corresponding influence vectors have the property $s_1(n) \to 0$: the influence of the most influential agent tends to 0 as the society grows infinitely.

Another obstacle to wisdom is the so-called *prominent* groups, which receive a disproportionate share of social attention and lead it astray. For a fixed network with n agents, a *group B* is a subset of the agents' set $\{1,\ldots, n\}$. A group B is *t-prominent* with respect to A (or prominent within t steps) if any agent $i \notin B$ is indirectly influenced by it: $A_{i,B}^t > 0$. The minimum weight of such an influence is called *the t-step prominence of B*: $\pi_B(A;t) = \min_{i\notin B} A_{i,B}^t$. A *family* is a sequence of groups (B_n), where $B_n \subset \{1,\ldots,n\}$ for each n. A family (B_n) is *uniformly prominent* with respect to $\left(A(n)\right)_{n=1}^{\infty}$ if there is a constant $a > 0$ such that, for each n, $\pi_{B_n}(A(n);\, t) \ge a$ at some step t. A family (B_n) is *finite* if it eventually stops growing, i.e., there exists a number q such that $\sup_n |B_n| \le q$. As was established, if there is a finite uniformly prominent family with respect to $(A(n))$, then the sequence is not wise. Thus, large societies with a "small" group affecting everyone in the network will never reach true beliefs about the state of the world.

The opinion dynamics models considered above describe the phenomenon of consensus (disagreements in the opinions of interacting agents decrease over time) and the accompanying phenomenon of social learning. In addition, the effect of medium-term disagreements is possible due to the network structure. In many cases, social and psychological phenomena are observed in social networks [49] as the result of long-term disagreements and stable information communities: the persistence of disagreements, group polarization (during a group discussion, any initially dominant point of view will strengthen), opinion polarization (disagreements between two opposition groups will increase), etc. The classical opinion dynamics models in the long term do not explain well the persistence of disagreements or even the strengthening of radical opinions in strongly connected networks. For these phenomena, new formal mathematical models are being developed [5, 6, 37, 50–53]. In particular, the formation of sets of agents with different beliefs is described by *bounded confidence* models [5, 6, 51] in which only sufficiently close agents can influence each other. (This rule of interaction is usually motivated by the phenomena of homophily and social identification.)

In the model [54], the opinion dynamics were described by a vector $x = (x_1, \ldots, x_n) \in R^n$. Agent i perceives the opinions of other agents only if they are sufficiently close to his opinion. In other words, the set of influence agents of agent i has the form $I(i, x) = \{1 \le j \le n \,|\, |x_i - x_j| \le \epsilon_i\}$, where $\epsilon_i > 0$ is the degree of confidence. (As a rule, $\epsilon_i = \epsilon$.) Then the opinion dynamics are given by

$$x_i(t+1) = |I(i, x(t))|^{-1} \sum_{j \in I(i, x(t))} x_j(t).$$

These dynamics match DeGroot's model with an influence matrix depending on the agents' opinions: $a_{ij}(x) = 1/|I(i, x)|$ if $j \in I(i, x)$, and $a_{ij}(x) = 0$ otherwise. The necessary and sufficient conditions for reaching a consensus were presented. In order to reach a consensus, it is necessary that the opinion vector at any step be an ϵ-profile. (A vector is an ϵ-profile if after sorting its elements in the ascending order, the distance between two neighbor elements will not exceed ϵ.) Otherwise, consensus is impossible: *the trust network splits into connected components, and groups of individuals with the same beliefs (information communities) appear in it*. In any case, the agents will come to equilibrium in a finite number of steps [55].

In [56], "cautious" agents were considered: their trust to received messages depends on the content of messages (opinions expressed in them). In order to reflect this dependence, a trust function $G(x, u)$ was introduced, where x and u denote the agent's opinion and message received by him, respectively. A series of assumptions were accepted regarding the properties of the trust function: different combinations of assumptions lead to different formalizations of the trust function corresponding to: an agent trusting to the received messages regardless of the content, a conservative agent, an innovator agent, a moderate conservative agent, and a moderate innovator agent. For example, the trust function of a moderate conservative agent has the form

$$G(x, u) = \beta\, [1 - (1 - \exp \times \\ \times (-\gamma\, |x - u|))\, \exp(-\gamma\, |x - u|)],$$

where β and γ are constants. In a practical interpretation, the agent selects and perceives information coinciding with his opinion until the disagreement becomes significant enough. Under very large deviations, the probability that he will notice such information increases. In the general case, the controlled opinion dynamics in a social network are described by

$$x_i^k = a_{ii} x_i^{k-1} + G_i\left(x_i^{k-1}, u^{k-1}\right) u^{k-1} + \\ + \sum_{j \in N_i} a_{ij} G_i\left(x_i^{k-1}, x_j^{k-1}\right) x_j^{k-1},\ k = 1, 2, \ldots,$$

where u denotes an external control (e.g., the media), and the individual trust functions $\{G_i(\cdot)\}_{i \in N}$ are such that the normalization condition holds. Within this model, the matrix A reflects the trust of agents to *information sources*; the trust functions, the trust of agents to *the content of information*. In a particular case of a homogeneous and regular network, an optimal informational control problem was stated: find a sequence of controls maximizing an efficiency criterion. This problem can be solved by standard methods. The dependence of the agent's degree of trust to the messages of other agents on their content led to the development of another model [8] of two interrelated processes: the propagation of actions through the network and the formation of agents' opinions. According to numerical simulations, various information communities can be formed in the network as a result of exchanging the beliefs.

We have discussed opinion dynamics models of "heuristic" individuals in a social network with continual beliefs. Generally speaking, these kinds of models—with gradually changing opinions—seem to be most natural. This fact was confirmed by studies in social psychology and behavioral economics. Particularly, as noted in [57], society (Indian villages) is divided into two types of agents: Bayesian agents and those acting by DeGroot's rule. Nevertheless, there are numerous publications where the opinions have ordinal or even nominal scales, etc. Models *with discrete beliefs* (*opinions*) also include voter models [58], majority models, and threshold models [59]. Many of these models, considering the network structure of interactions, are also known as models of the propagation of activity (information) in the network; for example, see the paper [60]. Here are some examples of voter models with discrete beliefs of individuals that illustrate the effect of disagreements in the network.

The paper [61] considered a set of N agents on an $L \times L$ regular lattice ($L^2 = N$). Agent $i \in \{1, 2, \ldots, N\}$ chooses an action $a^{(i)} \in A = \{a_1, \ldots, a_g, \ldots, a_G\}$, guided by his opinion according to a rule $r^{(i)} \in R = \{r_1, \ldots, r_k, \ldots, r_K\}$. The peculiarity of this model is the possibility of specifying multiple relations between opinions-rules and actions: rules are exclusive (one action is mandatory, the rest are prohibited) and inclusive (one of several actions may be performed with equal probability). The agents know the sets A and R and the relation matrix S of dimensions

$K \times G$. They can also observe the actions of neighbors in the network but not their opinions. At the initial step, each agent i randomly chooses a rule $r^{(i)} \in R$ with equal probability and acts according to it. At each next step τ, a randomly chosen agent i updates his beliefs about the rule followed by his neighbor $j \in M_i$ based on the observed action $a^{(j)}(\tau)$:

$$P^{(i)}\left(r^{(j)}(\tau) = r_k | a^{(j)}(\tau)\right) = \frac{P^{(i)}\left(a^{(j)}(\tau) | r_k\right)}{\sum_{k=1}^{K} P^{(i)}\left(a^{(j)}(\tau) | r_k\right)}.$$

Then—at the same step—agent i accepts an action rule in accordance with the probability of its use by the network neighbors.

As was established, *various information communities exist in the network if the agents apply inclusive rules*. If the agents apply only exclusive rules, the model reduces to the classical voter model.

A voting-based belief formation model richer by practical interpretations was proposed in [62]. As noted, long-term disagreements are rare in traditional economic models of social learning, although disagreements arise more often in reality than consensus. Many economic models of social learning rest on the assumption that each new piece of information received by an agent is true: the agent observes an element of the partitioned state space, containing the true state. This assumption (in some cases implicit) leads the agents to a consensus. The concept of information processing by agents was also introduced: agents can receive false information, making disagreements in such models a common outcome of network interactions. Two classes of axioms were presented for the belief updating rules of agents: the axioms of willingness-to-learn (the updating rule allows the agent to learn and reach the true estimate of the state of the world) and the axiom of non-manipulability (the updating rule leads the agent to the same belief when receiving the same information regardless of its form). Different updating rules are possible depending on the selected combinations of axioms. The author discussed two of them in detail:

– the agents adhering to the stubborn updating rule never give up their beliefs;

– the agents follow the stubborn updating rule, but they can completely change their beliefs if the new information completely eliminates the uncertainty about the state of the world.

The agents' interaction protocol is as follows: at each discrete time instant (step), an edge ij of the social network graph is selected randomly; agent i reports a randomly chosen statement P included in his belief B_j to agent j; then agent j updates his belief according to a rule $U_j(B_j, P)$, where the statement P is a subset of the states of the world Ω, i.e., $P \subseteq \Omega$. The network agents will reach a consensus if, at some step, there is a statement P^* such that $P(B_i) = P^*$ for each agent i, where $P(B_i)$ is the intersection of all statements from the beliefs of agent i. As it turned out, to reach a consensus in the network, rather strong assumptions about the truth of the initial beliefs and the number of stubborn agents have to be accepted. As was established, *the formation of different information communities in large networks is possible and even inevitable*.

CONCLUSIONS

Part II of the survey has considered the formation of information communities in societies with a nontrivial network structure. Individuals—the members of society—interact with each other within this structure. Observing the actions of his neighbors in the network, an individual (agent) can obtain additional information about the issue of interest.

Rational agents in such networks reach a consensus in the long run (come to a true or false agreement, depending on the conditions imposed on the topology and/or their initial beliefs). It is necessary to relax the agents' rationality requirement to obtain different information communities. In the first class of the models discussed, the agents are conditionally Bayesian: for example, they have "naive" beliefs about the composition and structure of the network or naively take into account the signals of neighbors. In the second class of the models, the agents are "simple" and forming their beliefs based on heuristic rules. In the case of individuals unconditionally trusting to the actions of their network neighbors, a consensus is a common outcome of network interactions: a single information community will appear. However, it is possible to specify the conditions for forming various medium-term (metastable) information communities. As has been demonstrated, stable information communities differing from each other may appear when: (a) in addition to social influence, individuals are affected by other socio-psychological factors (homophily, the inclination to confirm one's own point of view, etc.) and (b) along with the true information, false information propagates through the network.

Part III of the survey will consider empirical studies related to the existence of information communities in real social networks and their features.

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