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ADAPTIVE NEURAL-NETWORK-BASED CONTROL OF NONLINEAR UNDERACTUATED PLANTS: AN EXAMPLE OF A TWO-WHEELED BALANCING ROBOT[1]

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Abstract. This paper proposes a new method to control nonlinear underactuated plants for eliminating unmatched parametric uncertainties. The method is based on a model reference adaptive control. The controller consists of a basic LQ one and an adaptive compensator reducing the uncertainty norm under certain assumptions. The compensator involves a multilayer neural network due to its universal approximation properties. The network is trained online. The equations to tune the compensator's neural network parameters are derived using Lyapunov's second method and the backpropagation algorithm. The asymptotic convergence of the tracking error (the difference between the plant's and reference model's outputs) to a given domain is proved. The theoretical results are validated by numerical experiments with the developed control system for the mathematical model of a balancing LEGO EV3 robot in MATLAB.

Keywords: model reference adaptive control, balancing robot, suppression of unmatched parametric uncertainties, neural networks, online training, stability.

INTRODUCTION

In modern automatic control practice, the problems of controlling technical systems (plants) with a significant effect of parametric uncertainties are becoming increasingly important. Classical examples include manipulators [1, 2], unmanned and manned aerial vehicles in special operating modes [3, 4], industrial electric drives [5], and technological processes in chemical [6] and metallurgical [7, 8] industries.

Most modern methods for constructing control systems for plants with parametric uncertainties can be divided into robust approaches and model reference adaptive control methods. Robust systems are designed so that the performance criteria of the closed-loop control system (usually the phase and gain margins) satisfy a priori requirements in the worst operating conditions of the plant. On the other hand, adaptive systems estimate the uncertainty online and then form a control action to minimize the plant's deviation from a reference [3]. Compared to robust approaches, model reference adaptive systems need no a priori knowledge of the range of variations in the plant's parameters (the maximum value of parametric uncertainty); with a sufficient power margin for the control action, they yield a reference performance instead of a compromise one.

All methods of constructing model reference adaptive control systems can be divided into direct, indirect, and composite ones; see [3, 10, 11]. In the first case, the parameters of a preselected-structure controller are directly tuned; in the second case, the parameters of the plant and (or) parametric uncertainty are estimated, and this information is used to calculate the controller's parameters. Composite adaptive control systems combine direct and indirect adaptive control approaches to improve the adaptation process.

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The main problems of all three groups of model reference adaptive control methods are as follows [3, 10]:

– The tuned parameters converge to the ideal values only when satisfying the regressor's persistent excitation requirement, which is rather restrictive.

– From the practice viewpoint, the standard tuning loop yields an unsatisfactory quality of transients for tunable parameters, control, and tracking error (especially when the number of tunable parameters increases).

– The gain matrix of the adaptation loop is selected experimentally (manually).

– It is necessary to know the sign of the plant's gain matrix.

In recent years, much effort has been applied by domestic and foreign researchers to solve these problems. Among the last significant results, we mention the publications [12–15].

However, there is a fifth (more fundamental and less developed) problem in the model reference adaptive control theory: the assumption about the plant's adaptability [3, 10]. According to this assumption, model reference adaptive control methods in the general statement can be applied only if the plant's parametric uncertainty is matched with the control signal. It means the theoretical possibility of fully compensating the uncertainty by direct subtraction of the generated control. If the adaptability condition does not hold, direct compensation becomes impossible: the uncertainty is unmatched, and special methods are required for designing the control law and its tuning.

Generally speaking, there are two main classes of uncertainties unmatched with the control signal. The first class includes disturbances in the plant's description by autonomous differential equations. The second class includes disturbances arising in the plant's nonautonomous equations with a deficit of control channels (the so-called *underactuated systems with unmatched uncertainties*).

For a long time, unmatched uncertainties of the first class have been compensated using adaptive backstepping methods [16] and indirect methods based on tuning functions [17]. The disadvantages of these approaches include a high dynamic order of the control law and its tuning and higher complexity of the design process when increasing the plant's order. New methods have recently been proposed [18–20] to solve these problems—consider and compensate the effect of unmatched parametric uncertainties—in a different way. These solutions directly combine the theory of adaptive [10] and robust [9] control. In particular, the following procedure was proposed in [18, 20]. First, indirect model reference adaptive control methods were used to estimate the unmatched uncertainties. Then the resulting information was adopted to recalculate the parameters of the controller and the reference model using LMIs synthesis. With such an approach, the robustness of the closed loop system to arbitrary unmatched uncertainties is adaptively maintained, and the effect of matched uncertainties is compensated.

The literature suggests few model reference adaptive control methods to compensate the effect of unmatched uncertainties from the second group. The main difficulty here is that the deficit of control channels leads to the presence of one control signal in several equations. In the general case, this leads to a nontrivial problem of compensating control design. Various methods of changing coordinates [21–25] are well known [21] in geometric and nonlinear control theory to solve this problem. These methods allow passing from a plant's model with a deficit of control actions to an equivalent normal-form model. As a result, an appropriate control law can be chosen by the feedback linearization method [21]. The disadvantages of such methods are the complexity (or even impossibility) of calculating the exact transformation for nonlinear high-dimensional plants and the dependence of the transformation itself on the plant's parameters. (Hence, it should have adaptation.) Therefore, the problem under consideration is still not completely solved.

This paper proposes a new approach to compensate an unmatched parametric uncertainty for an underactuated plant. It rests on the assumed existence of an ideal compensating control for reducing the unmatched uncertainty by a norm. With this assumption, we obtain compensating control by solving an optimization problem. Most real underactuated plants are described by nonlinear differential equations. Hence, the unmatched uncertainty is nonlinear, and the optimization problem turns out to be nonlinearly parameterized and, in the general statement, rather difficult to solve. To solve it in the general statement, we use artificial neural networks with universal approximating properties to form a compensating control action [26]. In this case, the parameter tuning laws of the compensator's neural network are designed by combining Lyapunov's second method and the backpropagation method.

The main result is a new neural-network-based compensating control law and an online algorithm to tune its parameters, which ensure the asymptotic convergence of the tracking error of a nonlinear underactuated plant to a given domain with a chosen reference model.

A two-wheeled balancing robot is a classical example of a nonlinear underactuated plant. Therefore, we construct an adaptive neural-network-based control for such a robot without loss of generality as an illustrative example.

This paper uses the following notations: $(.)(i, j)$ or $(.)_{i,j}$ is an element standing at the junction of the *i*th row and *j*th column of a given matrix; $\det(.)$ is the matrix determinant; $\mathrm{tr}(.)$ is the matrix trace (the sum of all elements on the principal diagonal of a given matrix); $\mathrm{vec}(.)$ is the matrix vectorization (stacking the columns of a given matrix); $\mathrm{diag}\{a \quad b \quad \dots \quad c\}$ is a diagonal matrix with elements $a,b,\dots,c$ on the principal diagonal; $\|.\|_\infty$ is the L_∞ norm of a given matrix; $\|.\|$ is the Euclidean vector norm or Frobenius matrix norm, depending on the context; $(.)_{n\times n}$ is a matrix of dimensions $n\times n$.

1. THE MATHEMATICAL MODEL OF A TWO-WHEELED BALANCING ROBOT

The differential equations describing the dynamics of a two-wheeled balancing robot are derived using the Euler–Lagrange second method [27]. After reducing to the Cauchy form, they are given by

$$\begin{cases} \dot{x}_1 = x_3, \\ \dot{x}_2 = x_4, \\ \dot{x}_3 = -2(\beta + f_w)\left[E^{-1}(1,1) - E^{-1}(1,2)\right]x_3 - \\ 2\beta\left[E^{-1}(1,2) - E^{-1}(1,1)\right]x_4 + \\ \left[E^{-1}(1,1)MLRx_4{}^2 + E^{-1}(1,2)MgL\right]\sin(x_2) + \\ \alpha\left[E^{-1}(1,1) - E^{-1}(1,2)\right](u_1 + u_2), \\ \dot{x}_4 = -2(\beta + f_w)\left[E^{-1}(2,1) - E^{-1}(2,2)\right]x_3 - \\ 2\beta\left[E^{-1}(2,2) - E^{-1}(2,1)\right]x_4 + \\ \left[E^{-1}(2,1)MLRx_4{}^2 + E^{-1}(2,2)MgL\right]\sin(x_2) + \\ \alpha\left[E^{-1}(2,1) - E^{-1}(2,2)\right](u_1 + u_2), \end{cases} \qquad (1.1)$$

where

$$E^{-1} = \begin{bmatrix} \dfrac{J + ML^2 + 2n^2 J_m}{\det(\mathrm{E})} & -\dfrac{MLR\cdot\cos(x_2) - 2n^2 J_m}{\det(\mathrm{E})} \\ -\dfrac{MLR\cdot\cos(x_2) - 2n^2 J_m}{\det(\mathrm{E})} & \dfrac{2J_{\mathrm{w}} + (2m_{\mathrm{w}} + M)R^2 + 2n^2 J_m}{\det(\mathrm{E})} \end{bmatrix},$$

$$\det(E) = \left(2J_w + (2m_w + M)R^2 + 2n^2 J_m\right)\times$$
$$\left(J + ML^2 + 2n^2 J_m\right) - \left(MLR\cdot\cos(x_2) - 2n^2 J_m\right)^2,$$
$$\beta = \frac{nK_t K_b}{R_m} + f_m, \text{ and } \alpha = \frac{nK_t}{R_m}.$$

The model (1.1) is obtained by assuming the structural and parametric identity of the actuating motors. The parameters of the model (1.1) have the following physical interpretation: J_w is the wheel's moment of inertia; m_w is the wheel's mass; M is the robot's mass; R is the wheel's radius; n is the gear ratio of the gearbox; J_m is the motor's moment of inertia; L is the distance between the center of mass and the wheel axis; K_t is the motor torque constant; R_m is the resistance of the motor's armature circuit; K_b is the back emf constant; f_m is the coefficient of friction between the robot's body (further called the body) and the motor shaft; f_w is the coefficient of friction between the wheel and the motion surface; J is the body's moment of inertia; g is the acceleration of gravity. The robot's state variables are the average angle of rotation of the wheels (x_1), the body's angle of deflection from the normal (x_2), the wheel turning rate (x_3), and the body's rate of deflection from the normal (x_4). The voltages u_1 and u_2 applied to the left and right motors, respectively, are the control action.

For convenience, let us introduce the following additional notations for the model (1.1):

$$\begin{gathered} f_3(x_2, x_3, x_4) = \\ -2(\beta + f_w)\left[E^{-1}(1,1) - E^{-1}(1,2)\right]x_3 - \\ 2\beta\left[E^{-1}(1,2) - E^{-1}(1,1)\right]x_4 + \\ \left[E^{-1}(1,1)MLRx_4{}^2 + E^{-1}(1,2)MgL\right]\sin(x_2), \\ g_{31}(x_2) = g_{32}(x_2) = \alpha\left[E^{-1}(1,1) - E^{-1}(1,2)\right], \\ f_4(x_2, x_3, x_4) = \\ -2(\beta + f_w)\left[E^{-1}(2,1) - E^{-1}(2,2)\right]x_3 - \\ 2\beta\left[E^{-1}(2,2) - E^{-1}(2,1)\right]x_4 + \\ \left[E^{-1}(2,1)MLRx_4{}^2 + E^{-1}(2,2)MgL\right]\sin(x_2), \\ \text{and } g_{41}(x_2) = g_{42}(x_2) = \alpha\left[E^{-1}(2,1) - E^{-1}(2,2)\right]. \end{gathered} \qquad (1.2)$$

2. PROBLEM STATEMENT

Using the notations (1.2), we write the model (1.1) in the state space:

$$\dot{x} = A_0 x + B_3 f_3(x_2, x_3, x_4) + B_4 f_4(x_2, x_3, x_4) + g(x)u,$$
$$A_0 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix};\ B_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}; \tag{2.1}$$
$$B_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix};\ g(x_2) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ g_{31}(x_2) & g_{32}(x_2) \\ g_{41}(x_2) & g_{42}(x_2) \end{pmatrix}.$$

Here $x \in R^4$ is the measured state vector of the robot; $u = [u_1; u_2] \in R^2$ is the vector of voltages applied to the left and right motors; $f_3(x_2, x_3, x_4)$, $f_4(x_2, x_3, x_4)$, and $g(x_2)$ are the nonlinear functions given by (1.2), which satisfy the Lipschitz smoothness conditions.

Assumption 1. *The control action u is such that $u_1 = u_2$.*

This assumption is classical when the robot's rotation about the center of mass in the horizontal plane is not required to control.

Due to Assumption 1, the model (1.1) with two control actions can be reduced to an equivalent model with one control action $v = u_1$:

$$\dot{x} = A_0 x + B_3 f_3(x_2, x_3, x_4) + B_4 f_4(x_2, x_3, x_4) + Bv,$$
$$B = \begin{bmatrix} 0 & 0 & g_3(x_2) & g_4(x_2) \end{bmatrix}^{\mathrm{T}}; \tag{2.2}$$
$$\begin{cases} g_3(x_2) = g_{31}(x_2) + g_{32}(x_2), \\ g_4(x_2) = g_{41}(x_2) + g_{42}(x_2). \end{cases}$$

In addition, we will consider an auxiliary linear model obtained from (2.2) by linearizing the functions $f_3(x_2, x_3, x_4)$ and $f_4(x_2, x_3, x_4)$ and the nonlinear elements of the vector B in the neighborhood of the unstable equilibrium $x_2 = 0$:

$$\dot{x} = A_0 x + B_3 \bar{f}_3(x_2, x_3, x_4) + B_4 \bar{f}_4(x_2, x_3, x_4) + \bar{B}v = \bar{A}x + \bar{B}v. \tag{2.3}$$

Remark 1. To derive the model (2.3) from the model (1.1), we set some values of the physical and geometric parameters of the robot model (1.1) and use the equalities $\lim_{x_2 \to 0} \sin(x_2) = x_2$, $\lim_{x_2 \to 0} \cos(x_2) = 1$, and $x_4^2 = 0$ holding in the neighborhood of the linearization point $x_2 = 0$. When constructing the linearized model (2.3), the robot's parameters can be uncertain.

Based on the linear model (2.3), the LQ-optimal control law can be calculated by

$$v_{\mathrm{LQ}} = K_{\mathrm{LQ}}(r - x) = K_{\mathrm{LQ}} e, \tag{2.4}$$

where $r \in R^4$ is the vector of reference signals for the robot's state variables, and the matrix K_{LQ} is found by minimizing the criterion

$$J = \frac{1}{2}\int_0^{\infty} x^{\mathrm{T}} Q_{\mathrm{LQ}} x + R_{\mathrm{LQ}} v^2 \, d\tau \tag{2.5}$$

with positive definite diagonal matrices $Q_{\mathrm{LQ}} \in R^{4\mathrm{x}4}$ and $R_{\mathrm{LQ}} \in R$.

The desired control performance for the nonlinear plant (2.2) is given by the system (2.3) with the controller (2.4):

$$\dot{x}_{ref} = A_{ref} x_{ref} + B_{ref} r,$$
$$A_{ref} = \bar{A} - \bar{B}K_{\mathrm{LQ}};\ B_{ref} = \bar{B}K_{\mathrm{LQ}}. \tag{2.6}$$

To obtain an error equation between the nonlinear plant equations (2.2) and its linear reference model (2.6), we introduce the relations:

$$f_3(x_2, x_3, x_4) = \bar{f}_3(x_2, x_3, x_4) + \Delta_{f3},$$
$$f_4(x_2, x_3, x_4) = \bar{f}_4(x_2, x_3, x_4) + \Delta_{f4}, \tag{2.7}$$
$$B = \bar{B} + \Delta_B,$$

where Δ_{f3}, Δ_{f4}, and Δ_B are unknown Lipschitz smooth functions due to the parametric uncertainties and the nonlinearities for $x_2 \gg 0$.

Substituting the relations (2.7) into the model (2.2), we have

$$\dot{x} = A_0 x + B_3\left[\bar{f}_3(x_2, x_3, x_4) + \Delta_{f3}\right] + B_4\left[\bar{f}_4(x_2, x_3, x_4) + \Delta_{f4}\right] + \left(\bar{B} + \Delta_B\right)v. \tag{2.8}$$

Considering the expression (2.8), we choose the control law v in the form

$$v = v_{LQ} - v_{ad}. \tag{2.9}$$

Due to the expressions (2.3), (2.6), (2.7), and (2.9), equation (2.8) reduces to

$$\dot{x} = A_{ref} x + B_{ref} r + \underbrace{+B_3\Delta_{f3} + B_4\Delta_{f4} + \Delta_B v}_{\Lambda(z)} - \bar{B}v_{ad}, \tag{2.10}$$

where $\Lambda(z)$ is an unknown Lipschitz smooth function that describes the effect of parametric uncertainties and nonlinearities on the control performance of the balancing robot, and $z = \begin{bmatrix} x_2 & x_3 & x_4 & v \end{bmatrix} \in D \subset R^4$ is the variable of the function $\Lambda(z)$ defined in a compact domain D of the space R^4.

The error equation between (2.10) and (2.6) has the form

$$\dot{e}_{ref} = A_{ref} e_{ref} + \Lambda(z) - \bar{B} v_{ad}, \qquad (2.11)$$

where $e_{ref} = x - x_{ref}$ is the tracking error of the plant (2.10) with the reference model (2.6).

As is easily checked, the vector $\bar{B}$ has no Moore–Penrose pseudoinverse matrix such that $\bar{B}^{\dagger}\bar{B} = I$. Hence, $\Lambda(z)$ is a disturbance unmatched with the control signal. To design the control law v_{ad} in the adaptive problem statement, we accept the following assumption regarding compensation.

Assumption 2. *There exists a compensating signal* v_{ad}^* *of the variable z such that*

$$\left\|\Lambda(z) - \bar{B} v_{ad}^*\right\| \le \varepsilon_\Lambda < \Lambda_{\max},$$
$$v_{ad}^* = \arg\left[\min_{v_{ad}^*}\left\{\sup\left\|\Lambda(z) - \bar{B} v_{ad}^*\right\|\right\}\right], \qquad (2.12)$$

where $\Lambda_{\max} = \sup_{\forall z \in L_\infty}\left\|\Lambda(z)\right\|$, *and* ε_Λ *is the approximation error of* $\Lambda(z)$ *with the compensating signal* v_{ad}^*.

Remark 2. If Assumption 2 is not satisfied, then the adaptive compensation of the disturbance $\Lambda(z)$ is a fundamentally unsolvable problem in the class of smooth functions.

For clarity, we provide an illustrative example of when Assumption 2 holds. Let the difference $\Lambda(z) - \bar{B} v_{ad}^*$ have the form

$$\Lambda(z) - \bar{B} v_{ad}^* = \begin{bmatrix} 0 \\ 0 \\ a_{32}x_2 + a_{33}x_3 + a_{34}x_4 \\ a_{42}x_2 + a_{43}x_3 + a_{44}x_4 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ b_3 \\ b_4 \end{bmatrix} v_{ad}^*.$$

In this case, Assumption 2 is satisfied under at least one of the following relations:

$$\frac{a_{32}}{a_{42}} \propto \frac{b_3}{b_4};\ \frac{a_{33}}{a_{43}} \propto \frac{b_3}{b_4};\ \frac{a_{34}}{a_{44}} \propto \frac{b_3}{b_4}.$$

The part of the uncertainty $\Lambda(z)$ unmatched with the control signal can be compensated.

Thus, without any information about the structure of the function $\Lambda(z)$ and the impossibility to compensate it fully, we pose the following problem: minimize the error (2.12) and ensure the asymptotic convergence of the tracking error (2.11) to a bounded set:

$$\lim_{t\to\infty}\left\|e_{ref}(t)\right\| \le \bar{\varepsilon}_{e_{ref}}, \qquad (2.13)$$

where $\bar{\varepsilon}_{e_{ref}}$ is the limit tracking error.

Before presenting the main result, we introduce and justify a constraint on $\bar{\varepsilon}_{e_{ref}}$. For this purpose, we estimate the minorant and majorant of the tracking error e_{ref}. Letting $v_{ad} = v_{ad}^*$ and $v_{ad} = 0$ in equation (2.11) yields lower and upper bounds on the derivative $\dot{e}_{ref}$:

$$A_{ref}e_{ref} + \Lambda(z) - \bar{B}v_{ad}^* \le \dot{e}_{ref} < A_{ref}e_{ref} + \Lambda(z). \qquad (2.14)$$

For calculating the minorant and majorant of the error e_{ref}, consider the quadratic form

$$L = e_{ref}^{\mathrm{T}} P e_{ref},$$
$$\lambda_{\min}(P)\left\|e_{ref}\right\|^2 \le L \le \lambda_{\max}(P)\left\|e_{ref}\right\|^2, \qquad (2.15)$$

where P is the solution of the Lyapunov equation $A_{ref}^{\mathrm{T}}P + PA_{ref} = -Q,\ Q > 0.$

Due to (2.14), the derivative of the quadratic form (2.15) satisfies the two-sided inequality

$$e_{ref}^{\mathrm{T}}\left(A_{ref}^{\mathrm{T}}P + PA_{ref}\right)e_{ref} + $$
$$2e_{ref}^{\mathrm{T}}P\left[\Lambda(z) - \bar{B}v_{ad}^*\right] \le \dot{L} < \qquad (2.16)$$
$$e_{ref}^{\mathrm{T}}\left(A_{ref}^{\mathrm{T}}P + PA_{ref}\right)e_{ref} + 2e_{ref}^{\mathrm{T}}P\Lambda(z).$$

According to Assumption 2, from (2.16) we obtain:

$$-\lambda_{\min}(Q)\left\|e_{ref}\right\|^2 + 2\varepsilon_\Lambda\lambda_{\max}(P)\left\|e_{ref}\right\| \le$$
$$\dot{L} < -\lambda_{\min}(Q)\left\|e_{ref}\right\|^2 + 2\lambda_{\max}(P)\Lambda_{\max}\left\|e_{ref}\right\|. \qquad (2.17)$$

For any $a > 0$ and $b > 0$,

$$-a^2 + ab = \frac{1}{2}\left[-a^2 - (a-b)^2 + b^2\right] \le$$
$$-\frac{1}{2}a^2 + \frac{1}{2}b^2. \qquad (2.18)$$

Hence, the relations (2.17) imply

$$-\frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)}L + \frac{2\varepsilon_\Lambda^2\lambda_{\max}^2(P)}{\lambda_{\min}(Q)} \le \dot{L} <$$
$$-\frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)}L + \frac{2\lambda_{\max}^2(P)\Lambda_{\max}^2}{\lambda_{\min}(Q)}.$$

Considering (2.15), we have the following minorant and majorant of the tracking error:

$$\frac{1}{\lambda_{\min}(P)}e^{-\frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)}t}\left\|e_{ref}(0)\right\|^2 +$$
$$\frac{4\varepsilon_\Lambda^2\lambda_{\max}^3(P)}{\lambda_{\min}(P)\lambda_{\min}^2(Q)} \le \left\|e_{ref}\right\|^2 < \qquad (2.19)$$
$$\frac{1}{\lambda_{\min}(P)}e^{-\frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)}t}\left\|e_{ref}(0)\right\|^2 + \frac{4\lambda_{\max}^3(P)\Lambda_{\max}^2}{\lambda_{\min}(P)\lambda_{\min}^2(Q)}.$$

Letting $t \to \infty$ in (2.19), we finally estimate the limit tracking error as

$$2\varepsilon_{\Lambda} \frac{\lambda_{\max}(P)}{\lambda_{\min}(Q)} \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}} \le \bar{\varepsilon}_{e_{ref}} < 2\Lambda_{\max} \frac{\lambda_{\max}(P)}{\lambda_{\min}(Q)} \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}. \quad (2.20)$$

Well, according to the expressions (2.13) and (2.20), it is required to construct a compensating law v_{ad} that ensures the asymptotic convergence of the tracking error e_{ref} to a given set with the boundary $\bar{\varepsilon}_{e_{ref}}$.

3. AUXILIARY RESULTS FROM THE THEORY OF NEURAL-NETWORK-BASED CONTROL

To achieve this goal under an unknown structure of the nonlinear function $\Lambda(z)$ and a nonlinear parametrization of the optimization problem (2.12), we will construct the compensating control v_{ad} using neural networks with their universal approximation properties [26]. This section provides auxiliary results from the theory of neural-network-based control necessary for further considerations.

Proposition 1 [26]. *Any continuous function $f(z)$: $R^n \to R$ can be uniformly approximated in a compact domain $D \subset R^n$ using a neural network with one hidden layer with a sigmoidal activation function: for all $\bar{\varepsilon}_{NN} > 0$ and $z \in D$, there exist matrices V and W and values b^1 and b^2 such that*

$$\left\| f(z) - f_{NN}(z) \right\|_{\infty} = \left\| f(z) - W^{\mathrm{T}} \sigma\left(V^{\mathrm{T}} \bar{z}\right) \right\|_{\infty} \le \bar{\varepsilon}_{NN},$$

where

$$\bar{z} = \begin{bmatrix} b^1 & z \end{bmatrix}^{\mathrm{T}},\ \sigma(V \cdot \bar{z}) = \begin{bmatrix} b^2 & \sigma_1 & \sigma_2 & \cdots & \sigma_{N_2} \end{bmatrix}^{\mathrm{T}},$$

$$V = \begin{bmatrix} \theta_1^V & \cdots & \theta_{N_2}^V \\ v_{1,1} & \cdots & v_{1,N_2} \\ \vdots & \ddots & \vdots \\ v_{N_1,1} & \cdots & v_{N_1,N_2} \end{bmatrix}^{\mathrm{T}}, \text{ and } W = \begin{bmatrix} \theta_1^W & \cdots & \theta_{N_3}^W \\ w_{1,1} & \cdots & w_{1,N_3} \\ \vdots & \ddots & \vdots \\ w_{N_2,1} & \cdots & w_{N_2,N_3} \end{bmatrix}^{\mathrm{T}}.$$

The matrices $V \in R^{N_1 \times N_2}$ and $W \in R^{N_2 \times N_3}$ are the weight matrices of the hidden and output layers, respectively; b^1 and b^2 are the biases of the hidden and output layers, respectively; σ is the sigmoidal activation function of the hidden layer.

In other words, a neural network with a sigmoidal activation function approximates any continuous function of the variable z in a compact domain $D \in R^n$ with the error $\bar{\varepsilon}_{NN} = \; = \sup_{z \in D} \left\| \varepsilon(z) \right\|$:

$$f(z) = W^{\mathrm{T}} \sigma\left(V^{\mathrm{T}} \bar{z}\right) + \varepsilon(z). \quad (3.1)$$

In this case, the error $\bar{\varepsilon}_{NN}$ can be made arbitrarily small by increasing the number of neurons N_2 in the hidden layer.

Proposition 1 establishes the existence of ideal parameters of a neural network, not determining their values. Therefore, equation (3.1) is interpreted as describing the ideal output of a neural network. It is used to introduce the concept of a neural network with the parameters tuned by training:

$$\hat{f} = \hat{W}^{\mathrm{T}} \sigma\left(\hat{V}^{\mathrm{T}} \bar{z}\right). \quad (3.2)$$

The error between the current (3.2) and ideal (3.1) outputs of the neural network has the form

$$e = W^{\mathrm{T}} \sigma\left(V^{\mathrm{T}} \bar{z}\right) - \hat{W}^{\mathrm{T}} \sigma\left(\hat{V}^{\mathrm{T}} \bar{z}\right) + \varepsilon(z). \quad (3.3)$$

Hence, the ideal parameters V and W of the neural network can be found by optimizing the error function (3.3) with respect to the tuned parameters:

$$(V,\ W) = arg\left[\min_{(\hat{V}, \hat{W})} \left\{ \sup_{z \in D} |e| \right\} \right]. \quad (3.4)$$

Assumption 3. *The ideal weights of the neural network are bounded in a compact domain D:*

$$\|V\| \le V_M,\ \|W\| \le W_M.$$

The optimization problem (3.4) is nonlinearly parameterized due to the nonlinear activation function of the hidden layer. The error (3.3) is therefore rewritten in an approximate linearly parameterized form by expanding the activation function of the hidden layer into the Taylor series.

Proposition 2 [28]. *The linearly parameterized error e_{lin} of the neural-network-based approximation is given by*

$$e_{lin} = e = \tilde{W}^{\mathrm{T}} \left(\sigma\left(\hat{V}^{\mathrm{T}} \bar{z}\right) - \sigma'\left(\hat{V}^{\mathrm{T}} \bar{z}\right) \hat{V}^{\mathrm{T}} \bar{z} \right) + \hat{W}^{\mathrm{T}} \sigma'\left(\hat{V}^{\mathrm{T}} \bar{z}\right) \tilde{V}^{\mathrm{T}} \bar{z} + \varepsilon(z) - d, \quad (3.5)$$

where $\sigma'\left(\hat{V}^{\mathrm{T}} \bar{z}\right) = \mathrm{diag}\left\{ 0 \quad \sigma_1' \quad \sigma_2' \quad \cdots \quad \sigma_{N_2}' \right\}$ denotes the derivative of the activation function of the hidden layer; $\tilde{V} = V - \hat{V}$ is the parametric error of the hidden layer of the neural network; $\tilde{W} = W - \hat{W}$ is the parametric error of the output layer of the neural network;

d is the residual term. The difference ($\varepsilon(z) - d$) *in equation* (3.5) *is bounded* [28, 29] *due to the condition*

$$\left\|\varepsilon(z)-d\right\| \le \alpha_1\left\|\tilde{Z}\right\|+\alpha_2,$$
$$\alpha_1>0,\ \alpha_2>0,\ \tilde{Z}=\begin{bmatrix}\tilde{W} & 0\\ 0 & \tilde{V}\end{bmatrix}. \quad (3.6)$$

Then problem (3.4) is equivalent to the linearly parameterized problem

$$(V,W)=\arg\left[\min_{(\tilde{V},\tilde{W})}\left\{\sup_{z\in D}\left|e_{lin}\right|\right\}\right]. \quad (3.7)$$

The laws to tune the current parameters of the neural network [28] are obtained by solving (3.7):

$$\begin{cases}\dot{\hat{W}}=\Gamma_W\left(\sigma\left(\hat{V}^{\mathrm{T}}\bar{z}\right)-\sigma'\left(\hat{V}^{\mathrm{T}}\bar{z}\right)\hat{V}^{\mathrm{T}}\bar{z}\right)e,\\ \hat{W}(0)=0_{N_2\times N_3},\\ \dot{\hat{V}}=\Gamma_V\bar{z}e\hat{W}^{\mathrm{T}}\sigma'\left(\hat{V}^{\mathrm{T}}\bar{z}\right),\ \hat{V}(0)=0_{N_1\times N_2}.\end{cases} \quad (3.8)$$

These auxiliary results from the theory of neural-network-based control will be used below to obtain the compensating control v_{ad}.

4. THE MAIN RESULT

Under Assumption 2, adding and subtracting the value $\bar{B}v_{ad}^*$ from equation (2.11) yield

$$\dot{e}_{ref}=A_{ref}e_{ref}+\Lambda(z)-\bar{B}v_{ad}\pm\bar{B}v_{ad}^*=$$
$$A_{ref}e_{ref}+\bar{B}\left[v_{ad}^*-v_{ad}\right]+\Lambda(z)-\bar{B}v_{ad}^*. \quad (4.1)$$

By the problem statement, $z\in D\subset R^4$, and v_{ad}^* is a function of the variable z (Assumption 2). According to Proposition 1, the function v_{ad}^* can be approximated using an artificial neural network:

$$v_{ad}^*=W^{\mathrm{T}}\sigma\left(V^{\mathrm{T}}\bar{z}\right)+\varepsilon(z),$$
$$\bar{z}=\begin{bmatrix}b^1 & z\end{bmatrix}^{\mathrm{T}}=\begin{bmatrix}b^1 & x_2 & x_3 & x_4 & v\end{bmatrix}^{\mathrm{T}}. \quad (4.2)$$

Therefore, we choose v_{ad} in the form

$$v_{ad}=\hat{W}^{\mathrm{T}}\sigma\left(\hat{V}^{\mathrm{T}}\bar{z}\right). \quad (4.3)$$

Due to the expressions (4.3), (4.2), and (3.5), equation (4.1) reduces to

$$\dot{e}_{ref}=A_{ref}e_{ref}+\bar{B}\left[\tilde{W}^{\mathrm{T}}\left(\sigma\left(\hat{V}^{\mathrm{T}}\bar{z}\right)-\sigma'\left(\hat{V}^{\mathrm{T}}\bar{z}\right)\hat{V}^{\mathrm{T}}\bar{z}\right)+\right.$$
$$\left.\hat{W}^{\mathrm{T}}\sigma'\left(\hat{V}^{\mathrm{T}}\bar{z}\right)\tilde{V}^{\mathrm{T}}\bar{z}+\varepsilon(z)-d\right]+\Lambda(z)-\bar{B}v_{ad}^*. \quad (4.4)$$

Based on the laws (3.8), we introduce the following tuning laws for the weights of the hidden and output layers of the compensating neural network (4.3):

$$\begin{cases}\dot{\hat{W}}=\Gamma_W\left[\left(\sigma\left(\hat{V}^{\mathrm{T}}\bar{z}\right)-\sigma'\left(\hat{V}^{\mathrm{T}}\bar{z}\right)\hat{V}^{\mathrm{T}}\bar{z}\right)\times\right.\\ \left.e_{ref}^{\mathrm{T}}P\bar{B}-\sigma_W\hat{W}\right],\ \hat{W}(0)=0_{N_2\times N_3},\\ \dot{\hat{V}}=\Gamma_V\left[\bar{z}e_{ref}^{\mathrm{T}}P\bar{B}\hat{W}^{\mathrm{T}}\sigma'\left(\hat{V}^{\mathrm{T}}\bar{z}\right)-\sigma_V\hat{V}\right],\\ \hat{V}(0)=0_{N_1\times N_2},\end{cases} \quad (4.5)$$

where $\sigma_W > 0$ and $\sigma_V > 0$ are the coefficients of the sigma modifications [10].

Remark 3. Contrary to popular belief, the compensating neural network (4.3) needs no preliminary autonomous training: it can be tuned by formulas (4.5), starting from the zero parameters of the layers, directly during the plant's operation.

Based on equation (4.4), we introduce the generalized error vector $\zeta=\begin{bmatrix}e_{ref}^{\mathrm{T}} & \mathrm{vec}^{\mathrm{T}}\left(\tilde{W}\right) & \mathrm{vec}^{\mathrm{T}}\left(\tilde{V}\right)\end{bmatrix}^{\mathrm{T}}$ and study its properties.

Theorem 1. *Let the compensating law* v_{ad} *be given by* (4.3), *and let its parameters be tuned by formulas* (4.5). *Then the generalized error* ζ *is uniformly and ultimately bounded. Moreover, the steady-state tracking error* e_{ref} *can be reduced to satisfy inequalities* (2.13) *and* (2.20) *by increasing the number of neurons* N_2 *in the hidden layer and decreasing the values of the coefficients* σ_V *and* σ_W.

The proof of Theorem 1 is postponed to the Appendix.

Thus, we ensure the asymptotic convergence of the tracking error e_{ref} to a given domain using the neural-network-based compensating law (4.3) and tuning its parameters by formulas (4.5).

Remark 4. These recommendations for increasing the number of neurons N_2 in the hidden layer and decreasing the values of the coefficients σ_V and σ_W have rather simple interpretations:

– Increasing the number of neurons N_2 in equation (4.4) allows satisfying the inequality $\Lambda(z)+\bar{B}\left[-v_{ad}^*+\varepsilon(z)-d\right]\le\left\|\varepsilon_\Lambda+\bar{B}\left[\varepsilon(z)-d\right]\right\|<\Lambda_{\max}$. In other words, the uncertainty $\bar{B}\left[\varepsilon(z)-d\right]$ introduced by the neural network into the closed loop does not increase the system uncertainty after compensating ε_Λ compared to the initial value $\Lambda_{\max}$.

– Choosing small values of the coefficients σ_V and σ_W allows the tunable neural network to better approximate the ideal compensating signal $W^{\mathrm{T}}\sigma\left(V^{\mathrm{T}}\bar{z}\right)$, thereby reducing more the error $\tilde{W}^{\mathrm{T}}\left(\sigma\left(\hat{V}^{\mathrm{T}}\bar{z}\right)-\sigma'\left(\hat{V}^{\mathrm{T}}\bar{z}\right)\hat{V}^{\mathrm{T}}\bar{z}\right)+\hat{W}^{\mathrm{T}}\sigma'\left(\hat{V}^{\mathrm{T}}\bar{z}\right)\tilde{V}^{\mathrm{T}}\bar{z}$. However, decreasing the values of the coefficients σ_W and σ_V also reduces the robustness of the tuning laws (4.5) to the uncompensated uncertainty $\left\|\varepsilon_\Lambda+\bar{B}\left[\varepsilon(z)-d\right]\right\|$ (under σ_V and σ_W close to 0, the value γ_1 in (A.7) can be negative). Hence, the values σ_V and σ_W should be assigned by the classical tradeoff between the quality of tracking the ideal trajectory x_{ref} and system robustness to the uncompensated uncertainty $\left\|\varepsilon_\Lambda+\bar{B}\left[\varepsilon(z)-d\right]\right\|$.

5. EXPERIMENTAL VALIDATION OF THE RESULTS

The developed control system was applied during experiments to the mathematical model of a LEGO EV3 balancing robot in Matlab/Simulink. The nominal values of the robot's parameters in the model (1.1) are given below.

Nominal values of robot's parameters

Parameter	Value	Parameter	Value
J_w, kg·m^2	$8.75\cdot10^{-6}$	K_b, V·s/rad	0.468
m_w, kg	0.024	R_m, Ω	6.69
R, m	0.027	f_m	0.0022
n	1	f_w	0
J_m, kg·m^2	10^{-5}	g, m/s^2	9.81
L, m	0.105	M, kg	0.8
K_t, N·m/A	0.317		

The body's moment of inertia J was calculated by the formula $J = ML^2/3 = 0.0029$ kg · m^2. The gain matrix K_{LQ} of the LQ controller was obtained using the robot's linearized model (2.3) with the nominal parameter values (see the table) by optimizing the criterion (2.5) with the matrices $Q = I$ and $R = 1$:

$$K_{\mathrm{LQ}}=\left[-0.7071\quad -77.0619\quad -1.5816\quad -9.3949\right].$$

The experiments involved a neural network with a sigmoidal activation function with four neurons on the input layer ($N_1 = 5$), forty neurons on the hidden layer ($N_2 = 40$), and one neuron on the output layer ($N_3 = 1$). In all experiments, the variable parameters of the neural network's tuning loop were as follows: $\Gamma_W=10^5 I_{N_2\times N_2}$, $\Gamma_V=10^{-3} I_{N_2\times N_2}$, $\sigma_W=0.1$, $\sigma_V=0.001$. For switching from the control v back to control u, the relation $v = u_1 = u_2$ was used in the experiments (see Assumption 1).

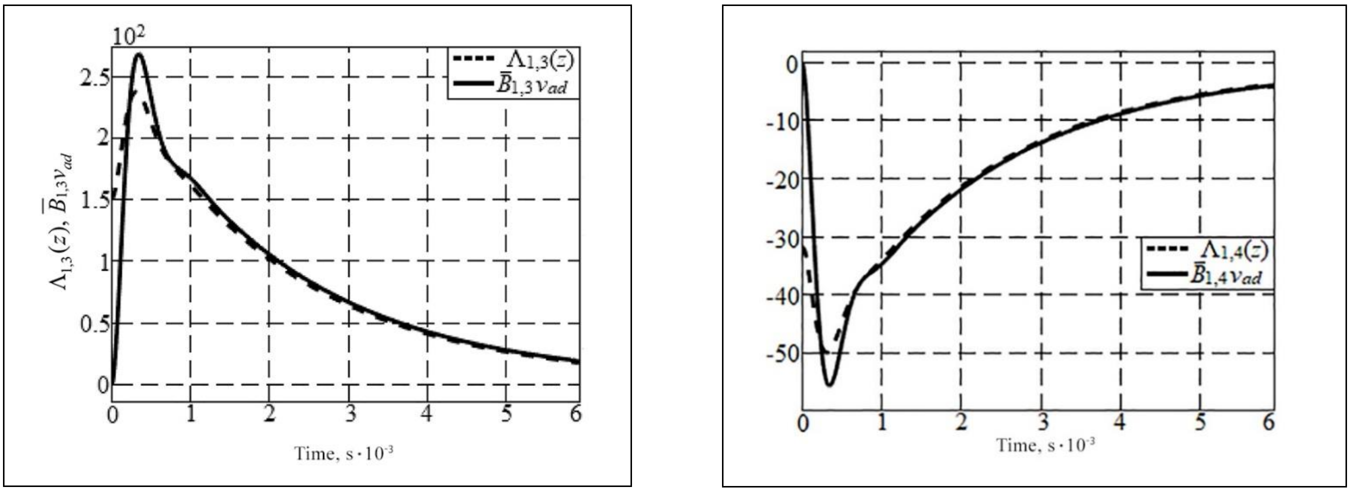
Two experiments were carried out in total. The first experiment was intended to check the compensator (4.3) when approximating the uncertainties caused by changes in the robot's parameters during operation in the neighborhood of the linearization point $x_2 = 0$. The second experiment was intended to check the compensator (4.3) when approximating the uncertainties caused by changes in the robot's parameters and nonlinearities during operation in a domain out of the neighborhood mentioned. The initial conditions of the plant (2.1) and the reference model (2.6) were the same in all experiments, and the zero vector was used as the reference signal r (the stabilization mode of the balancing robot).

In the first experiment, the models (2.1) and (2.6) began to move from the state-space point $x(0)=\left[0\quad 0.01\quad 0\quad 0\right]^{\mathrm{T}}$, and the function $\Lambda(z)$ was caused by doubling the robot's nominal mass M. Figure 1 shows the elements of the vector v_{ad} and function $\Lambda(z)$ in this experiment.

The transients in Fig. 1 demonstrate the high accuracy and sufficiently fast approximation of the disturbance $\Lambda(z)$ using the neural network.

Figures 2*a* and 2*b* show the norms of the tracking errors e_{ref} and the control actions v, respectively, obtained in the first experiment using the control system with the neural network (LQ-NN) and without it (LQ).

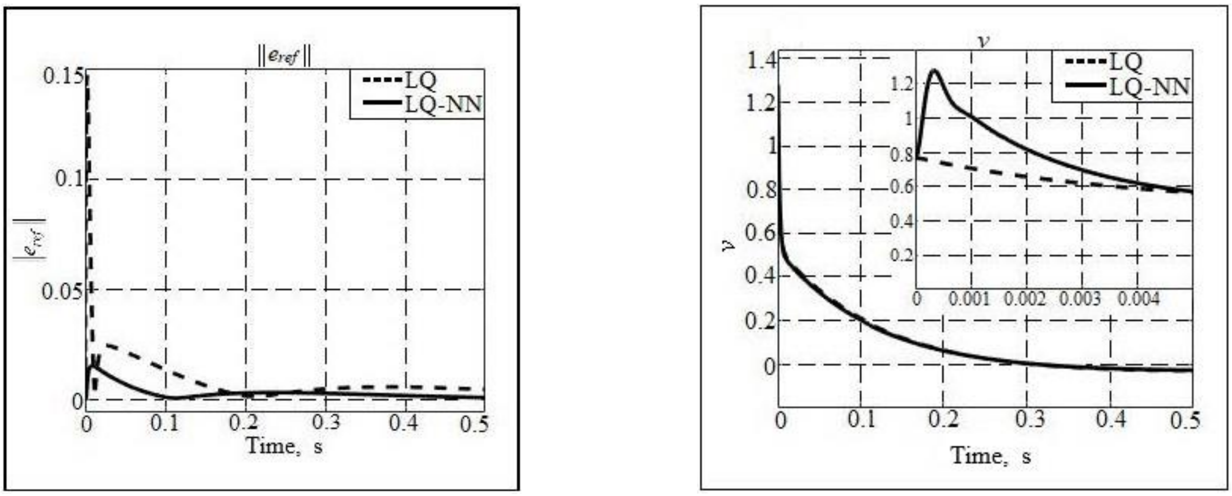
The upper bound on the target set (2.13), (2.20) is given by the upper bound on the trajectory of the closed loop system with the LQ controller (for $v_{ad} = 0$). Hence, Fig. 2*a* confirms the uniform and ultimate boundedness of the tracking error e_{ref} by the target set (2.20). This result validates the conclusions of Theorem 1. According to Fig. 2*b*, the costs of the total control action v to compensate the uncertainty $\Lambda(z)$ are not significant.



(a)

(b)

Fig. 1. The results of neural-network-based approximation: (*a*) $\Lambda_{13}(z)$ and (*b*) $\Lambda_{14}(z)$.



(a)

(b)

Fig. 2. (*a*) Norms of the tracking errors e_{ref} in systems LQ-NN and LQ, (*b*) control actions *v* in systems LQ-NN and LQ.

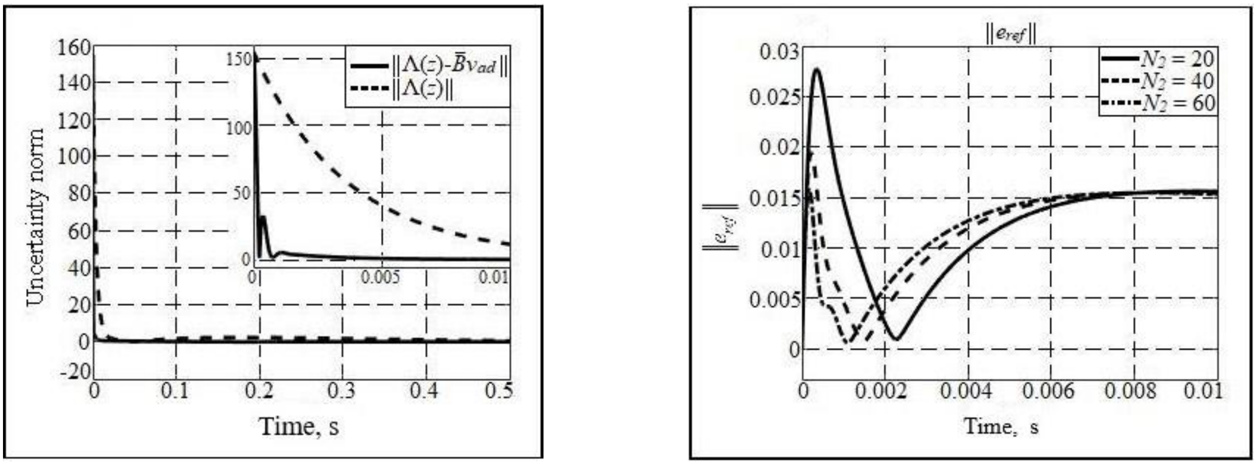
Figure 3*a* shows the uncertainty $\left\|\Lambda(z)\right\|$ in the control system with the LQ controller and the uncertainty $\left\|\Lambda(z)-\bar{B}v_{ad}\right\|$ after compensation in the control system with the neural-network-based compensator LQ-NN.

According to Fig. 3*a*, during the entire experiment, the area under the system uncertainty curve after compensation, $\left\|\Lambda(z)-\bar{B}v_{ad}\right\|$, is smaller than the area under the system uncertainty curve without compensation, $\left\|\Lambda(z)\right\|$. Hence, the following inequality holds:

$$\forall t>0:\ \dot{e}_{ref} \le A_{ref}e_{ref} + \Lambda(z) - \bar{B}v_{ad} < A_{ref}e_{ref} + \Lambda(z).$$

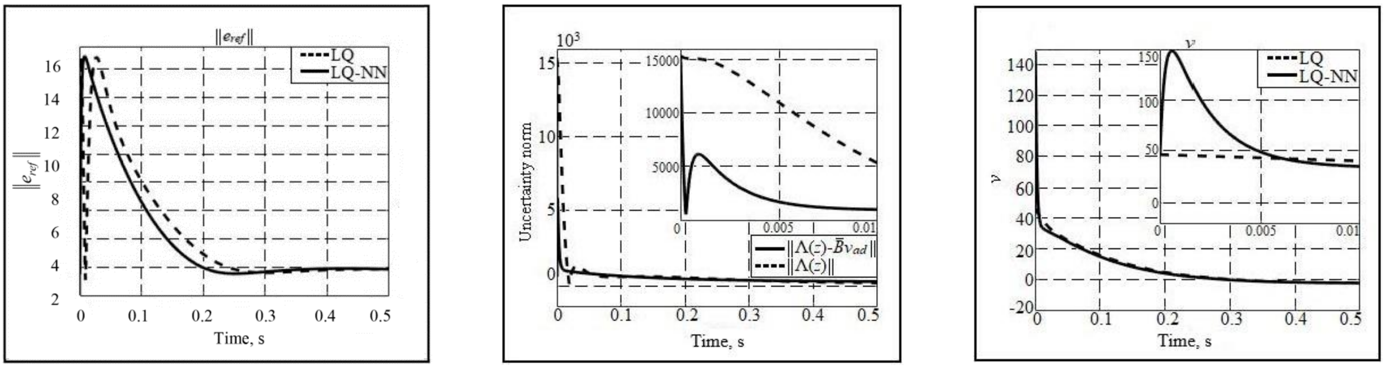
Using this result and considerations similar to (2.15)–(2.20), we validate the convergence of the tracking error e_{ref} to the domain specified by inequality (2.20); see Theorem 1. Figure 3*b* confirms the possibility of further reducing the tracking error e_{ref} by increasing the number of neurons N_2 in the hidden layer.

In the second experiment, the robot's mass was also doubled, but the robot (2.1) started moving from the initial state $x(0)=\begin{bmatrix}0 & 0.8 & 0 & 0\end{bmatrix}^{\mathrm{T}}$. Therefore, the uncertainty $\Lambda(z)$ was caused by the robot's nonstationary parameters and the nonlinearities. Figure 4 shows the norm of the error $\left(\Lambda(z)-\bar{B}v_{ad}\right)$ and the norms of the error e_{ref} and control actions obtained in the second experiment using the control system with the neural network (LQ-NN) and without it (LQ).



(a) (b)

Fig. 3. (*a*) Uncertainties $\|\Lambda(z)\|$ and $\|\Lambda(z)-\bar{B}v_{ad}\|$, (*b*) norms of the tracking errors e_{ref} under different values N_2.



(a) (b) (c)

Fig. 4. (*a*) Norms of the tracking errors e_{ref} in systems LQ-NN and LQ, (*b*) uncertainties $\|\Lambda(z)\|$ and $\|\Lambda(z)-\bar{B}v_{ad}\|$, (*c*) control actions *v* in systems LQ-NN and LQ.

According to Fig. 4, the developed system can compensate the disturbance $\Lambda(z)$ caused by parametric uncertainty and robot's nonlinearities. Moreover, since the upper bound on the target set (2.13), (2.20) is given by the trajectories of the closed loop system with the LQ controller (for $v_{ad} = 0$), Fig. 4*a* confirms the convergence of the tracking error e_{ref} to the given domain in the case of nonlinear uncertainty $\Lambda(z)$. Comparing the costs of the control action *v* for compensating the linear (Fig. 2*b*) and nonlinear uncertainties (Fig. 4*b*), we arrive at the following result: the costs of the control action *v* grow proportionally with increasing the complexity of the function $\Lambda(z)$.

CONCLUSIONS

This paper has proposed an adaptive neural-network-based control system for a two-wheeled balancing robot with a rigorously proved stability of a closed control loop and a neural network compensator trained online.

The new procedure for designing an adaptive neural-network-based control system can be applied not only to a two-wheeled balancing robot but also to other nonlinear underactuated plants (e.g., industrial cranes [30, 31], manipulators [32], underwater vehicles [33], vertical and/or short take-off and landing (V/STOL) aircrafts [34, 35], and other mechanical systems [35, 36]).

APPENDIX

P r o o f of Theorem 1.

Consider the bounded quadratic form

$$V = e_{ref}^{\mathrm{T}} P e_{ref} + \mathrm{tr}\left(\tilde{W}^{\mathrm{T}} \Gamma_W^{-1} \tilde{W}\right) + \mathrm{tr}\left(\tilde{V}^{\mathrm{T}} \Gamma_V^{-1} \tilde{V}\right),$$
$$\lambda_m \|\zeta\|^2 \le V\left(\|\zeta\|\right) \le \lambda_M \|\zeta\|^2, \tag{A.1}$$
$$\lambda_{\mathrm{m}} = \min\left\{\lambda_{\min}(P), \lambda_{\min}\left(\Gamma_W^{-1}\right), \lambda_{\min}\left(\Gamma_V^{-1}\right)\right\},\ \lambda_M = \max\left\{\lambda_{\max}(P), \lambda_{\max}\left(\Gamma_W^{-1}\right), \lambda_{\max}\left(\Gamma_V^{-1}\right)\right\}.$$

We calculate the derivative of (A.1) taking into account (4.4) and (4.5):

$$\dot{V} = -e_{ref}^{\mathrm{T}} Q e_{ref} + 2e_{ref}^{\mathrm{T}} P\bar{B}\left[\tilde{W}^{\mathrm{T}}\left(\sigma\left(\hat{V}^{\mathrm{T}}\bar{z}\right) - \sigma'\left(\hat{V}^{\mathrm{T}}\bar{z}\right)\hat{V}^{\mathrm{T}}\bar{z}\right) + \hat{W}^{\mathrm{T}}\sigma'\left(\hat{V}^{\mathrm{T}}\bar{z}\right)\tilde{V}^{\mathrm{T}}\bar{z} + \varepsilon(z) - d\right] +$$
$$2e_{ref}^{\mathrm{T}} P\left[\Lambda(z) - \bar{B}v_{ad}^{*}\right] - 2\mathrm{tr}\left(\tilde{W}^{\mathrm{T}}\Gamma_W^{-1}\Gamma_W\left[\left(\sigma\left(\hat{V}^{\mathrm{T}}\bar{z}\right) - \sigma'\left(\hat{V}^{\mathrm{T}}\bar{z}\right)\hat{V}^{\mathrm{T}}\bar{z}\right)e_{ref}^{\mathrm{T}} P\bar{B} - \sigma_W\hat{W}\right]\right) - \tag{A.2}$$
$$2\mathrm{tr}\left(\tilde{V}^{\mathrm{T}}\Gamma_V^{-1}\Gamma_V\left[\bar{z}e_{ref}^{\mathrm{T}} P\bar{B}\hat{W}^{\mathrm{T}}\sigma'\left(\hat{V}^{\mathrm{T}}\bar{z}\right) - \sigma_V\hat{V}\right]\right) =$$
$$-e_{ref}^{\mathrm{T}} Q e_{ref} + 2e_{ref}^{\mathrm{T}} P\bar{B}\left[\varepsilon(z) - d\right] + 2e_{ref}^{\mathrm{T}} P\left[\Lambda(z) - \bar{B}v_{ad}^{*}\right] + 2\mathrm{tr}\left(\tilde{W}^{\mathrm{T}}\sigma_W\hat{W}\right) + 2\mathrm{tr}\left(\tilde{V}^{\mathrm{T}}\sigma_V\hat{V}\right).$$

Due to the expression (2.12) и (3.6), an upper bound on the derivative (A.2) is given by

$$\dot{V} \le -\lambda_{\min}(Q)\|e_{ref}\|^2 + 2\|e_{ref}\|\lambda_{\max}(P)\|\bar{B}\|\left(\alpha_1\|\tilde{Z}\| + \alpha_2\right) + 2\|e_{ref}\|\lambda_{\max}(P)\varepsilon_\Lambda +$$
$$2\sigma_W\|\tilde{W}\|\|\hat{W}\| + 2\sigma_V\|\tilde{V}\|\|\hat{V}\| = \tag{A.3}$$
$$-\lambda_{\min}(Q)\|e_{ref}\|^2 + 2\|e_{ref}\|\lambda_{\max}(P)\|\bar{B}\|\left(\alpha_1\|\tilde{Z}\| + \alpha_2 + \alpha_3\right) + 2\sigma_W\|\tilde{W}\|\|\hat{W}\| + 2\sigma_V\|\tilde{V}\|\|\hat{V}\|,$$

where $\alpha_3 = \frac{\varepsilon_\Lambda}{\|\bar{B}\|}$.

According to inequality (2.18), the terms in (A.3) satisfy the following upper bounds:

$$-\lambda_{\min}(Q)\|e_{ref}\|^2 + 2\|e_{ref}\|\lambda_{\max}(P)\|\bar{B}\|\left(\alpha_1\|\tilde{Z}\| + \alpha_2 + \alpha_3\right) \le$$
$$\frac{1}{2}\left[-\lambda_{\min}(Q)\|e_{ref}\|^2 - \left(\sqrt{\lambda_{\min}(Q)}\|e_{ref}\| - \frac{2\lambda_{\max}(P)\|\bar{B}\|}{\sqrt{\lambda_{\min}(Q)}}\left(\alpha_1\|\tilde{Z}\| + \alpha_2 + \alpha_3\right)\right)^2 + \right.$$
$$\left.\frac{4\lambda_{\max}^2(P)\|\bar{B}\|^2\left(\alpha_1\|\tilde{Z}\| + \alpha_2 + \alpha_3\right)^2}{\lambda_{\min}(Q)}\right] \le \frac{-\lambda_{\min}(Q)\|e_{ref}\|^2}{2} + \frac{2\lambda_{\max}^2(P)\|\bar{B}\|^2}{\lambda_{\min}(Q)}\left(\alpha_1\|\tilde{Z}\| + \alpha_2 + \alpha_3\right)^2, \tag{A.4}$$
$$\|\tilde{W}\|\|\hat{W}\| = -\|\tilde{W}\|^2 + \|\tilde{W}\|\|W\| \le -\frac{1}{2}\|\tilde{W}\|^2 + \frac{1}{2}W_M^2,\ \|\tilde{V}\|\|\hat{V}\| = -\|\tilde{V}\|^2 + \|\tilde{V}\|\|V\| \le -\frac{1}{2}\|\tilde{V}\|^2 + \frac{1}{2}V_M^2.$$

Considering (A.4), the upper bound (A.3) takes the form

$$\dot{V} \le \frac{-\lambda_{\min}(Q)\|e_{ref}\|^2}{2} - \sigma_W\|\tilde{W}\|^2 - \sigma_V\|\tilde{V}\|^2 + \frac{2\lambda_{\max}^2(P)\|\bar{B}\|^2}{\lambda_{\min}(Q)}\left(\alpha_1\|\tilde{Z}\| + \alpha_2 + \alpha_3\right)^2 + \sigma_W W_M^2 + \sigma_V V_M^2. \tag{A.5}$$

Since

$$\sigma_W \left\|\tilde{W}\right\|^2 + \sigma_V \left\|\tilde{V}\right\|^2 \geq \underbrace{\min\{\sigma_W, \sigma_V\}}_{\sigma_Z} \left\|\tilde{Z}\right\|^2,$$

the expression (A.5) reduces to

$$\dot{V} \leq \frac{-\lambda_{\min}(Q)\left\|e_{ref}\right\|^2}{2} - \sigma_Z\left\|\tilde{Z}\right\|^2 + \frac{2\lambda_{\max}^2(P)\left\|\bar{B}\right\|^2}{\lambda_{\min}(Q)}\left[\alpha_1^2\left\|\tilde{Z}\right\|^2 + 2\alpha_1(\alpha_2+\alpha_3)\left\|\tilde{Z}\right\| + (\alpha_2+\alpha_3)^2\right] +$$

$$\sigma_W W_M^2 + \sigma_V V_M^2 \leq \frac{-\lambda_{\min}(Q)\left\|e_{ref}\right\|^2}{2} - \left(\sigma_Z - \frac{2\lambda_{\max}^2(P)\left\|\bar{B}\right\|^2}{\lambda_{\min}(Q)}\alpha_1^2\right)\left\|\tilde{Z}\right\|^2 + \tag{A.6}$$

$$\frac{2\lambda_{\max}^2(P)\left\|\bar{B}\right\|^2}{\lambda_{\min}(Q)}\left(2\alpha_1(\alpha_2+\alpha_3)\left\|\tilde{Z}\right\| + (\alpha_2+\alpha_3)^2\right) + \sigma_W W_M^2 + \sigma_V V_M^2.$$

By analogy with (A.4), completing the square for the terms containing $\left\|\tilde{Z}\right\|$ in (A.6), we write

$$\dot{V} \leq \frac{-\lambda_{\min}(Q)\left\|e_{ref}\right\|^2}{2} - \frac{1}{2}\underbrace{\left(\sigma_Z - \frac{2\lambda_{\max}^2(P)\left\|\bar{B}\right\|^2}{\lambda_{\min}(Q)}\alpha_1^2\right)}_{\gamma_1}\left\|\tilde{Z}\right\|^2 +$$

$$\underbrace{\frac{2\lambda_{\max}^4(P)\left\|\bar{B}\right\|^4}{\lambda_{\min}^2(Q)}\gamma_1^{-1}\left(2\alpha_1(\alpha_2+\alpha_3)\right)^2 + \frac{2\lambda_{\max}^2(P)\left\|\bar{B}\right\|^2}{\lambda_{\min}(Q)}(\alpha_2+\alpha_3)^2 + \sigma_W W_M^2 + \sigma_V V_M^2}_{\gamma_2} = \tag{A.7}$$

$$-\frac{1}{2}\left(\lambda_{\min}(Q)\left\|e_{ref}\right\|^2 + \gamma_1\left\|\tilde{W}\right\|^2 + \gamma_1\left\|\tilde{V}\right\|^2\right) + \gamma_2 \leq -\kappa V + \gamma_2,\ \kappa = \frac{1}{2\lambda_M}\min\{\lambda_{\min}(Q), \gamma_1\}.$$

Here γ_1 and γ_2 are special notations for the compact form of (A.7); see above.

Applying the comparison lemma [21], we obtain a solution of inequality (A.7):

$$\left\|\zeta(t)\right\| \leq \sqrt{\frac{\lambda_M}{\lambda_m}e^{-\kappa t}\left\|\zeta(0)\right\| + \frac{\gamma_2}{\kappa\lambda_m}}. \tag{A.8}$$

Hence, the error ζ is uniformly and ultimately bounded [3, 10, 21].

Let us prove the asymptotic convergence of the tracking error e_{ref} to a given domain from inequality (A.8). Letting $t \to \infty$ and using the value γ_2, we arrive at the following limit estimate of the tracking error:

$$\left\|e_{ref}\right\| \leq \sqrt{\frac{2\lambda_{\max}(P)}{\lambda_{\min}(Q)\lambda_{\min}(P)}\gamma_2} \leq 2\varepsilon_\Lambda \frac{\lambda_{\max}(P)}{\lambda_{\min}(Q)}\sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}} +$$

$$\underbrace{\sqrt{\frac{2\lambda_{\max}(P)}{\lambda_{\min}(Q)\lambda_{\min}(P)}}\left[\frac{\sqrt{2}\lambda_{\max}^2(P)\left\|\bar{B}\right\|^2}{\lambda_{\min}(Q)}\gamma_1^{-\frac{1}{2}}\left(2\alpha_1(\alpha_2+\alpha_3)\right) + \frac{\sqrt{2}\lambda_{\max}(P)\left\|\bar{B}\right\|}{\lambda_{\min}^{\frac{1}{2}}(Q)}\alpha_2 + \sqrt{\sigma_W}W_M + \sqrt{\sigma_V}V_M\right]}_{\alpha_4}. \tag{A.9}$$

Substituting this expression into (2.20), we check the inequality

$$2\varepsilon_\Lambda \frac{\lambda_{\max}(P)}{\lambda_{\min}(Q)}\sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}} \leq 2\varepsilon_\Lambda \frac{\lambda_{\max}(P)}{\lambda_{\min}(Q)}\sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}} + \alpha_4 < 2\left\|\Lambda(z)\right\|\frac{\lambda_{\max}(P)}{\lambda_{\min}(Q)}\sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}. \tag{A.10}$$

Inequality (A.10) holds if and only if the value α_4 is sufficiently small:

$$0 \le \alpha_4 < 2\left[\left\|\Lambda(z)\right\| - \varepsilon_\Lambda\right]\frac{\lambda_{\max}(P)}{\lambda_{\min}(Q)}\sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}.$$

By the definition (A.9), the value of the coefficient α_4 depends on those of the coefficients σ_V, σ_W, α_1, and α_2. In turn, the coefficients α_1 and α_2 (see formulas (3.1) and (3.6)) are inversely proportional to the number of neurons N_2 in the hidden layer. Therefore, the value of the coefficient α_4 can be reduced (thereby, ensuring inequality (A.10) and the asymptotic convergence of the tracking error e_{ref} to the given domain (2.13)) by increasing the number N_2 and decreasing the values of the coefficients σ_W and σ_V. The proof of Theorem 1 is complete.

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