DOI: http://doi.org/10.25728/cs.2021.1.1

STATE ESTIMATION METHODS FOR FUZZY INTEGRAL MODELS. PART I: APPROXIMATION METHODS

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Abstract. The existing and newly proposed methods for estimating the state of integral models with fuzzy uncertainty are reviewed. A fuzzy integral model with the limit transition defined in the Hausdorff metric is introduced. This model is used to formulate the state estimation problem for the models described by fuzzy Fredholm–Volterra integral equations. Several fuzzy methods for solving this problem are considered as follows: the fuzzy Laplace transform, the method of "embedding" models (transforming an original system into a higher dimension system and solving the resulting problem by traditional linear algebra methods), the Taylor estimation of the degenerate kernels under the integral sign that are represented by power polynomials, and the estimation of the nondegenerate kernels by degenerate forms using the Taylor approximation. As shown below, in some cases, the estimation results are related to the solution of fuzzy systems of linear algebraic equations. Test examples are solved for them.

Keywords: fuzzy Riemann integral, fuzzy integral model, fuzzy methods for estimating integral models.

INTRODUCTION

The models described by integral equations, further referred to as the integral models, are widespread in different branches of applied physics, mechanics, economics, and other areas dealing with mathematical descriptions of various objects. In the theory of differential equations, the existence and uniqueness of a solution is proved using the principle of contraction mappings, when an original problem is written as an equivalent integral model [1].

In control theory, integral models often represent control systems with feedback [2]. The integral Wiener–Hopf models are used to describe the perturbations affecting a system, an approach to model uncertainty in the processing of current information from an object [3]. The Fredholm and Volterra integral equations are used in the theory of elasticity, gas dynamics and electrodynamics, and ecology, i.e., in all areas obeying the laws of conservation of mass, momentum, and energy. In all cases mentioned, the unknown variables are under the integral sign.

In real conditions, control systems are subjected to various kinds of perturbations. They are represented by various mathematical models, which are being intensively developed on the theoretical basis and actively used in various applications. Among the most widespread theoretical approaches for these purposes, we mention the theory of intervals [4, 5], the theory of fuzzy sets [6], the theory of possibilities [7], hybrid probability theory, the theory of fuzzy mathematical statistics and fuzzy random processes [8], etc.

This paper describes the uncertainties within the theory of fuzzy sets, which is the most adequate and universal representation for various kinds of perturbations. As is easily demonstrated, the models discussed above follow from the general model of the theory of fuzzy sets. For example, in the paper [9], a fuzzy system of linear equations (FSLE) was solved, and one of the solution's coordinates was obtained in the form of a fuzzy membership function. However, fixing its base, we obtain a solution for this coordinate in the interval form.

Similar reasoning can be adopted to construct solution intervals for fuzzy differential equations. In the theory of possibilities, the membership function is interpreted as a certain probability density that, however, does not satisfy the probability axioms accepted in the traditional statistical theory. Therefore, the theory of possibilities is supposed to describe not mass phenomena but the possibilities of an individual object.

Hybrid probability theory represents the traditional probability space for random variables with traditional probability densities with initial or central moments in the form of fuzzy variables with given membership functions (usually triangles). Concerning traditional stochastic processes, the hybrid theory for fuzzy Markov stochastic processes operates with fuzzy states obtained by enlarging crisp states. This approach reduces the dimension of the transition matrix and, consequently, the corresponding computational difficulties during its inversion.

Generally speaking, the modern theory of fuzzy sets is a kind of core that groups various models of uncertainties.

Based on the foregoing, this paper aims to present various methods, both the existing and newly proposed ones, for estimating integral models under fuzzy uncertainty.

The scientific novelty of this paper consists in new state estimation methods developed by the authors for integral models, such as the method of degenerate kernels, the fuzzy least squares method, and the fuzzy Galerkin method. As shown below, "strong/weak" estimation results can occur when the estimation procedure yields fuzzy systems of linear algebraic equations. The authors first investigated this effect when solving the FSLE and then applied to estimate integral models.

Below, fuzzy integral models in the form of the Fredholm–Volterra equations are introduced, and some methods to solve them are considered.

1. BASIC DEFINITIONS

The basic definitions of the theory of fuzzy sets were given in [6]. Let us introduce the definitions used in this paper. The notations are the following: fuzzy variables (numbers) have the subscript "fuz," e.g., x_{fuz} is a fuzzy variable (element), $y_{\text{fuz}}(\boldsymbol{x})$ is a fuzzy function of many variables, where $\boldsymbol{x} = (x_1, x_2,\ldots, x_n)^{\mathrm{T}}$, $y'_{\text{fuz}\,x_i}(\boldsymbol{x})$ is the fuzzy derivative with respect to the variable x_i, and $\dot{\boldsymbol{x}}_{\text{fuz}}(t)$ is the fuzzy time derivative of a vector $\boldsymbol{x}_{\text{fuz}}$.

The belonging of an element x to some set X ($x \in X$) is formalized by *a membership function* $r(x)$, $r \in [0,1]$, $x = x_{\text{fuz}} \in X$ for a fuzzy element x_{fuz}:

$$r(x) = \begin{cases} \underline{r}(x) \in [0, 1], \\ \overline{r}(x) \in [0, 1], \end{cases}$$

where $r(x)$ is a multivalued function with left $\underline{r}(x)$ and right $\overline{r}(x)$ branches with respect to $r(x) = 1$.

The function $r(x)$ is often written in the level representation – the inverse mapping $r^{-1}(x) = x(r) = (\underline{x}(r),\ \overline{x}(r) \mid r \in [0, 1])$. A collective $\{x_{\text{fuz}}\}$ defines a fuzzy set X_{fuz}. For x_{fuz}, the chain of equivalent representations is sometimes used: $x_{\text{fuz}} \Leftrightarrow r(x)$, $r \in [0, 1]$ $\Leftrightarrow (\underline{r}(x),\ \overline{r}(x) \mid \underline{r},\ \overline{r} \in [0, 1] \Leftrightarrow (\underline{x}(r),\ \overline{x}(r) \mid r \in [0, 1])$, etc.

Fuzzy function (*mapping*) $y_{\text{fuz}}(x)$ of fuzzy variables. Let E be the set of all fuzzy variables with a given membership function $r(x)$, $r \in [0, 1]$, $x \in R$. Then $y_{\text{fuz}}(x): R \to E$ defines a fuzzy-valued function, and the following parametric representation holds:

$$y_{\text{fuz}}(x) = y(x, r) = (\underline{y}(x, r),\ \overline{y}(x, r) \mid r \in [0, 1]).$$

The Banach space of fuzzy variables is introduced using the conventional approach of functional analysis [10]. A collection $\{x_{\text{fuz}}\}$ with the addition and multiplication operations and the existence of an inverse element forms a vector (linear) space E. In the space E, the following metric and norm are defined:

$$d(x_{\text{fuz}\,i}, x_{\text{fuz}\,j}) = \sup_{r\in[0,1]} \times$$

$$\times\{\max[|\underline{x}_i(r) - \overline{x}_j(r)|, |\overline{x}_i(r) - \underline{x}_j(r)|]\},$$

$$\|x_{\text{fuz}\,i} - x_{\text{fuz}\,j}\| = d(x_{\text{fuz}\,i}, x_{\text{fuz}\,j}).$$

A fuzzy Cauchy sequence is a sequence of the form

$$\{x_{\text{fuz}\,n}\}\colon \{d(x_{\text{fuz}\,n}, x_{\text{fuz}\,m}) \to 0\}_{n,m\to\infty},$$

and the space E is complete if

$$x_{\text{fuz}\,n} \underset{n\to\infty}{\to} x_{\text{fuz}},\ x_{\text{f}} \in E.$$

These definitions lead to the Banach space of fuzzy variables (E, d). The pair (E, d) forms a complete metric space.

Fuzzy continuity at a point is defined using the local limit at this point, which is treated in the Hausdorff metric. Fuzzy continuity on an interval is defined as fuzzy continuity for all values of the interval.

According to the general approach, *the fuzzy derivative* of a function with respect to its crisp variable is found by defining the following operations for some fuzzy function described above: subtraction or the existence of an opposite element, multiplication by a constant, passage to the limit in a given metric. This paper uses two types of fuzzy derivatives: the Seikkala derivative $y'^{S}_{\text{fuz}}(x)$ and the Buckley–Feuring derivative $y'^{BF}_{\text{fuz}}(x)$. The following statement holds: if the fuzzy derivatives exist and are continuous at a point $x = x_*$, then they are equal to each other at this point.

A fuzzy integral is understood in the fuzzy Riemann sense [11].

Consider a fuzzy mapping $f_{\text{fuz}}:[a,b]\subset R\to E$, where E is a fuzzy set. If for each partition $P=\{t_0,\dots,t_n\}\in[a,b]$ and $\forall\xi_i\in[t_{i-1},t_i]$, $i=\overline{1,n}$, there exists the representation $R_p=\sum_{i=1}^{n}f_{\text{fuz}}(\xi_i)(t_i-t_{i-1})$ and $\Delta=\max\left\{\left|t_i-t_{i-1}\right|,i=\overline{1,n}\right\}$, then the fuzzy Riemann integral of $f_{\text{fuz}}(t)$ is given by

$$\int_a^b f_{\text{fuz}}(t)dt=\lim_{\Delta\to0}R_p, \tag{1}$$

where the limit is defined in the Hausdorff metric $d(u,v)$: for $u,v\in E\Rightarrow d(u,v)=\sup\{\max\left[\left|\underline{u}(r)-\underline{v}(r)\right|,\ \left|\overline{u}(r)-\overline{v}(r)\right|\right]\}$, where $r\in[0,1]\subset R$, and $\underline{u},\overline{u},\underline{v},\overline{v}$ are the nonparametric representations of the fuzzy variables u,v.

Under (1), a function $f_{\text{fuz}}(t)=f(t,r)=(\underline{f}(t,r),\ \overline{f}(t,r)\,|\,r\in[0,1])$ continuous in the Hausdorff satisfies the relations

$\underline{\int_a^b f(t,r)dt}=\int_a^b \underline{f}(t,r)dt$ and $\overline{\int_a^b f(t,r)dt}=\int_a^b \overline{f}(t,r)dt$, $r\in[0,1]\subset R$, where underline and overline indicate the lower and upper value objects, respectively.

If a fuzzy variable $z_{\text{fuz}}(t)$, $t\in[a,b]\subset R$, is under the fuzzy integral sign, it satisfies the fuzzy integral equation

$$z_{\text{fuz}}(t)+\int_a^b K(t,\tau)z_{\text{fuz}}(\tau)d\tau=f(t).$$

By analogy with the traditional classification, there are *fuzzy integral models* described by the Fredholm–Volterra equations of the first and second kinds:

$\int_{t_1}^{t_2}K(t,\tau)z_{\text{fuz}}(\tau)d\tau=u_{\text{fuz}}(t)$ is a fuzzy integral model described by the Fredholm equation of the first kind, where $t\in[t_1,t_2]\subset R$, and $K(t,\tau)$ is a crisp or fuzzy kernel;

$z_{\text{fuz}}(t)-\lambda\int_{t_1}^{t_2}K(t,\tau)z_{\text{fuz}}(\tau)d\tau=u_{\text{fuz}}(t)$ is a fuzzy integral model described by the Fredholm equation of the second kind, where $\lambda\in K$ is a parameter.

The limits of integration can be finite or infinite. The variables satisfy the inequality $t_1\le t,\tau\le t_2$, whereas the kernel $K(t,\tau)$ and the free term $u_{\text{fuz}}(t)$ must be continuous or satisfy the Fredholm conditions.

In the general case, the fuzzy Fredholm equations of the first and second kinds imply the fuzzy Volterra equations of the first and second kinds. The Volterra equations differ from the Fredholm equations by a variable limit of integration:

$\int_{t_1}^{t}K(t,\tau)z_{\text{fuz}}(\tau)d\tau=u_{\text{fuz}}(t)$, $t_1\le t\le t_2$, is a fuzzy integral model described by the Volterra equation of the first kind, where $K(t,\tau)$ is a crisp or fuzzy kernel;

$z_{\text{fuz}}(t)-\lambda\int_{t_1}^{t}K(t,\tau)z_{\text{fuz}}(\tau)d\tau=u_{\text{fuz}}(t)$ is a fuzzy integral model described by the Volterra equation of the second kind.

The Volterra integral equation can be considered a special case of the Fredholm equation with a properly completed kernel. The Volterra equations have several important properties that are not inherent in the Fredholm equations and cannot be derived from them. In view of this aspect, we will use only the general properties of the Fredholm and Volterra equations below.

Sufficient conditions for the existence of a unique solution of fuzzy Fredholm–Volterra integral equations of the second kind were given in [12–14]. For the sake of definiteness, consider a fuzzy Volterra equation of the second kind. For the existence of a fuzzy solution, the method of successive fuzzy approximations is used under the assumption that fuzzy approximations are defined in the rectangle $\Pi=[\tau,t]$, on which they have fuzzy continuity and a bounded Seikkala derivative. Then the sequence of fuzzy approximations converges in the Hausdorff metric to a fuzzy solution. Moreover, due to the boundedness of the derivative, convergence in t follows: the sequence of fuzzy approximations converges uniformly to the desired fuzzy variable, which is taken as a fuzzy solution of the original fuzzy integral equation. The uniqueness of a fuzzy solution is proved by contradiction.

The fuzzy Fredholm–Volterra equations of the first and second kinds (see above) can be represented in a short (operator) form [15]:

$$\lambda(Kz_{\text{fuz}})(t)z_{\text{fuz}}(t)=u_{\text{fuz}}(t)\ \text{and}$$
$$[I-\lambda(Kz_{\text{fuz}})(t)]z_{\text{fuz}}(t)=u_{\text{fuz}}(t), \tag{2a}$$

where

$$(Kz_{\text{fuz}})(t)=\int_{t_1}^{t_2}K(t,\tau)z_{\text{fuz}}(\tau)d\tau \tag{2b}$$

is the operator for the fuzzy Fredholm equations,

$$(Kz_{\text{fuz}})(t)=\int_{t_1}^{t}K(t,\tau)z_{\text{fuz}}(\tau)d\tau \qquad (2c)$$

is the operator for the fuzzy Volterra equations of the first and second kinds, and I is an identity operator.

2. PROBLEM STATEMENT

There is a fuzzy model described by the integral equation (2a), (2b), or (2c). It is required to consider different fuzzy estimation methods for its state.

3. FUZZY ESTIMATION METHODS

3.1. Estimation by fuzzy Laplace transform

The definition of the fuzzy Laplace transform and its properties were described in detail in the papers [16, 17]. Also, some examples of applying this transformation to find solutions of various fuzzy Volterra integral equations of the convolution type with crisp and fuzzy kernels were considered therein. The problem was generalized to the case of a fuzzy partial differential component in a fuzzy integral equation. As a result, the fuzzy Laplace transform method was extended to the case of fuzzy linear second-order partial differential equations of the parabolic and hyperbolic types.

3.2. Estimation by embedding

Consider a fuzzy model described by the Fredholm integral equation of the second kind:

$$x_{\text{fuz}}(s)=f_{\text{fuz}}(s)+\int_a^b K(s,\tau)x_{\text{fuz}}(\tau)d\tau. \qquad (3)$$

The existence of a unique solution, the conditions imposed on the functions f_{fuz} and $K(s,\tau)$, the definition of a solution for the equation with fuzzy parameters, the space of functions to find a solution, and the conditions under which equation (3) exists were thoroughly considered in [13, 14].

The exact fuzzy solution of equation (4) is constructed in the form of the infinite series [13]

$$x_{\text{fuz}}(s)=\sum_{i=1}^{\infty}a_{\text{fuz}\,i}h_i(s), \qquad (4)$$

where $\{h_i(\cdot)\}$ is a sequence of functions in the space $L_2(a,b)$, and $a_{\text{fuz}\,i}$ are fuzzy coefficients.

An approximate solution of equation (4) can be represented as the finite series

$$x_{\text{fuz}}(s)\simeq\tilde{x}_{\text{fuz}\,n}(s)=\sum_{i=1}^{n}\tilde{a}_{\text{fuz}\,i}h_i(s),$$

where $\tilde{a}_{\text{fuz}\,i}$ are the fuzzy coefficients for estimation, and $h_i(s)$ are known functions. To find them, we substitute the expression for $\tilde{x}_{\text{fuz}\,n}(s)$ into equation (3) instead of x_{fuz}. Proceeding in this way, we obtain an equation of the form (3), and the solution is

$$\sum_{i=1}^{n}\tilde{a}_{\text{fuz}\,i}h_i(s)=f_{\text{fuz}}(s)+\sum_{i=1}^{n}\tilde{a}_{\text{fuz}\,i}\int_a^b K(s_j,\tau)h_i(\tau)d\tau. \qquad (5)$$

Equation (5) contains n unknown fuzzy variables $\tilde{a}_{\text{fuz}\,1},\ldots,\ \tilde{a}_{\text{fuz}\,n}$. To calculate them, we need n equations and therefore use n points $s_1,\ldots,s_n\in[a,b]$. The resulting fuzzy system of linear equations for the coefficients $\tilde{a}_{\text{fuz}\,i}$ is

$$\sum_{i=1}^{n}h_i(s_j)\tilde{a}_{\text{fuz}\,i}=f_{\text{fuz}\,i}(s_j)+\sum_{i=1}^{n}\left(\int_a^b K(s_j,\tau)h_i(\tau)d\tau\right)\tilde{a}_{\text{fuz}\,i},\ j=\overline{1,n}.$$

In the matrix form, it can be written as

$$A\tilde{a}_{\text{fuz}}=f_{\text{fuz}}+B\tilde{a}_{\text{fuz}}, \qquad (6)$$

where $A=(a_{ij})$ and $B=(b_{ij})$ are matrices with the crisp elements $a_{ij}=h_i(s_j)$ and $b_{ij}=\int_a^b K(s_j,\tau)h_i(\tau)d\tau$, $i,j=\overline{1,n}$; $\tilde{a}_{\text{fuz}}=(\tilde{a}_{\text{fuz}\,1},\ldots,\ \tilde{a}_{\text{fuz}\,n})^T$ and $f_{\text{fuz}}=(f_{\text{fuz}}(s_1),\ldots,\ f_{\text{fuz}}(s_n))^{\text{T}}$ are vectors with fuzzy components.

The matrix equation (6) reduces to the standard form

$$\tilde{A}a_{\text{fuz}}=f_{\text{fuz}},\ \tilde{A}=A-B, \qquad (7)$$

and the resulting system is solved by the method of embedding [9, 18].

According to this method, equation (7) is transformed to the extended (embedded) system:

$$S_{(2n\times2n)}\cdot X_{\text{fuz}\,(2n\times2n)}=Y_{\text{fuz}\,(2n\times2n)},$$

where $X_{\text{fuz}}=(\underline{\tilde{a}}_1,\ldots,\ \underline{\tilde{a}}_n\,|\,\bar{\tilde{a}}_1,\ldots,\ \bar{\tilde{a}}_n)^{\text{T}}$ and $Y_{\text{fuz}}=(\underline{f}_1,\ldots,\ \underline{f}_n\,|\,\bar{f}_1,\ldots,\ \bar{f}_n)^{\text{T}}$.

The matrix S has a block structure: $S=\begin{pmatrix}S_1 & S_2\\ S_2 & S_1\end{pmatrix}$. The matrix S_1 is obtained from the matrix (A – B) by zeroizing all negative elements. To construct the matrix S_2, we should replace all negative elements in the matrix (A – B) by their absolute values and all other elements by zeros:

$$s_{ij}=a_{ij}-b_{ij},\ s_{i+n,j+n}=a_{ij}-b_{ij},\ a_{ij}-b_{ij}>0;$$

$$s_{i,j+n} = -\left(a_{ij} - b_{ij}\right),\ s_{i+n,j} = -\left(a_{ij} - b_{ij}\right),$$
$$a_{ij} - b_{ij} < 0 .$$

If $|S| \neq 0$ (S is nonsingular), then

$X_{\text{fuz}} = S^{-1} Y_{\text{fuz}}$,

where $S^{-1} = \begin{pmatrix} U & V \\ V & U \end{pmatrix}$, $U = 0.5[(S_1 + S_2)^{-1} +$ $+(S_1 - S_2)^{-1}]$, and $V = 0.5[(S_1 + S_2)^{-1} - (S_1 - S_2)^{-1}]$.

The case of a singular matrix S was considered in detail in the papers [19, 20].

The accuracy of the approximate solution can be estimated as follows.

The residual r_n and error ε_n vectors are determined via the Hausdorff metric:

$$r_n = D(f_{\text{fuz}}, L\tilde{x}_{\text{fuz}}) = (d(f_{\text{fuz}}, L\tilde{x}_{\text{fuz}\,1}), ..., d(f_{\text{fuz}}, L\tilde{x}_{\text{fuz}\,n}))^{\mathrm{T}},$$
$$\varepsilon_n = D(\tilde{x}_{\text{fuz}}, x_{\text{fuz}}) = (d(\tilde{x}_{\text{fuz}}, x_{\text{fuz}\,1}), ..., d(\tilde{x}_{\text{fuz}}, x_{\text{fuz}\,n}))^{\mathrm{T}},$$

where

$L = I - K$ and $K = (Kx_{\text{fuz}})(s) = \int_a^b K(s, \tau) x_{\text{fuz}}(\tau) d\tau.$

The following upper bound on the approximate solution accuracy was derived in [21]:

$$\|\varepsilon_n\| \leq \|r_n\| \cdot [1 - \|K\|]^{-1} \text{ for } \|K\| < 1.$$

Example 1. Consider an integral equation of the form

$$x_{\text{fuz}}(s) = f_{\text{fuz}}(s) + \int_{-1}^{1} (s+1) x_{\text{fuz}}(\tau) d\tau ,$$

where $a = -1$, $b = 1$, and

$$f_{\text{fuz}}(s) = f(s, r) = (\underline{f}(s, r) = s^3(r^2 + r),\ \overline{f}(s, r) =$$
$$= s^3(4 - r^3 - r) | r \in [0, 1]),\ -1 \leq s,\ \tau \leq 1.$$

It is required to estimate the state by the method of embedding.

Solution. We choose $h_1(s) = 1$ and $h_2(s) = s^3$, assuming that $s_1 = -1$ and $s_2 = 1$. Then the elements of equation (5) take the following form:

$$\underline{f}(s_1) = -(r^2 + r),\ \underline{f}(s_2) = (r^2 + r),\ \overline{f}(s_1) =$$
$$= -(4 - r^3 - r),\ \overline{f}(s_2) = (4 - r^3 - r);$$
$$A = \begin{pmatrix} h_1(s_1) & h_2(s_1) \\ h_1(s_2) & h_2(s_2) \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix};$$
$$B = \begin{pmatrix} b_{11} = \int_{-1}^{1} (-1+1) 1 d\tau & b_{12} = \int_{-1}^{1} (-1+1)(-1)^3 d\tau \\ b_{21} = \int_{-1}^{1} (1+1) 1 d\tau & b_{22} = \int_{-1}^{1} (1-1) 1^3 d\tau \end{pmatrix},$$

$$B = \begin{bmatrix} 0 & 0 \\ 4 & 0 \end{bmatrix} \Rightarrow \tilde{A} = A - B = \begin{bmatrix} 1 & -1 \\ -3 & 1 \end{bmatrix} \Rightarrow$$
$$\Rightarrow S = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 3 & 0 & 0 & 0 \end{bmatrix} \Rightarrow$$
$$\Rightarrow (S_1 + S_2)^{-1} = \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix}^{-1} = 0.5 \begin{bmatrix} 1 & -1 \\ -3 & 1 \end{bmatrix},$$
$$(S_1 - S_2)^{-1} = \begin{bmatrix} 1 & -1 \\ -3 & 1 \end{bmatrix}^{-1} = 0.5 \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix} \Rightarrow$$
$$\Rightarrow \begin{bmatrix} \underline{a}_1 \\ \underline{a}_2 \\ -\overline{a}_1 \\ -\overline{a}_2 \end{bmatrix} = \begin{bmatrix} -0.5 & 0 & 0 & 0.5 \\ 0 & -0.5 & 1.5 & 0 \\ 0 & 0.5 & -0.5 & 0 \\ 1.5 & 0 & 0 & -0.5 \end{bmatrix} \begin{bmatrix} -(r^2 + r) \\ (r^2 + r) \\ -(4 - r^3 - r) \\ (4 - r^3 - r) \end{bmatrix};$$

$$V = 0.5[(S_1 + S_2)^{-1} - (S_1 - S_2)^{-1}] = \begin{bmatrix} 0 & 0.5 \\ 1.5 & 0 \end{bmatrix}.$$

Therefore,

$$\begin{bmatrix} \underline{a}_1 \\ \underline{a}_2 \\ -\overline{a}_1 \\ -\overline{a}_2 \end{bmatrix} = \begin{bmatrix} -0.5 & 0 & 0 & 0.5 \\ 0 & -0.5 & 1.5 & 0 \\ 0 & 0.5 & -0.5 & 0 \\ 1.5 & 0 & 0 & -0.5 \end{bmatrix} \begin{bmatrix} -(r^2 + r) \\ (r^2 + r) \\ -(4 - r^3 - r) \\ (4 - r^3 - r) \end{bmatrix},$$

$$\begin{cases} a_{\text{fuz}\,1} = (\underline{a}_1(r), \overline{a}_1(r)) = (0.25r^3 + 0.25r^2 + 0.5r - 1, \\ -0.25r^3 - 0.25r^2 - 0.5r + 1 \,|\, r \in [0,1]), \\ a_{\text{fuz}\,2} = (\underline{a}_2(r), \overline{a}_2(r)) = (0.25r^3 + 0.75r^2 + 0.5r + 1, \\ \quad -0.75r^3 + 0.25r^2 - 0.5r + 3 \,|\, r \in [0,1]). \end{cases} \tag{8}$$

The approximate estimate of the state is

$$x_{\text{fuz}}(s) \simeq x_{\text{fuz}\,n=2}(s) = a_{\text{fuz}\,1} h_1(s)$$
$$+ a_{\text{fuz}\,2} h_2(s) \Big|_{\substack{h_1(\cdot)=1 \\ h_2(\cdot)=s^3}} = a_{\text{fuz}\,1} + a_{\text{fuz}\,2} s^3,$$

where $a_{\text{fuz}\,1}$ and $a_{\text{fuz}\,2}$ are given by (8).

This estimate can be strong or weak; see the method proposed by the authors in the paper [19, 20]. ♦

3.3. Taylor estimation

In the general form, this method is often considered for a fuzzy system of integral equations [21]. For the sake of simplicity, we will implement a particular case of this system described by a single Fredholm equation of the second kind:

$$x_{\text{fuz}}(s) = f_{\text{fuz}}(s) + \int_a^b K(s, \tau) x_{\text{fuz}}(\tau) d\tau , \tag{9}$$

where $a \leq s, \tau \leq b$; $K(s, \tau)$ is a given crisp kernel differentiable by both variables on the interval $[a, b] \subset R$; $x_{\text{fuz}}(s)$ is a fuzzy unknown found from equation (9).

Let $f_{\text{fuz}}(s)$ and $x_{\text{fuz}}(s)$ have the parametric representations

$$f_{\text{fuz}}(s) = f(s,r) = (\underline{f}(s,r), \overline{f}(s,r) \,|\, r \in [0,1]),$$

$$x_{\text{fuz}}(s) = x(s,r) = (\underline{x}(s,r), \overline{x}(s,r) \,|\, r \in [0,1]).$$

Then equation (9) can be written in the parametric form

$$\begin{cases} \underline{x}(s,\, r) = \underline{f}(s,\, r) + \int_a^b \underline{U}(\tau,\, r)d\tau, \\ \overline{x}(s,\, r) = \overline{f}(s,\, r) + \int_a^b \overline{U}(\tau,\, r)d\tau, \end{cases} \quad (10)$$

$r \in [0,1]$,

where

$$\underline{U}(\tau,\, r) = \begin{cases} K(s,\, \tau)\underline{x}(\tau,\, r),\ K(s,\tau) \ge 0, \\ K(s,\, \tau)\overline{x}(\tau,\, r),\ K(s,\tau) < 0; \end{cases}$$

$$\overline{U}(\tau,\, r) = \begin{cases} K(s,\, \tau)\overline{x}(\tau,\, r),\ K(s,\tau) \ge 0, \\ K(s,\, \tau)\underline{x}(\tau,\, r),\ K(s,\tau) < 0. \end{cases}$$

Assume that the following inequalities hold on the interval $[a, b] \subset R$:

$$\begin{cases} K(s,\, \tau) \ge 0,\ a \le \tau \le c, \\ K(s,\, \tau) < 0,\ c < \tau \le b. \end{cases}$$

Then the system of equations (10) can be reduced to

$$\begin{cases} \underline{x}(s,\, r) = \underline{f}(s,\, r) + \int_a^c K(s,\, \tau)\underline{x} \times \\ \times(\tau,\, r)d\tau + \int_c^b K(s,\, \tau)\overline{x}(\tau,\, r)d\tau, \\ \overline{x}(s,\, r) = \overline{f}(s,\, r) + \int_a^c K(s,\, \tau)\overline{x} \times \\ \times(\tau,\, r)d\tau + \int_c^b K(s,\, \tau)\underline{x}(\tau,\, r)d\tau. \end{cases} \quad (11)$$

Now we expand the integrand functions $\underline{x}(\tau,\, r)$, $\overline{x}(\tau,\, r)$ in (11) into the Taylor polynomials of degree n. For a fixed point $\tau = z$, we obtain

$$\begin{cases} \underline{x}(s,\, r) = \underline{f}(s,\, r) + \int_a^c K(s,\tau)\sum_{i=0}^{N}\frac{1}{i!}\underline{x}_\tau^{(i)}(\tau,\, r)(\tau - z)^i d\tau + \\ \quad + \int_c^b K(s,\, \tau)\sum_{i=0}^{N}\frac{1}{i!}\overline{x}_\tau^{(i)}(\tau,\, r)(\tau - z)^i d\tau. \\ \overline{x}(s,r) = \overline{f}(s,r) + \int_a^c K(s,\, \tau)\sum_{i=0}^{N}\frac{1}{i!}\overline{x}_\tau^{(i)}(\tau,\, r)(\tau - z)^i d\tau + \\ \quad + \int_c^b K(s,\, \tau)\sum_{i=0}^{N}\frac{1}{i!}\underline{x}_\tau^{(i)}(\tau,\, r)(\tau - z)^i d\tau, \end{cases} \quad (12)$$

$$\underline{x}_\tau^{(i)}(\tau,r) = \left.\frac{\partial^i \underline{x}(\tau,r)}{\partial \tau^i}\right|_{\tau=z} \quad \text{and} \quad \overline{x}_\tau^{(i)}(\tau,r) = \left.\frac{\partial^i \overline{x}(\tau,r)}{\partial \tau^i}\right|_{\tau=z}.$$

Differentiating both of equations (12) $p = \overline{0, n}$ times with respect to the variable s yields:

$$\begin{aligned} &\underline{x}_s^{(p)}(s,r) = \underline{f}_s^{(p)}(s,r) + \sum_{i=0}^{N}\frac{1}{i!}\int_a^c K_s^{(p)}(s,\tau)\underline{x}_\tau^{(i)}(\tau,r) \times \\ &\times(\tau - z)^i d\tau + \sum_{i=0}^{N}\frac{1}{i!}\int_c^b K_s^{(p)}(s,\tau)\overline{x}_\tau^{(i)}(\tau,r)(\tau - z)^i d\tau, \\ &\overline{x}_s^{(p)}(s,r) = \overline{f}_s^{(p)}(s,r)+)\sum_{i=0}^{N}\frac{1}{i!}\int_a^c K_s^{(p)}(s,\tau\overline{x}_\tau^{(i)}(\tau,r) \times \\ &\times(\tau - z)^i d\tau + \sum_{i=0}^{N}\frac{1}{i!}\int_c^b K_s^{(p)}(s,\tau)\underline{x}_\tau^{(i)}(\tau,r)(\tau - z)^i d\tau, \end{aligned} \quad (13)$$

where $\underline{x}_s^{(p)}(s,r) = (\frac{\partial^p \underline{x}(s,r)}{\partial s^p})$, $\overline{x}_s^{(p)}(s,r) = (\frac{\partial^p \overline{x}(s,r)}{\partial s^p})$,

and $K_s^{(p)}(s,\, \tau) = \left(\frac{\partial^p K(s,\, \tau)}{\partial s^p}\right)$, $p = \overline{0, n}$.

Interchanging the integral and sum signs, we write the system of equations (13) as

$$\begin{aligned} &\underline{x}_s^{(p)}(s,r) = \underline{f}_s^{(p)}(s,r) + \sum_{i=0}^{N}\frac{1}{i!}\int_a^c K_s^{(p)}(s,\tau)\underline{x}_\tau^{(i)}(\tau,\, r) \times \\ &\times(\tau - z)^i d\tau + \sum_{i=0}^{N}\frac{1}{i!}\int_c^b K_s^{(p)}(s,\tau)\overline{x}_\tau^{(i)}(\tau,\, r)(\tau - z)^i d\tau, \\ &\overline{x}_s^{(p)}(s,r) = \overline{f}_s^{(p)}(s,r)+)\sum_{i=0}^{N}\frac{1}{i!}\int_a^c K_s^{(p)}(s,\, \tau)\overline{x}_\tau^{(i)}(\tau,\, r) \times \\ &\times(\tau - z)^i d\tau + \sum_{i=0}^{N}\frac{1}{i!}\int_c^b K_s^{(p)}(s,\tau)\underline{x}_\tau^{(i)}(\tau,\, r)(\tau - z)^i d\tau. \end{aligned} \quad (14)$$

Denoting

$$S_{pi}^1 = \frac{1}{i!}\int_a^c K_s^{(p)}(s,\, \tau)(\tau - z)^i d\tau,$$

$$S_{pi}^2 = \frac{1}{i!}\int_c^b K_s^{(p)}(s,\, \tau)(\tau - z)^i d\tau,$$

we reduce equations (14) to

$$\begin{aligned} &\underline{x}_s^{(p)}(s,\, r) = \underline{f}_s^{(p)}(s,\, r) + \sum_{i=0}^{N} S_{pi}^1(s,\, \tau)\underline{x}_\tau^{(i)} \times \\ &\quad \times(\tau,\, r) + \sum_{i=0}^{N} S_{pi}^2 \overline{x}_\tau^{(i)}(\tau,\, r), \\ &\overline{x}_s^{(p)}(s,\, r) = \overline{f}_s^{(p)}(s,\, r) + \sum_{i=0}^{N} S_{pi}^1(s,\, \tau)\overline{x}_\tau^{(i)} \times \\ &\quad \times(\tau,\, r) + \sum_{i=0}^{N} S_{pi}^2 \underline{x}_\tau^{(i)}(\tau,\, r), \end{aligned} \quad p = \overline{0, n}. \quad (15)$$

Introducing the vectors

$$\underline{X}(s,\, r) = (\underline{x}_s^{(0)}, \ldots,\, \underline{x}_s^{(n)})^{\mathrm{T}},\ \overline{X}(s,\, r) = (\overline{x}_s^{(0)}, \ldots,\, \overline{x}_s^{(n)})^{\mathrm{T}},$$

$\underline{F}(s, r) = (\underline{f}_s^{(0)}, ..., \underline{f}_s^{(n)})^{\mathrm{T}}$, $\bar{F}(s, r) = (\bar{f}_s^{(0)}, ..., \bar{f}_s^{(n)})^{\mathrm{T}}$

and the matrices

$$S^1 = \left(S_{pi}^1\right),\ S^2 = \left(S_{pi}^2\right),\ \left(p, i\right) = \overline{0, n},$$

for s, $z = a^* \in [a, b]$ we finally write (15) in the matrix form

$$X = F + \left(S^1 + S^2\right)X,\ X = (\underline{X}, \bar{X})^{\mathrm{T}},\ F = (\underline{F}, \bar{F})^{\mathrm{T}}. \quad (16)$$

The solution is

$$S = \begin{pmatrix} S^1 & S^2 \\ S^2 & S^1 \end{pmatrix} \Rightarrow (S - I)X = -F \Rightarrow X^* =$$

$$= -(S - I)^{-1}F, \left|S - I\right| \neq 0.$$

The convergence of the solution X^* to the exact solution $\tilde{X}\left(X^* \underset{n\to\infty}{\to} \tilde{X}\right)$ was proved in [22].

Example 2. Consider an integral equation of the form

$$x_{\mathrm{fuz}}(s) = f_{\mathrm{fuz}}(s) + \int_0^2 s^2(1+\tau)x_{\mathrm{fuz}}(\tau)d\tau, \quad (17)$$

where

$$f_{\mathrm{fuz}}(s) = f(s, r) = \left(\underline{f}(s, r) = sr, \bar{f}(s, r) = \left(\frac{14}{3}\right)s^2(r-2)\right)$$

$$\text{and } r \in [0,1] \subset R_1.$$

It is required to determine $x_{\mathrm{fuz}}(s) = x(s, r) = \left(\underline{x}(s, r), \bar{x}(s, r)\right)$.

To find the solution, we write equation (17) in the parametric form:

$$\begin{aligned} \underline{x}(s, r) &= sr + \int_0^2 s^2(1+\tau)\underline{x}(\tau, r)d\tau, \\ \bar{x}(s, r) &= \left(\frac{14}{3}\right)s^2(r-2)\int_0^2 s^2(1+\tau)\bar{x}(\tau, r)d\tau. \end{aligned} \quad (18)$$

In these expressions, the kernel is $K(s, \tau) = s^2 (1 + \tau) \geq$ $\geq 0\ \forall\tau \in [0,2]$, where the interval [0,2] defines the limits of integration in equation (18). Therefore, this interval does not contain the partition point c present in the system of equations (12).

We expand the unknown integrand functions $\underline{x}(\tau, r), \bar{x}(\tau, r)$ in (17) into the Taylor polynomials of degree n, letting $n = 1$ for simplicity. For $\tau = z \in [0, 2]$, we obtain:

$$\underline{x}(\tau, r) = 1 + \left(\frac{\partial\underline{x}}{\partial\tau}\right)\Bigg|_{\tau=z}(\tau - z),$$

$$\bar{x}(\tau, r) = 1 + \left(\frac{\partial\bar{x}}{\partial\tau}\right)\Bigg|_{\tau=z}(\tau - z),\ 0 \leq \tau,\ z \leq 2,\ r \in [0,1].$$

Since $n = 1$, each of equations (18) should be differentiated with respect to s, p = 0 and p = 1 times:

$$p = 0 \Rightarrow \frac{\partial^0}{\partial s^0} \Rightarrow$$

$$\Rightarrow \begin{cases} \underline{x}(s, r) = sr + \int_0^2 s^2(1+\tau)\underline{x}(\tau, r)dr, \\ \bar{x}(s, r) = \dfrac{14}{3}s^2(r-2) + \int_0^2 s^2(1+\tau)\bar{x}(\tau, r)d\tau, \end{cases}$$

$$p = 1 \Rightarrow \frac{\partial^1}{\partial s^1} \Rightarrow$$

$$\Rightarrow \begin{cases} \dot{\underline{x}}_s(s, r) = r + \int_0^2 2s(1+\tau)\underline{x}(\tau, r)dr, \\ \dot{\bar{x}}_s(s, r) = \dfrac{28}{3}s(r-2) + \int_0^2 s^2(1+\tau)\bar{x}(\tau, r)d\tau. \end{cases}$$

Next, we consider the vectors X and F and the elements of the matrix S:

$$X = (\underline{x}\big|s = a^*, r),\ \dot{\underline{x}}_s(s = a^*, r);$$

$$(\bar{x}\big|s = a^*, r),\ \dot{\bar{x}}_s(s = a^*, r)^{\mathrm{T}}$$

is the vector of fuzzy variables to be determined; $a^* \in [a, b]$;

$$F = \Big(\underline{f}(s = a^*, r) = a^*r,\ \dot{\underline{f}}_s = 1 \cdot r;\ \bar{f}(s = a^*,$$

$$r) = \frac{14}{3}a^{*2}(r-2),\ \dot{\bar{f}}_s = \frac{28}{3}a^*(r-2)\Big)^{\mathrm{T}}$$

is a given vector of fuzzy variables.

The elements S_{pi}^2 of the matrix S^2 are 0 since the interval [0,2] does not contain the partition point c. Therefore, $S^2 = \left(S_{pi}^2 = 0\right)$. The elements S_{pi}^1 of the matrix S^1 are

$$S_{pi}^1 = \frac{1}{i!}\int_0^2 K_s^{(p)}(s, \tau)(\tau - z)^i d\tau =$$

$$\frac{1}{i!}\int_0^2 [(s = a^* = 0)^2(1+\tau)]_s^{(p)}(\tau - z = a^* = 0)^i d\tau = 0,$$

$$p, i = 0, 1.$$

Hence, $S^1 = \left(S_{pi}^1\right) = 0$, and consequently,

$$S = \begin{pmatrix} S^1 = 0 & S^2 = 0 \\ S^2 = 0 & S^1 = 0 \end{pmatrix}.$$

Then equation (16) gives

$$S - I\big|_{s=0} = -I.$$

As a result, the matrix equation

$$\left(S-I\right)\Big|_{S=0}\cdot X=-F\Rightarrow IX=F$$

takes the form

$$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}\underline{x}\\ \dot{\underline{x}}_s\\ \overline{x}\\ \dot{\overline{x}}\end{pmatrix}=\begin{pmatrix}ar\\ 1-r\\ \left(\frac{14}{3}\right)a^2(r-2)\\ \left(\frac{28}{3}\right)a(r-2)\end{pmatrix}\Rightarrow x_{\text{fuz}}=x(s,\,r)=$$

$$=\left(\underline{x}(s,\,r)\Big|_{a=s}=sr,\ \overline{x}(s,\,r)=\left(\frac{14}{3}\right)s^2(r-2)\Big|r\in[0,\,1]\right).$$

In the general case, it is necessary to define the fuzzy set of integral equations [22, 23]

$$x_{\text{fuz}\,i}(s)=f_{\text{fuz}\,i}(s)+\sum_{j=1}^{m}\int_a^b K_{ij}(s,\,\tau)x_{\text{fuz}\,i}(\tau)d\tau\,,$$

$$i=\overline{1,m}\,. \tag{19}$$

In this system, $a\le s,\tau\le b$, and $K_{ij}(s,\,\tau),\,i,\ j=$, $=\overline{1,\ m}$, are given crisp kernels differentiable by both variables on the interval $[a,b]$; $f_{\text{fuz}\,i}$ are given fuzzy functions; $x_{\text{fuz}\ i}(s)=(x_{\text{fuz}\ 1}(s),\ \dots,\ x_{\text{fuz}\ m}(s))^{\text{T}}$ is the fuzzy vector to be determined. The fuzzy variables $f_{\text{fuz}\,i}(s)$ and $x_{\text{fuz}\,i}(s)$ are written in the parametric form

$$f_{\text{fuz}\,i}(s)=f_i(s,r)=(\underline{f}(s,r),\,\overline{f}(s,r)\big|r\in[0,1]),$$

$$x_{\text{fuz}\,i}(s)=x_i(s,\,r)=(\underline{x}_i(s,r),\,\overline{x}_i(s,\,r)\big|r\in[0,1]),\ i=\overline{1,\,m}.$$

Next, the sequential transformations described above (see the one-dimensional case) are used: the interval of integration $[a,b]$ is partitioned using the points c_{ij}, $i,j=\overline{1,m}$; the unknown variables $\underline{x}_i(s,r),\overline{x}_i(s,r)$ are expanded into the Taylor polynomials of degree n at an arbitrary point $\tau=z\in[a,\,b]\subset R$; each of equations (19), written in the parametric form, is differentiated $p=\overline{0,n}$ times with respect to s; the symbols $\int$ and Σ are interchanged; the corresponding notations are introduced for the vectors and matrices involved.

These transformations yield fuzzy systems of linear equations of the form (16):

$$SX=F\,, \tag{20}$$

where $X(\cdot)=\left(\underline{x}_1^{(n)}(\cdot),\,\overline{x}_1^{(n)}(\cdot),\dots,\underline{x}_m^{(n)}(\cdot),\,\overline{x}_m^{(n)}(\cdot)\right)^{\text{T}}$ is the unknown vector of fuzzy variables; $(\cdot)$ indicates $(s=a,r)$; (n) is the number of the derivative and the degree of the Taylor polynomial; $\underline{x}_k^{(n)}(\cdot)=$ $=\left(\underline{x}_k(\cdot),\dots,\ \underline{x}_k^{(n)}(\cdot)\right)^{\text{T}}$; $\overline{x}_k^{(n)}(\cdot)=\left(\overline{x}_k(\cdot),\dots,\overline{x}_k^{(n)}(\cdot)\right)^{\text{T}}$, $k=\overline{1,\,m}$, are the components of the vector X;

$F(\cdot)=\left(\underline{f}_1^{(n)}(\cdot),\ \overline{f}_1^{(n)}(\cdot),\dots,\ \underline{f}_m^{(n)}(\cdot),\ \overline{f}_m^{(n)}(\cdot)\right)^{\text{T}}$ is a given vector of fuzzy variables;

$\underline{f}_k^{(n)}=-\left(\underline{f}_k(\cdot),\dots,\ \underline{f}_k^{(n)}(\cdot)\right)^{\text{T}}$; $\overline{f}_k^{(n)}=-\left(\overline{f}_k(\cdot),\dots,\ \overline{f}_k^{(n)}(\cdot)\right)^{\text{T}}$;

$S=\begin{pmatrix}S^{(1,1)}&\dots&S^{(1,m)}\\ \vdots&\ddots&\vdots\\ S^{(m,1)}&\dots&S^{(m,m)}\end{pmatrix}$ is a matrix with matrix elements

$$W^{(i,j)}=\begin{pmatrix}S_{11}^{(ij)}&S_{12}^{(ij)}\\ S_{21}^{(ij)}&S_{22}^{(ij)}\end{pmatrix};\ S_{11}^{(ij)}=$$

$$=S_{22}^{(ij)}=\begin{pmatrix}S_{00}^{(ij)}-1&\dots&S_{0n}^{(ij)}\\ \vdots&\ddots&\vdots\\ S_{n0}^{(ij)}&\dots&S_{nn}^{(ij)}-1\end{pmatrix};$$

$$S_{12}^{(ij)}=S_{21}^{(ij)}=\begin{pmatrix}S_{00}^{*(ij)}&\dots&S_{0n}^{*(ij)}\\ \vdots&\ddots&\vdots\\ S_{n0}^{*(ij)}&\dots&S_{nn}^{*(ij)}\end{pmatrix}.\ \blacklozenge$$

Example 3. Let:

$$\underline{f}_1(s,\,r)=s\cdot r-\frac{27}{4}s^2(r^3-2)-\frac{14}{3}s^2r-\frac{1}{4}s^2r(r^4+2);$$

$$\overline{f}_1(s,\,r)=\frac{14}{3}s^2(r-2)+\frac{3}{4}s^2(r^3-2)-$$

$$-s(r-2)+\frac{9}{4}s^2r(r^4+2);$$

$$\underline{f}_2(s,\,r)=s(s^5+2r)-14.1(s-2)^2(r^3-2)-$$

$$-\frac{8}{3}(s^2+1)r-0.3(s-2)^3r(r^4+2);$$

$$\overline{f}_2(s,\,r)=\frac{8}{3}(s^2+1)(r-2)-s(3r^3-6)+$$

$$+0.9(s-2)^2(r^3-2)+4.7(s-2)r(r^4+2).$$

The collection of kernels is

$$K_{11}(s,\,\tau)=s^2(1+\tau);\ K_{12}(s,\,\tau)=$$

$$=s^2(1-\tau^2),\ K_{21}(s,\,\tau)=(1+s^2)\tau,$$

and $K_{22}(s,\,\tau)=(s-2)(1-\tau^3)$, where $0\le s$ and $\tau\le 2$.

Solution. Choosing the point $z=0$ for the Taylor expansion, we obtain:

$$W^{(1,1)}=\begin{pmatrix}-1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix};W^{(1,2)}=(0)_{i,j}\,,i,j=\overline{1,4};$$

$$W^{(2,1)} = \begin{pmatrix} 2 & \frac{8}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & \frac{8}{3} \\ 0 & 0 & 0 & 0 \end{pmatrix};$$

$$W^{(2,1)} = \begin{pmatrix} 2 & 1.2 & -11 & -\frac{94}{5} \\ -3 & -2.2 & 11 & \frac{94}{5} \\ -11 & -\frac{94}{5} & 2 & 1.2 \\ 11 & \frac{94}{5} & -3 & -2.2 \end{pmatrix}.$$

Solving the fuzzy system of linear equations (20) gives

$$X = S^{-1}F,$$

where

$$X = X(a=0, r) = (\underline{x}_1,\ \underline{x}_1^*,\ \overline{x}_1,\ \overline{x}_1^*,\ \underline{x}_2,\ \underline{x}_2^*,\ \overline{x}_2,\ \overline{x}_2^*)^{\mathrm{T}};$$

$$S^{-1}F = (0,\ r,\ 0,\ 2-r,\ 0,\ r^5+2r,\ 0,\ 6-r^3)^{\mathrm{T}}. \blacklozenge$$

3.4. Estimation by method of degenerate kernels

Let the equation kernel be a finite sum in which each term is the product of some function of τ by some function of s. In this case, equation (19) with the kernel $K(s, \tau) = \sum_{i=1}^{n} a_i(s) b_i(\tau)$ is a fuzzy Fredholm integral equation with a crisp nondegenerate kernel [22, 23]. Like before (see subsection 3.2), the following assumptions are made to ensure the existence of a unique fuzzy solution by the method of successive approximation: $a_i(s)$ is defined, piecewise continuous in the Hausdorff sense, and bounded by the first Seikkala derivative for $s \in [a, b] \subset R$; $b_i(\tau)$ satisfies the same constraint for $\tau \in [0, t] \subset R$.

We modify the well-known method for solving traditional (crisp) equations with degenerate kernels to solve the corresponding fuzzy equation [24].

Consider a fuzzy integral equation (19) with the kernel

$$K(s, \tau) = \sum_{i=1}^{n} a_i(s) b_i(\tau).$$

Assume that the following inequalities hold on the interval of integration $[a, b]$:

$$(i): \begin{cases} \sum_{i=1}^{n} a_i(s) b_i(\tau) \ge 0, \\ a \le \tau \le b; \end{cases} \quad (ii): \begin{cases} \sum_{i=1}^{n} a_i(s) b_i(\tau) < 0, \\ a \le \tau \le b; \end{cases}$$

$$(iii): \begin{cases} \sum_{i=1}^{n} a_i(s) b_i(\tau) \ge 0, \\ a \le \tau \le c, \\ \sum_{i=1}^{n} a_i(s) b_i(\tau) < 0, \\ c \le \tau \le b. \end{cases}$$

In case (i), the system of equations (19) satisfies the relations

$$x_{\text{fuz}}(s) = f_{\text{fuz}}(s) + \int_a^b K(s, \tau) x_{\text{fuz}}(\tau) d\tau \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \underline{x}(s, r) = \underline{f}(s, r) + \sum_{i=1}^{n} a_i(s) \underline{x}_i, \\ \overline{x}(s, r) = \overline{f}(s, r) + \sum_{i=1}^{n} a_i(s) \overline{x}_i, \\ \underline{x}_i = \int_a^b b_i(\tau) \underline{x}(\tau) d\tau,\ \overline{x}_i = \\ = \int_a^b b_i(\tau) \overline{x}(\tau) d\tau. \end{cases} \quad (21)$$

Multiplying the expressions (21) by $b_i(\tau)$ and integrating them on the interval $[a, b]$, we obtain:

$$\underline{x}_i = \underline{f}_i + \sum_{i=1}^{n} a_{ij}^1 \underline{x}_i,\ \underline{f}_i = \int_a^b \underline{f}(\tau) b_i(\tau) d\tau,\ a_{ij}^1 = \int_a^b b_i(\tau) a_j(\tau) d\tau,$$

$$\overline{x}_i = \overline{f}_i a_{ij}^1 \overline{x}_i,\ \ \overline{f}_i = \int_a^b \overline{f}(\tau) b_i(\tau) d\tau,\ i = \overline{1, n}.$$

These relations lead to the fuzzy system of linear equations

$$X_{\text{fuz}} = AX_{\text{fuz}} + F_{\text{fuz}},$$

where

$$X_{\text{fuz}} = (\underline{X} \,|\, \overline{X})^{\mathrm{T}};\ \underline{X} = (\underline{x}_1, \ldots, \underline{x}_n); \overline{X} = (\overline{x}_1, \ldots, \overline{x}_n);$$

$$F_{\text{fuz}} = (\underline{F} \,|\, \overline{F})^{\mathrm{T}},$$

$$F_{\text{fuz}} = (\underline{F} \,|\, \overline{F});\ \underline{F} = (\underline{f}_1, \ldots, \underline{f}_n),\ \overline{F} = (\overline{f}_1, \ldots, \overline{f}_n);$$

$$A = \begin{pmatrix} A^1 & 0 \\ 0 & A^1 \end{pmatrix};\ A^1 = (a_{ij}^1),$$

$$a_{ij}^1 = \int_a^b b_i(\tau) a_j(\tau) d\tau,\ i, j = \overline{1, n}.$$

It can be written in the traditional fuzzy calculus form [19, 20]:

$(I - A) X_{\text{fuz}} = F_{\text{fuz}}$, where I denotes an identity matrix. (22)

The case $|I - A| = 0$ was studied in the papers [19, 20].

Example 4. Consider an integral equation of the form (21):

$$x_{\text{fuz}}(s) = f_{\text{fuz}}(s) + \int_0^{0.5} s \cdot \tau \cdot x_{\text{fuz}}(\tau) d\tau, \ s \in [0, \ 0.5],$$

where

$$f_{\text{fuz}}(s) = f(s, r) = (\underline{f}(s, r), \ \overline{f}(s,r) | r \in [0,1]);$$

$$K(s, \tau) = s\tau \geq 0, \ a_i(s) = s, \ b_i(\tau) = \tau.$$

The solution is found from the fuzzy matrix equation (22) with the matrix elements

$$X_{\text{fuz}} = X(s, r) = (\underline{x}_1 | \overline{x}_1)^{\mathrm{T}}; \ F_{\text{fuz}} = F(s, r) =$$

$$= \left(\underline{f}_1 = \int_0^{0.5} \underline{f}(\tau)\tau d\tau \overline{f}_1 = \int_0^{0.5} \overline{f}(\tau)\tau d\tau \right)^{\mathrm{T}};$$

$$a_{ij}^1 = \int_0^{0.5} \tau^2 d\tau = \frac{\tau^3}{3}\Big|_0^{0.5} = \frac{1}{24}; \ I - A = 1 - a_{11}^1 = 1 - \frac{1}{24} = \frac{23}{24}.$$

Hence,

$$\underline{x}_1 = \frac{24}{23} \int_0^{0.5} \underline{f}(\tau)\tau d\tau, \ \overline{x}_1 = \frac{24}{23} \int_0^{0.5} \overline{f}(\tau)\tau d\tau,$$

and the solution has the form

$$x_{\text{fuz}}(s) = (\underline{x}(s,r), \overline{x}(s,r) \,|\, r \in [0,1]),$$

where

$$\underline{x}(s,r) = \underline{f}(s,r) + s\underline{x}_1, \ \overline{x}(s,r) = \overline{f}(s,r) + s\overline{x}_1.$$

In case (ii), the calculations similar to case (i) yield

$$(\overline{X} | \underline{X})^{\mathrm{T}} = (-I\underline{X} | -I\overline{X})^{\mathrm{T}},$$

due to the multiplication rule of fuzzy variables x,

$$kx = \begin{cases} (k\underline{x}, \ k\overline{x}), \ k \geq 0, \ k \in R, \\ (k\overline{x}, \ k\underline{x}), \ k < 0. \end{cases}$$

After trivial transformations, we finally obtain:

$$\begin{pmatrix} \underline{X} \\ \overline{X} \end{pmatrix} = \begin{pmatrix} A_1 & 0 \\ 0 & A_1 \end{pmatrix} \begin{pmatrix} \underline{X} \\ \overline{X} \end{pmatrix} + \begin{pmatrix} \underline{F} \\ \overline{F} \end{pmatrix} \Leftrightarrow \begin{pmatrix} \underline{X} \\ \overline{X} \end{pmatrix} =$$

$$= \begin{pmatrix} A_1 & 0 \\ 0 & A_1 \end{pmatrix} \begin{pmatrix} -I\underline{X} \\ -I\overline{X} \end{pmatrix} + \begin{pmatrix} -I\underline{F} \\ -I\overline{F} \end{pmatrix} \Leftrightarrow$$

$$\Rightarrow (I + A) X_{\text{fuz}} = -F_{\text{fuz}}. \ ♦$$

3.5. Estimation by method of nondegenerate kernel approximated by degenerate one

Consider the relation (21), and let the kernel be $K(s, \tau) = K(s \cdot \tau)$. According to the Taylor expansion, for $(s \cdot \tau) \simeq 0$ we obtain

$$K(s \cdot \tau) \sum_{i=1}^n e_i (s \cdot \tau)^i = \sum_{i=1}^n a_i(s) b_i(\tau).$$

Hence, the equation

$$x_{\text{fuz}}(s) = f_{\text{fuz}}(s) + \int_a^b K(s, \tau) x_{\text{fuz}}(\tau) d\tau$$

is solved using the method described in subsection 3.4.

Example 5. Consider the integral equation

$$x_{\text{fuz}}(s) = f_{\text{fuz}}(s) + \int_0^{0.5} \sin(s \cdot \tau) x_{\text{fuz}}(\tau) d\tau.$$

Applying the Taylor approximation of the kernel $K(s \cdot \tau) \simeq s \cdot \tau,$ we can use the results of Example 4.

Under the approximation

$$K(s \cdot \tau) \cdot s \cdot \tau - \frac{1}{3}(s \cdot \tau)^3 = a_1 b_1 \Big|_{\substack{a_1 = 3 \\ b_1 = \tau}} + a_2 b_2 \Big|_{\substack{a_2 = 3^3 \\ b_2 = -\frac{1}{3}\tau^3}},$$

this equation can be also solved using the method from subsection 3.4. ♦

CONCLUSIONS

Based on the definition of a fuzzy Riemann integral, the problem of estimating the states of models described by fuzzy Fredholm–Volterra integral equations has been formulated under the assumed existence of their unique solutions.

Various state estimation methods for fuzzy integral equations have been considered, namely, the fuzzy Laplace transform, the method of "embedding" models, the Taylor estimation of the degenerate kernels, and the estimation of the nondegenerate kernels by degenerate forms. Test examples have been solved for them. As shown above, in some cases, the estimation results are related to the solution of fuzzy systems of linear algebraic equations.

In part II of the survey, other state estimation methods for linear and nonlinear fuzzy integral models will be considered, namely, the least squares method and its modifications, the Galerkin and Chebyshev methods, and sinc functions.

REFERENCES

1. Agafonova, S.A., German, A.D., and Muratova, T.V., *Differentsial'nye uravneniya* (Differential Equations), Moscow: Bauman Moscow State Technical University, 2011. (In Russian.)
2. Volkov, I.K., and Kanatnikov, A.N., *Integral'nye preobrazovaniya i operatsionnoe ischislenie* (Integral Transformations and Operational Calculus), Moscow: Bauman Moscow State Technical University, 2015. (In Russian.)
3. Volkov, I.K., Zuev, S.M., and Tsvetkova, G.M., *Sluchainye protsessy* (Random Processes), Moscow: Bauman Moscow State Technical University, 2006. (In Russian.)
4. Levin, V.I., Interval Equations in Problems of Research of Uncertain Systems, *Information Technologies*, 2018, vol. 24, no. 3, pp. 147–153. (In Russian.)
5. Altunin, A.E. and Semukhin, M.V., *Modeli i algoritmy prinyatiya reshenii v nechetkikh usloviyakh* (Models and Algorithms of Decision-Making in Uncertain Conditions), Tyumen: Tyumen State University, 2000. (In Russian.)
6. *Metody robastnogo, neiro-nechetkogo i adaptivnogo upravleniya* (Methods of Robust, Neural-Fuzzy, and Adaptive Control), Pupkov, K.A., Ed., Moscow: Bauman Moscow State Technical University, 2001. (In Russian.)

7. Dubois, D. and Prade, H., *Possibility Theory: An Approach to Computerized Processing of Uncertainty*, Boston: Springer-Verlag, 1988.
8. Mochalov, I.A., Petrunin, N.G., Red'kin, A.S., and Tsegel'skii, S.V., Fuzzy Probabilistic and Statistical Methods, *Information Technologies*, Appendix no. 4, 2003, pp. 1–24. (In Russian.)
9. Mochalov, I.A. and Khrisat, M.S., Estimation Parameter Model Using Fuzzy Random Data, *Information Technologies*, 2014, no. 2 (210), pp. 14–22. (In Russian.)
10. Kolmogorov, A.N. and Fomin, S.V., *Elements of the Theory of Functions and Functional Analysis*, Dover, 1999.
11. Hochstadt, H., *Integral Equations. Pure and Applied Mathematics*, New York–London–Sydney: John Wiley & Sons, 1973.
12. Congxin, W., and Ming, M. On the Integrals, Series and Integral Equations of Fuzzy Set-Valued Functions, *Journal of Harbin Institute of Technology*, 1990, vol. 21, pp. 9–11.
13. Park, J.Y. and Jeong, J.V., On Existence and Uniqueness of Solution of Fuzzy Volterra–Fredholm Integral Equations, *Fuzzy Sets & Systems*, 2000, no. 115, pp. 425–431.
14. Friedman, M., Ma, M., and Kandel, A., Numerical Solutions of Fuzzy Differential and Integral Equations, *Fuzzy Sets and Systems*, 1999, vol. 106, no. 1, pp. 35–48.
15. Lizorkin, P.I., *Kurs differentsial'nykh i integral'nykh uravnenii s dopolnitel'nymi glavami analiza* (A Course in Differential and Integral Equations with Additional Analysis Chapters), Moscow: Nauka, 1981. (In Russian.)
16. Demenkov, N.P., Mikrin, E.A., and Mochalov, I.A. Fuzzy Transformation of Laplace in Tasks of Fuzzy Mathematical Modelling. Part 1, *Information Technologies*, 2017, vol. 23, no. 4, pp. 251–257. (In Russian.)
17. Demenkov, N.P., Mikrin, E.A., and Mochalov, I.A., Fuzzy Transformation of Laplace in Tasks of Fuzzy Mathematical Modelling. Part II, *Information Technologies*, 2017, vol. 23, no. 5, pp. 362–369. (In Russian.)
18. Friedman, M., Ming, M., and Kandel, A., Fuzzy Linear Systems, *Fuzzy Sets and Systems*, 1998, no. 96, pp. 201–209.
19. Demenkov, N.P., Mikrin, E.A., and Mochalov, E.A., Methods of Solving Fuzzy Systems of Linear Equations. Part 1. Complete Systems, *Control Sciences*, 2019, no. 4, pp. 3–14. (In Russian.) DOI: http://doi.org/1025728/pu.2019.4.1
20. Demenkov, N.P., Mikrin, E.A., and Mochalov, E.A., Methods of Solving Fuzzy Systems of Linear Equations. Part 2. Incomplete Systems, *Control Sciences*, 2019, no. 5, pp. 19–28. (In Russian.) DOI: http://doi.org/1025728/pu.2019.5.2
21. Jahantigh, M., Allahviaranloo, T., and Otadi, M., Numerical Solution of Fuzzy Integral Equations, *Applied Mathematical Sciences*, 2008, vol. 2, no. 1, pp. 33–46.
22. Jafarian, A., Measoomy Nia, S., Tavan, S., and Banifazel, M., Solving Linear Fredholm Fuzzy Integral Equations Systems by Taylor Expansion Method, *Applied Mathematical Sciences*, 2012, vol. 6, no. 83, pp. 4103–4117.
23. Jafarian, A., Measoomy Nia, S., and Tavan, S., A Numerical Scheme to Solve Fuzzy Linear Volterra Integral Equations Systems, *Journal of Applied Mathematics*, 2012, Article ID 216923.
24. Tikhonov, A.N., and Arsenin, V.Y., Solutions of Ill-Posed Problems, Winston, 1977.

This paper was recommended for publication by F.F. Pashchenko, a member of the Editorial Board.

Received February 21, 2020, and revised November 18, 2020. Accepted November 24, 2020.

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Cite this article

Demenkov, N.P., Mikrin, E.A., Mochalov, I.A. State Estimation Methods for Fuzzy Integral Models. Part I: Approximation Methods. *Control Sciences* **1,** 2–12 (2021).
http://doi.org/10.25728/cs.2021.1.1