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A METHOD FOR CONSTRUCTING NONELEMENTARY LINEAR REGRESSIONS BASED ON MATHEMATICAL PROGRAMMING

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Abstract. This paper is devoted to constructing nonelementary linear regressions consisting of explanatory variables and all possible combinations of their pairs transformed using binary minimum and maximum operations. Such models are formalized through a 0-1 mixed integer linear programming problem. By adjusting the constraints on binary variables, we control the structural specification of a nonelementary linear regression, namely, the number of regressors, their types, and the composition of explanatory variables. In this case, the model parameters are approximately estimated using the ordinary least squares method. The formulated problem has advantages: the number of constraints does not depend on the sample size, and the signs of the estimates for the explanatory variables are consistent with the signs of their correlation coefficients with the dependent variable. Regressors are eliminated at the initial stage to reduce the time for solving the problem and make the model quite interpretable. A nonelementary linear regression of rail freight in Irkutsk oblast is constructed, and its interpretation is given.

Keywords: nonelementary linear regression, ordinary least squares method, 0-1 mixed integer linear programming problem, subset selection, coefficient of determination, interpretation, rail freight.

INTRODUCTION

In regression analysis [1, 2] based on economic data, special attention is paid to constructing production functions (PFs), i.e., mathematical relationships between production volumes (outputs) and production factors. Published in 1986, the monograph [3] was entirely devoted to the theory, methods, and application of PFs. It considered the following PFs: linear, multi-mode, Cobb–Douglas, Leontief, Allen, CES (*Constant Elasticity of Substitution*), LES (*Linear Elasticity of Substitution*), and Solow. At present, new modifications of PFs appear; they are investigated and are actively used in econometric studies [4–6]. In this paper, we construct nonelementary regression models specified on the basis of the well-known Leontief PF

$$y_i = \min\{\alpha_1 x_{i1}, \alpha_2 x_{i2}, ..., \alpha_l x_{il}\} + \varepsilon_i\,,\ i = \overline{1, n}, \qquad (1)$$

with the following notations: n is the sample size; l is the number of explanatory variables; y_i, $i = \overline{1, n}$, are the values of the independent variable y; x_{ij}, $i = \overline{1, n}$, $j = \overline{1, l}$, are the values of the explanatory variables x_1, $x_2, ..., x_l$; α_j, $j = \overline{1, l}$, are unknown parameters; finally, ε_i, $i = \overline{1,n}$, are approximation errors. In the theory of PFs, the variable y in equation (1) is interpreted as the output, whereas $x_1, ..., x_n$ are interpreted as the indicators of production factors.

Note that the monograph [3] also identified the "parallel" Leontief function

$$y_i = \min\{\alpha_{11} x_{i1},\ \alpha_{12} x_{i2}, ..., \alpha_{1l} x_{il}\} + \ldots$$
$$\min\{\alpha_{k1} x_{i1}, \alpha_{k2} x_{i2}, ..., \alpha_{kl} x_{il}\} + \varepsilon_i,\ i = \overline{1, n}.$$

This function reflects a process where the output is composed of the outputs of k parallel production processes with fixed proportions of factors using common resources. For two production factors x_1 and x_2, the "parallel" Leontief function is called the linear programming function.

According to the monograph [7], the parameters of the Leontief PF (1) can be estimated using non-smooth optimization methods [8–10], which are often difficult to implement. Therefore, the exact estimation of the PF (1) was reduced in [7] to a 0-1 mixed integer linear programming problem (MILPP) using the least absolute deviations (LAD) method. Note that 0-1 MILPPs are also called partially Boolean linear programming problems. At the same time, the author [7] proposed an approximate estimation method for the Leontief PF based on enumerating the estimates from a preformed domain.

The paper [11] introduced a function with the opposite meaning to the PF (1):

$$y_i = \max\left\{\alpha_1 x_{i1}, \alpha_2 x_{i2}, ..., \alpha_l x_{il}\right\} + \varepsilon_i, \quad i = \overline{1, n}, \quad (2)$$

The paper [12] considered the symbiosis of the functions (1) and (2):

$$\begin{aligned} y_i = \min\left\{\alpha_1 x_{i1}, \alpha_2 x_{i2}, ..., \alpha_l x_{il}\right\} + \\ \max\left\{\beta_1 x_{i1}, \beta_2 x_{i2}, ..., \beta_l x_{il}\right\} + \varepsilon_i, \; i = \overline{1, n}. \end{aligned} \quad (3)$$

In [11] and [12], the exact estimation of the parameters of the regressions (2) and (3) was reduced to corresponding 0-1 MILPPs using the LAD method. In the modern scientific literature, there is increased attention to regression models based on mathematical programming; for example, see the papers [13–15]. An explanation is recent advances in the technology for solving 0-1 MILPPs.

This paper deals with estimating regression models specified based on the Leontief PF using the ordinary least squares (OLS) method [1, 2]. Such a problem was first formulated in [16] for the regression (1) with two explanatory variables. The paper [17] proposed a nonelementary linear regression (NLR) of the form

$$\begin{aligned} y_i = \alpha_0 + \sum_{j=1}^{l} \alpha_j x_{ij} + \\ \sum_{j=1}^{C_l^2} \alpha_{j+l} \min\{x_{i,\mu_{j1}}, \lambda_j x_{i,\mu_{j2}}\} + \varepsilon_i, \; i = \overline{1, n}, \end{aligned} \quad (4)$$

with the following notations: μ_{j1} and μ_{j2}, $j = \overline{1, C_l^2}$, are elements of the first and second columns of the index matrix $\mathbf{M}_{C_l^2 \times 2}$ (its rows contain all possible combinations of index pairs of the variables); α_j, $j = \overline{0, l + C_l^2}$, and λ_j, $j = \overline{1, C_l^2}$, are unknown parameters. By assumption, all variables in equation (4) have strictly positive values.

Obviously, NLR belongs to the class of nonlinear parametric models. But if all parameters λ_j, $j = \overline{1, C_l^2}$, are assigned definite values, the regression becomes linear, and its parameters α_j, $j = \overline{0, l + C_l^2}$, can be easily estimated using the OLS method. As established in the paper [17], the OLS-optimal estimates of the NLR parameters λ_j, $j = \overline{1, C_l^2}$, belong to the intervals

$$\lambda_j \in \left(\lambda_{\min}^{(j)}, \lambda_{\max}^{(j)}\right), \; j = \overline{1, l}, \quad (5)$$

where $\lambda_{\min}^{(j)} = \min\left\{\frac{x_{1,\mu_{j1}}}{x_{1,\mu_{j2}}}, \frac{x_{2,\mu_{j1}}}{x_{2,\mu_{j2}}}, ..., \frac{x_{n,\mu_{j1}}}{x_{n,\mu_{j2}}}\right\}$ and $\lambda_{\max}^{(j)} = \max\left\{\frac{x_{1,\mu_{j1}}}{x_{1,\mu_{j2}}}, \frac{x_{2,\mu_{j1}}}{x_{2,\mu_{j2}}}, ..., \frac{x_{n,\mu_{j1}}}{x_{n,\mu_{j2}}}\right\}$. The points $\lambda_j = \lambda_{\min}^{(j)}$ and $\lambda_j = \lambda_{\max}^{(j)}$ cannot be used because of the perfect multicollinearity of the variables.

Due to these properties, an approximate OLS estimation method was proposed in [17] for the NLR (4). The method enumerates the values of the parameters λ_j, $j = \overline{1, C_l^2}$, from the intervals (5).

Unfortunately, the total number of regressors grows significantly with increasing the number l of explanatory variables in the NLR (4). Therefore, it becomes necessary to select a certain number of the most "informative" regressors [7]. Two strategies were developed for this purpose in [18]. Each strategy forms a set of alternative regressions according to a special algorithm; then the approximate OLS estimation method [17] is implemented for each regression; finally, the model with the smallest sum of the squared residuals is selected. The main disadvantage of the NLR construction approach proposed in [18] is the exhaustive search of all possible alternatives: it can take too much time to select the most informative regressors. A more promising approach involves 0-1 MILPPs; see below.

In the paper [19], the selection of the most informative regressors in linear regression estimation using the OLS method was reduced to a 0-1 MILPP. An open issue in [19] was choosing a large positive number M affecting both the speed and solution of the problem. It was successfully settled in the next publication [20]: the 0-1 MILPP formulated therein allows constructing a linear regression with a given number of explanatory variables, in which the signs of the OLS estimates are consistent with the signs of the correlation coefficients between the variables y and x_j, $j = \overline{1, l}$. In the course of computational experiments, the conclusion of the paper [21] was confirmed: such a problem with constraints on the signs of the coefficients is solved an order of magnitude faster than without them. In this paper, the main goal is to reduce the construction of the NLR to the 0-1 MILPP considered in the paper [20], which is efficiently solvable.

1. A METHOD FOR CONSTRUCTING NONELEMENTARY LINEAR REGRESSIONS

The NLR equation (4) contains only one binary operation, the minimum. Hereinafter, the binary minimum (maximum) is a mathematical operation with two arguments that returns their minimum (maximum). Let us supplement this regression model with regressors with the binary maximum:

$$y_i=\alpha_0+\sum_{j=1}^{l}\alpha_j x_{ij}+\sum_{j=1}^{C_l^2}\alpha_{j+l}\min\{x_{i,\mu_{j1}},\lambda_j x_{i,\mu_{j2}}\}+\sum_{j=1}^{C_l^2}\alpha_{j+l+C_l^2}\max\{x_{i,\mu_{j1}},\lambda_j x_{i,\mu_{j2}}\}+\varepsilon_i,\ i=\overline{1,n}. \tag{6}$$

The total number of regressors in equation (6), $l+2C_l^2$, is much greater than in equation (4).

Equation (6) is introduced for the first time. Therefore, we pose the following problem: formalize the procedure of constructing this model as a 0-1 MILPP. This can be done as follows.

For each parameter λ_j, $j=\overline{1,C_l^2}$, from equation (6), we determine the intervals (5). Then we evenly divide each of these intervals by p points, forming a matrix $\Lambda=\left(\lambda_{jk}^*\right)$, $j=\overline{1,C_l^2}$, $k=\overline{1,p}$. The element λ_{jk}^* of this matrix shows the k th value of the parameter λ_j for the j th pair of the variables. Replacing the unknown parameters λ_j in equation (6) with the known elements of the matrix Λ yields

$$y_i=\alpha_0+\sum_{j=1}^{l}\alpha_j x_{ij}+\sum_{j=1}^{C_l^2}\sum_{k=1}^{p}\alpha_{jk}^-\min\left\{x_{i,\mu_{j1}},\lambda_{jk}^* x_{i,\mu_{j2}}\right\}+\sum_{j=1}^{C_l^2}\sum_{k=1}^{p}\alpha_{jk}^+\max\left\{x_{i,\mu_{j1}},\lambda_{jk}^* x_{i,\mu_{j2}}\right\}+\varepsilon_i,\ i=\overline{1,n}, \tag{7}$$

where α_{jk}^-, $j=\overline{1,C_l^2}$, $k=\overline{1,p}$, are the unknown parameters for regressors with the binary minimum and α_{jk}^+, $j=\overline{1,C_l^2}$, $k=\overline{1,p}$, are the unknown parameters for regressors with the binary maximum. In model (7), the total number of regressors is $l+2pC_l^2$, even exceeding that in model (6). For example, for $l=100$ variables and $p=10$ partitions, the regression (7) has 99 100 regressors.

Substituting $z_{ijk}^-=\min\left\{x_{i,\mu_{j1}},\lambda_{jk}^* x_{i,\mu_{j2}}\right\}$ and $z_{ijk}^+=\max\left\{x_{i,\mu_{j1}},\lambda_{jk}^* x_{i,\mu_{j2}}\right\}$, $i=\overline{1,n}$, $j=\overline{1,C_l^2}$, $k=\overline{1,p}$, into equation (7) gives the multiple linear regression model

$$y_i=\alpha_0+\sum_{j=1}^{l}\alpha_j x_{ij}+\sum_{j=1}^{C_l^2}\sum_{k=1}^{p}\alpha_{jk}^- z_{ijk}^-+\sum_{j=1}^{C_l^2}\sum_{k=1}^{p}\alpha_{jk}^+ z_{ijk}^++\varepsilon_i,\ i=\overline{1,n}. \tag{8}$$

Following [19], let us reduce the selection of the most informative regressors for the linear regression (8) with OLS estimation to a 0-1 MILPP. First, we normalize all variables of equation (8) using the well-known rule: subtract from each value of the variable its arithmetic mean and divide the result by the standard deviation.

For model (8), we write the standardized regression equation

$$w_i=\sum_{j=1}^{l}\beta_j q_{ij}+\sum_{j=1}^{C_l^2}\sum_{k=1}^{p}\beta_{jk}^- h_{ijk}^-+\sum_{j=1}^{C_l^2}\sum_{k=1}^{p}\beta_{jk}^+ h_{ijk}^++\xi_i,\ i=\overline{1,n}, \tag{9}$$

where: w is the normalized variable y; q_j, $j=\overline{1,l}$, are the normalized variables x_j, $j=\overline{1,l}$; h_{jk}^- and h_{jk}^+, $j=\overline{1,C_l^2}$, $k=\overline{1,p}$, are the normalized variables z_{jk}^- and z_{jk}^+, $j=\overline{1,C_l^2}$, $k=\overline{1,p}$, respectively; β_j, $j=\overline{1,l}$, and β_{jk}^- and β_{jk}^+, $j=\overline{1,C_l^2}$, $k=\overline{1,p}$, are unknown standardized coefficients; finally, ξ_i, $i=\overline{1,n}$, are new approximation errors.

For model (9), the OLS estimates are given by

$$\tilde{\beta}=R_{XX}^{-1}\cdot R_{YX}, \tag{10}$$

where $R_{XX}=\begin{pmatrix} R_{xx} & R_{xz^-} & R_{xz^+}\\ R_{z^-x} & R_{z^-z^-} & R_{z^-z^+}\\ R_{z^+x} & R_{z^+z^-} & R_{z^+z^+}\end{pmatrix}$ is a correlation block matrix of dimensions $(l+2pC_l^2)\times(l+2pC_l^2)$. This matrix consists of the following blocks:

$$R_{xx}=\left(r_{x_jx_k}\right),\ j=\overline{1,l},\ k=\overline{1,l};$$

$$R_{xz^-}=\left(r_{x_sz_{jk}^-}\right),\ s=\overline{1,l},\ j=\overline{1,C_l^2},\ k=\overline{1,p};$$

$$R_{xz^+}=\left(r_{x_sz_{jk}^+}\right),\ s=\overline{1,l},\ j=\overline{1,C_l^2},\ k=\overline{1,p};$$

$$R_{z^-x}=\left(r_{z_{jk}^-x_s}\right),\ j=\overline{1,C_l^2},\ k=\overline{1,p},\ s=\overline{1,l};$$

$$R_{z^-z^-}=\left(r_{z_{s_1s_2}^-z_{kj}^-}\right),\ s_1=C_l^2,\ s_2=\overline{1,p},\ j=\overline{1,C_l^2},\ k=\overline{1,p};$$

$$R_{z^-z^+}=\left(r_{z_{s_1s_2}^-z_{kj}^+}\right),\ s_1=C_l^2,\ s_2=\overline{1,p},\ j=\overline{1,C_l^2},\ k=\overline{1,p};$$

$$R_{z^+x}=\left(r_{z_{jk}^+x_s}\right),\ j=\overline{1,C_l^2},\ k=\overline{1,p},\ s=\overline{1,l};$$

$$R_{z^+z^-}=\left(r_{z_{s_1s_2}^+z_{kj}^-}\right),\ s_1=C_l^2,\ s_2=\overline{1,p},\ j=\overline{1,C_l^2},\ k=\overline{1,p};$$

$R_{z^+z^+} = \left(r_{z^+_{s_1s_2} z^+_{kj}} \right)$, $s_1 = C_l^2$, $s_2 = \overline{1,\ p}$, $j = \overline{1,\ C_l^2}$, $k = \overline{1,\ p}$;

$R_{YX} = \left(R_{yx} \quad R_{yz^-} \quad R_{yz^+} \right)^{\mathrm{T}}$ is the correlation block vector of dimensions $\left(l + 2pC_l^2 \right) \times 1$ consisting of the blocks

$R_{yx} = \left(r_{yx_j} \right)$, $j = \overline{1,\ l}$; $R_{yz^-} = \left(r_{yz^-_{jk}} \right)$, $j = \overline{1,\ C_l^2}$, $k = \overline{1,\ p}$; $R_{yz^+} = \left(r_{yz^+_{jk}} \right)$, $j = \overline{1,\ C_l^2}$, $k = \overline{1,\ p}$.

The coefficient of determination of model (9) is given by

$$R^2 = \sum_{j=1}^{l} r_{yx_j} \beta_j + \sum_{j=1}^{C_l^2} \sum_{k=1}^{p} r_{yz^-_{jk}} \beta^-_{jk} + \sum_{j=1}^{C_l^2} \sum_{k=1}^{p} r_{yz^+_{jk}} \beta^+_{jk}. \quad (11)$$

Then, using formulas (10) and (11), we state the problem of selecting the most informative regressors for the linear regression (8):

$$R^2 \to \max, \quad (12)$$

$$-\left(1-\delta_j\right) M \le \sum_{k=1}^{l} r_{x_j x_k} \beta_k + \sum_{s=1}^{C_l^2} \sum_{k=1}^{p} r_{x_j z^-_{sk}} \beta^-_{sk} + \sum_{s=1}^{C_l^2} \sum_{k=1}^{p} r_{x_j z^+_{sk}} \beta^+_{sk} - r_{yx_j} \le (1-\delta_j) M,\ j = \overline{1,\ l}, \quad (13)$$

$$-\left(1-\delta^-_{jk}\right) M \le \sum_{s=1}^{l} r_{x_s z^-_{jk}} \beta_s + \sum_{s_1=1}^{C_l^2} \sum_{s_2=1}^{p} r_{z^-_{s_1s_2} z^-_{jk}} \beta^-_{s_1s_2} + \sum_{s_1=1}^{C_l^2} \sum_{s_2=1}^{p} r_{z^+_{s_1s_2} z^-_{jk}} \beta^+_{s_1s_2} - r_{yz^-_{jk}} \le (1-\delta^-_{jk}) M,$$
$$j = \overline{1,\ C_l^2},\ k = \overline{1,\ p}, \quad (14)$$

$$-\left(1-\delta^+_{jk}\right) M \le \sum_{s=1}^{l} r_{x_s z^+_{jk}} \beta_s + \sum_{s_1=1}^{C_l^2} \sum_{s_2=1}^{p} r_{z^-_{s_1s_2} z^+_{jk}} \beta^-_{s_1s_2} + \sum_{s_1=1}^{C_l^2} \sum_{s_2=1}^{p} r_{z^+_{s_1s_2} z^+_{jk}} \beta^+_{s_1s_2} - r_{yz^+_{jk}} \le (1-\delta^+_{jk}) M,$$
$$j = \overline{1,\ C_l^2},\ k = \overline{1,\ p}, \quad (15)$$

$$-\delta_j M \le \beta_j \le \delta_j M,\ j = \overline{1,\ l}, \quad (16)$$

$$-\delta^-_{jk} M \le \beta^-_{jk} \le \delta^-_{jk} M,\ j = \overline{1,\ C_l^2},\ k = \overline{1,\ p}, \quad (17)$$

$$-\delta^+_{jk} M \le \beta^+_{jk} \le \delta^+_{jk} M,\ j = \overline{1,\ C_l^2},\ k = \overline{1,\ p}, \quad (18)$$

$$\delta_j \in \{0, 1\},\ j = \overline{1,\ l}, \quad (19)$$

$$\delta^-_{jk} \in \{0,1\},\ j = \overline{1,\ C_l^2},\ k = \overline{1,\ p}, \quad (20)$$

$$\delta^+_{jk} \in \{0, 1\},\ j = \overline{1,\ C_l^2},\ k = \overline{1,\ p}, \quad (21)$$

$$\sum_{j=1}^{l} \delta_j + \sum_{j=1}^{C_l^2} \sum_{k=1}^{p} \delta^-_{jk} + \sum_{j=1}^{C_l^2} \sum_{k=1}^{p} \delta^+_{jk} = m, \quad (22)$$

where: m is a given number of regressors; δ_j, $j = \overline{1,\ l}$, are the Boolean variables specified by the rule

$$\delta_j = \begin{cases} 1 \text{ if the } j\text{th variable enters into the regression,} \\ 0 \text{ otherwise;} \end{cases}$$

δ^-_{jk}, $j = \overline{1,\ C_l^2}$, $k = \overline{1, p}$, are Boolean variables specified by the rule

$$\delta^-_{jk} = \begin{cases} 1 \text{ if the } j\text{th binary minimum with} \\ \text{the } k\text{th transformation enters into the regression,} \\ 0 \text{ otherwise;} \end{cases}$$

δ^+_{jk}, $j = \overline{1,\ C_l^2}$, $k = \overline{1, p}$, are Boolean variables specified by the rule

$$\delta^+_{jk} = \begin{cases} 1 \text{ if the } j\text{th binary maximum with} \\ \text{the } k\text{th transformation enters into the regression,} \\ 0 \text{ otherwise;} \end{cases}$$

finally, M is a large positive number.

An advantage of the 0-1 MILPP (12)–(22) is that the number of its constraints does not depend on the sample size n.

In the 0-1 MILPP (12)–(22), the strategy for constructing the NLR is regulated by constraints on the binary variables. Consider the following strategies.

Strategy 1. Selecting m regressors in the linear regression (7).

Here, we simply need to solve problem (12)–(22). In this case, the final model may contain several regressors with the same binary operation and with the same pair of variables but with different values of the parameter λ_j.

Strategy 2. Estimating the NLR (6) approximately using the OLS method (without selecting regressors).

Here, we need to solve the problem with the objective function (12), the constraints (13)–(21) and

$$\sum_{k=1}^{p} \delta^-_{jk} = 1,\ \sum_{k=1}^{p} \delta^+_{jk} = 1,\ j = \overline{1,\ C_l^2}.$$

(In other words, for each pair of the variables, each binary operation enters into the model with only one value of the parameter λ_j.)

Strategy 3. Selecting m regressors in the NLR (6).

Here, we need to solve the problem with the objective function (12), the constraints (13)–(22) and

$$\sum_{k=1}^{p} \delta^-_{jk} \le 1,\ \sum_{k=1}^{p} \delta^+_{jk} \le 1,\ j = \overline{1,\ C_l^2}. \quad (23)$$

Note that by adjusting the constraints on the binary variables, we can control the type of regressors in the

NLR (6). For example, adding the constraints $\sum_{j=1}^{C_l^2}\sum_{k=1}^{p}\delta_{jk}^{-}=0$ and $\sum_{j=1}^{C_l^2}\sum_{k=1}^{p}\delta_{jk}^{+}=0$ to problem (12)–(22) yields the problem of selecting the most informative regressors for the linear regression; the constraints $\sum_{j=1}^{l}\delta_j=0$ and $\sum_{j=1}^{C_l^2}\sum_{k=1}^{p}\delta_{jk}^{+}=0$, the same problem for the regression with binary minimum operations only; the constraints $\sum_{j=1}^{l}\delta_j=0$ and $\sum_{j=1}^{C_l^2}\sum_{k=1}^{p}\delta_{jk}^{-}=0$, the same problem for the regression with binary maximum operations only.

Also, it is possible to control the composition of the variables in the model. For this purpose, we introduce a binary matrix $V=\{v_{ij}\}$, $i=\overline{1,\, l+2pC_l^2}$, $j=\overline{1,\, l}$, in which

$$v_{ij}=\begin{cases}1 \text{ if the } j\text{th variable enters into} \\ \text{the } i\text{th regressor of model (7),} \\ 0 \text{ otherwise.}\end{cases}$$

Then integrating the linear constraints

$$\sum_{j=1}^{l}v_{ij}\delta_j+\sum_{j=1}^{C_l^2}\sum_{k=1}^{p}v_{i,l+k+p(j-1)}\delta_{jk}^{-}+\sum_{j=1}^{C_l^2}\sum_{k=1}^{p}v_{i,l+pC_l^2+k+p(j-1)}\delta_{jk}^{+}\le 1,\ i=\overline{1,\, l}, \tag{24}$$

into problem (12)–(22) allows constructing the NLR with m regressors into which each explanatory variable enters at most once. In this case, conditions (23) naturally hold.

Unfortunately, for problem (12)–(22), it is not completely clear how to specify large numbers M. To settle this issue, we adopt the approach from [20]. Let us replace the constraints (13)–(18) by the following ones:

$$-\left(1-\delta_j\right)M_{u_j}^{-}\le\sum_{k=1}^{l}r_{x_jx_k}\beta_k+\sum_{s=1}^{C_l^2}\sum_{k=1}^{p}r_{x_jz_{sk}^{-}}\beta_{sk}^{-}+\sum_{s=1}^{C_l^2}\sum_{k=1}^{p}r_{x_jz_{sk}^{+}}\beta_{sk}^{+}-r_{yx_j}\le(1-\delta_j)M_{u_j}^{+},\ j=\overline{1,\, l}, \tag{25}$$

$$-\left(1-\delta_{jk}^{-}\right)M_{u_{jk}^{-}}^{-}\le\sum_{s=1}^{l}r_{x_sz_{jk}^{-}}\beta_s+\sum_{s_1=1}^{C_l^2}\sum_{s_2=1}^{p}r_{z_{s_1s_2}^{-}z_{jk}^{-}}\beta_{s_1s_2}^{-}+\sum_{s_1=1}^{C_l^2}\sum_{s_2=1}^{p}r_{z_{s_1s_2}^{+}z_{jk}^{-}}\beta_{s_1s_2}^{+}-r_{yz_{jk}^{-}}\le(1-\delta_{jk}^{-})M_{u_{jk}^{-}}^{+}, \tag{26}$$

$$j=\overline{1,\, C_l^2},\ k=\overline{1,\, p},$$

$$-\left(1-\delta_{jk}^{+}\right)M_{u_{jk}^{+}}^{-}\le\sum_{s=1}^{l}r_{x_sz_{jk}^{+}}\beta_s+\sum_{s_1=1}^{C_l^2}\sum_{s_2=1}^{p}r_{z_{s_1s_2}^{-}z_{jk}^{+}}\beta_{s_1s_2}^{-}+\sum_{s_1=1}^{C_l^2}\sum_{s_2=1}^{p}r_{z_{s_1s_2}^{+}z_{jk}^{+}}\beta_{s_1s_2}^{+}-r_{yz_{jk}^{+}}\le(1-\delta_{jk}^{+})M_{u_{jk}^{+}}^{+}, \tag{27}$$

$$j=\overline{1,\, C_l^2},\ k=\overline{1,\, p},$$

$$0\le\beta_j\le\delta_jM_{\beta_j},\ j\in J_\beta^{+}, \tag{28}$$

$$\delta_jM_{\beta_j}\le\beta_j\le 0,\ j\in J_\beta^{-}, \tag{29}$$

$$0\le\beta_{jk}^{-}\le\delta_{jk}^{-}M_{\beta_{jk}^{-}},\ j,k\in J_{\beta^-}^{+}, \tag{30}$$

$$\delta_{jk}^{-}M_{\beta_{jk}^{-}}\le\beta_{jk}^{-}\le 0,\ j,k\in J_{\beta^-}^{-}, \tag{31}$$

$$0\le\beta_{jk}^{+}\le\delta_{jk}^{+}M_{\beta_{jk}^{+}},\ j,k\in J_{\beta^+}^{+}, \tag{32}$$

$$\delta_{jk}^{+}M_{\beta_{jk}^{+}}\le\beta_{jk}^{+}\le 0,\ j,k\in J_{\beta^+}^{-}, \tag{33}$$

where: J_β^{+} and J_β^{-} are the index sets constructed from the set $\{1, 2, ..., l\}$ so that their elements satisfy the conditions $r_{yx_j}>0$ and $r_{yx_j}<0$, respectively; $J_{\beta^-}^{+}$ and $J_{\beta^-}^{-}$ are the index sets constructed from the set $\{\{1, 2\}, ..., \{1, p\}, \{2, 1\}, ..., \{2, p\}, ..., \{C_l^2, 1\}, ..., \{C_l^2, p\}\}$ so that their elements satisfy the conditions $r_{yz_{jk}^{-}}>0$ and $r_{yz_{jk}^{-}}<0$, respectively; $J_{\beta^+}^{+}$ and $J_{\beta^+}^{-}$ are the index sets constructed from the set $\{\{1, 2\}, ..., \{1, p\}, \{2, 1\}, ..., \{2, p\}, ..., \{C_l^2, 1\}, ..., \{C_l^2, p\}\}$ so that their elements satisfy the conditions $r_{yz_{jk}^{+}}>0$ and $r_{yz_{jk}^{+}}<0$; finally, $M_{\beta_j}=1/r_{yx_j}$, $j=\overline{1,\, l}$, and $M_{\beta_{jk}^{-}}=1/r_{yz_{jk}^{-}}$ and $M_{\beta_{jk}^{+}}=1/r_{yz_{jk}^{+}}$, $j=\overline{1,\, C_l^2}$, $k=\overline{1,\, p}$.

To find $M_{u_j}^{-}$ in the constraints (25), we need to solve a series of l linear programming problems with the objective functions $u_j\to\min$ subject to the constraints

$$0\le\beta_j\le M_{\beta_j},\ j\in J_\beta^{+}, \tag{34}$$

$$M_{\beta_j}\le\beta_j\le 0,\ j\in J_\beta^{-}, \tag{35}$$

$$0\le\beta_{jk}^{-}\le M_{\beta_{jk}^{-}},\ j,k\in J_{\beta^-}^{+}, \tag{36}$$

$$M_{\beta_{jk}^{-}}\le\beta_{jk}^{-}\le 0,\ j,k\in J_{\beta^-}^{-}, \tag{37}$$

$$0\le\beta_{jk}^{+}\le M_{\beta_{jk}^{+}},\ j,k\in J_{\beta^+}^{+}, \tag{38}$$

$$M_{\beta_{jk}^{+}}\le\beta_{jk}^{+}\le 0,\ j,k\in J_{\beta^+}^{-}, \tag{39}$$

$$\sum_{k=1}^{l} r_{x_j x_k} \beta_k + \sum_{s=1}^{C_l^2}\sum_{k=1}^{p} r_{x_j z_{sk}^-} \beta_{sk}^- + \sum_{s=1}^{C_l^2}\sum_{k=1}^{p} r_{x_j z_{sk}^+} \beta_{sk}^+ - r_{yx_j} = u_j,\ j=\overline{1,\ l}, \quad (40)$$

$$\sum_{s=1}^{l} r_{x_s z_{jk}^-} \beta_s + \sum_{s_1=1}^{C_l^2}\sum_{s_2=1}^{p} r_{z_{s_1 s_2}^- z_{jk}^-} \beta_{s_1 s_2}^- + \sum_{s_1=1}^{C_l^2}\sum_{s_2=1}^{p} r_{z_{s_1 s_2}^+ z_{jk}^-} \beta_{s_1 s_2}^+ - r_{yz_{jk}^-} = u_{jk}^-,\ j=\overline{1,\ C_l^2},\ k=\overline{1,\ p}, \quad (41)$$

$$\sum_{s=1}^{l} r_{x_s z_{jk}^+} \beta_s + \sum_{s_1=1}^{C_l^2}\sum_{s_2=1}^{p} r_{z_{s_1 s_2}^- z_{jk}^+} \beta_{s_1 s_2}^- + \sum_{s_1=1}^{C_l^2}\sum_{s_2=1}^{p} r_{z_{s_1 s_2}^+ z_{jk}^+} \beta_{s_1 s_2}^+ - r_{yz_{jk}^+} = u_{jk}^+,\ j=\overline{1,\ C_l^2},\ k=\overline{1,\ p}, \quad (42)$$

$$\sum_{j=1}^{l} r_{yx_j} \beta_j + \sum_{j=1}^{C_l^2}\sum_{k=1}^{p} r_{yz_{jk}^-} \beta_{jk}^- + \sum_{j=1}^{C_l^2}\sum_{k=1}^{p} r_{yz_{jk}^+} \beta_{jk}^+ \le 1. \quad (43)$$

To find $M_{u_j}^+$, we need to solve a series of l linear programming problems with the objective functions $u_j \to \max$ subject to the constraints (34)–(43). Similarly, the numbers $M_{u_{jk}^-}^-$, $M_{u_{jk}^-}^+$, $M_{u_{jk}^+}^-$, and $M_{u_{jk}^+}^+$ are obtained by solving a series of pC_l^2 linear programming problems with the objective functions $u_{jk}^- \to \min$, $u_{jk}^- \to \max$, $u_{jk}^+ \to \min$, and $u_{jk}^+ \to \max$, respectively, subject to the constraints (34)–(43).

Thus, by solving the 0-1 MILPP with the objective function (12) and the constraints (19)–(22), (25)–(33), we construct the linear regression (7) with m regressors in which the signs of the estimates of the parameters β are consistent with those of the corresponding correlation coefficients of the regressors with the variable y. In other words, the following inequalities hold: $\beta_j r_{yx_j} > 0$, $j=\overline{1,\ l}$; $\beta_{jk}^- r_{yz_{jk}^-} > 0$, $\beta_{jk}^+ r_{yz_{jk}^+} > 0$, $j=\overline{1,\ C_l^2}$, $k=\overline{1,\ p}$. The NLR construction strategy in this problem is still regulated, e.g., by constraints (23) and (24) on the binary variables.

As experimentally established in [20, 21], the 0-1 MILPP (12), (19)–(22), (25)–(33) is solved an order of magnitude faster than problem (12)–(22). Moreover, since the signs of the estimates of the parameters β are consistent with those of the corresponding correlation coefficients, the absolute contributions of the variables to the total determination R^2 are given by

$$C_{x_j}^{\text{abs}} = r_{yx_j} \beta_j,\ j=\overline{1,\ l},\ C_{z_{jk}^-}^{\text{abs}} = r_{yz_{jk}^-} \beta_{jk}^-,\quad C_{z_{jk}^+}^{\text{abs}} = r_{yz_{jk}^+} \beta_{jk}^+,\ j=\overline{1,\ C_l^2},\ k=\overline{1,\ p}. \quad (44)$$

They can be used to assess the effect of each regressor on the variable y.

We make two important remarks about the solution of problem (12), (19)–(22), (25)–(33).

Remark 1. As mentioned, the signs of the estimates of the parameters β in the solution are consistent with those of the corresponding correlation coefficients. Hence, all signs of the correlation coefficients r_{yx_j} must match the physical meaning of the variables. For this purpose, experts from the relevant subject area can be involved. Inconsistent variables should be excluded from consideration. Otherwise, the resulting regression will be difficult to interpret.

Remark 2. For example, suppose that model (8) contains the regressor $z_{11}^- = \min\{x_1, 8x_2\}$ at the parameter α_{11}^-. After the transition to the piecewise representation, the parameter α_{11}^- will have either the variable x_1 or the variable $8x_2$. If $r_{yz_{11}^-} > 0$, the estimate of the parameter α_{11}^- will surely be positive, and the variables x_1 and $8x_2$ will affect y with the plus sign. In this case, the correlation coefficients r_{yx_1} and r_{yx_2} must be positive. (Otherwise, there is a problem with interpreting the model.) On the other hand, if $r_{yz_{11}^-} < 0$, the estimate of the parameter α_{11}^- will surely be negative, and the variables x_1 and $8x_2$ will affect y with the minus sign. In this case, the correlation coefficients r_{yx_1} and r_{yx_2} must be negative. Therefore, after agreeing on the signs of the correlation coefficients r_{yx_j}, $j=\overline{1,\ l}$, with the experts, it is necessary to form the variables z_{jk}^- and z_{jk}^+, $j=\overline{1,\ C_l^2}$, $k=\overline{1,\ p}$, find their correlation coefficients with the variable y, and eliminate those not satisfying the conditions

$$(r_{yz_{jk}^-} > 0 \text{ and } r_{yx_{\mu_{j1}}} > 0 \text{ and } r_{yx_{\mu_{j2}}} > 0) \text{ or } (r_{yz_{jk}^-} < 0 \text{ and } r_{yx_{\mu_{j1}}} < 0 \text{ and } r_{yx_{\mu_{j2}}} < 0),\quad j=\overline{1,\ C_l^2},\ k=\overline{1,\ p}, \quad (45)$$

$$(r_{yz_{jk}^+} > 0 \text{ and } r_{yx_{\mu_{j1}}} > 0 \text{ and } r_{yx_{\mu_{j2}}} > 0) \text{ or } (r_{yz_{jk}^+} < 0 \text{ and } r_{yx_{\mu_{j1}}} < 0 \text{ and } r_{yx_{\mu_{j2}}} < 0),\quad j=\overline{1,\ C_l^2},\ k=\overline{1,\ p}. \quad (46)$$

Removing the contradictory variables will naturally decrease the time to construct the NLR. This time can be considerably reduced further if we supplement the expressions (45) and (46) with the conditions

$$\left| r_{yz_{jk}^-} \right| \ge r,\ \left| r_{yz_{jk}^+} \right| \ge r,\ j=\overline{1,\ C_l^2},\ k=\overline{1,\ p}, \quad (47)$$

where r is a number chosen from the interval [0, 1]. The greater the number r is, the smaller the number of variables will be, and the less time it will take to solve the problem.

2. MODELING

To construct an NLR, we collected annual statistical data on the horizon 2000–2020 for the dependent variable y (freight forward by public railway transport in Irkutsk oblast, million rubles) and 62 variables x_1, x_2, ..., x_{62}, presumably affecting y. First, 6 variables with the absolute value of the correlation coefficient with y not exceeding 0.2 were removed from the list. Then the values of correlation coefficients for the remaining 56 variables were given to 2 experts representing the East Siberian Department of the Russian Railways. They were asked to eliminate the variables for which the signs of the correlation coefficients with y did not correspond to the economic meaning of the problem. After the expertise procedure, 8 factors remained under consideration:

– the percentage of the working-age population, x_2;

– labor force (thousand people), x_3;

– the number of pensioners (thousand people), x_5;

– the number of private cars per 1000 people, x_8;

– the number of enterprises and organizations, x_{18};

– organizations' accounts payable (million rubles), x_{20};

– electricity production (billion kWh), x_{22};

– rail freight tariffs (c. u.), x_{58}.

The value of the variable x_{58} for 2001 was set equal to 1000 c. u. It was used to find the other values of the variable x_{58} using the known tariff indices.

For the selected variables, the correlation coefficients with the variable y were $r_{yx_2} = 0.785$, $r_{yx_3} = 0.543$, $r_{yx_5} = -0.483$, $r_{yx_8} = -0.446$, $r_{yx_{18}} = 0.538$, $r_{yx_{20}} = -0.204$, $r_{yx_{22}} = 0.476$, and $r_{yx_{58}} = -0.465$.

These variables affect the variable y as follows:

- The growth of the labor force x_2 and x_3, as well as the growth of the number of enterprises and organizations x_{18} and electricity production x_{22}, increases the output of products in the region, causing a higher demand for rail freight. On the other hand, an increase in the variable x_5 hinders economic development, reducing the demand for rail freight.
- The surplus of private cars x_8 reduces the demand for rail transportation (passenger and freight).
- The growth of organizations' accounts payable x_{20} has a negative impact on the regional economy: for example, it can lead to imposing various penalties.
- Higher freight tariffs x_{58} naturally reduce the demand for rail freight.

Then, the intervals (5) of the parameters λ_j were determined for each pair of the selected variables. To form the matrix Λ, we uniformly divided each interval by four points. As a result, $4C_8^2 = 112$ variables z_{jk}^-, $j = \overline{1,28}$, $k = \overline{1,4}$, were obtained with the binary minimum operation, and the same number of the variables z_{jk}^-, $j = \overline{1,28}$, $k = \overline{1,4}$, were obtained with the binary maximum operation. From the 224 variables, we excluded those not satisfying conditions (45)–(47) with $r = 0.2$ (140 variables in total). Thus, the final list included 92 variables, of which 8 were explanatory and 84 were transformed using the minimum and maximum operations.

The NLR was constructed by solving the 0-1 MILPP with the objective function (12) and the constraints (19)–(21), (25)–(33). We emphasize that the constraint (22) on the number of regressors was not applied. The constraints (24) were considered to ensure that each explanatory variable entered into the final model at most once. The LPSolve IDE solver was used to solve the 0-1 MILPPs, and a special program in the Delphi environment was developed to form mathematical models of the problems for the solver. First, the unknown numbers in the constraints (25)–(27) were calculated by the program. For that purpose, 184 linear programming problems with the corresponding objective functions and the linear constraints (34)–(43) were solved. Then, the 0-1 MILPP problem (12), (19)–(21), (24)–(33) with 284 constraints, 92 real and 92 binary variables was formulated using the calculated numbers and the developed program for the LPSolve solver. It was solved on a PC with an Intel Core i5 processor (3.40 GHz, 4 cores) and 8 GB RAM. As a result, the following NLR was constructed in approximately 30 s:

$$
\begin{aligned}
\tilde{y} = -24.5274 + \underset{(13.98)}{\overset{(0.6427)}{1.1895}} \min\{x_2,\ 0.000933x_{18}\} - \\
\underset{(-3.361)}{\overset{(0.1129)}{0.0196}} \min\{x_5,\ 0.006754x_{20}\} - \\
\underset{(-2.182)}{\overset{(0.0843)}{0.0323}} \min\{x_8,\ 0.11725x_{58}\} + \\
\underset{(3.859)}{\overset{(0.1063)}{0.0254}} \max\{x_3,\ 23.079x_{22}\}.
\end{aligned}
\tag{48}
$$

Here, the numbers in parentheses below the coefficients are Student's t-test values, and the numbers in parentheses above the coefficients are the absolute contributions of the regressors to the total determination (formulas (44)). All regressors were significant by Student's t-test with the significance level $\alpha = 0.05$.

The mathematical apparatus proposed in this paper does not control the significance of NLR coefficients by Student's t-test or the absolute contributions of the variables during the regression construction procedure. For significance control, we expect to integrate special linear constraints into the 0-1 MILPP in the future.

The coefficient of determination of the NLR (48) is $R^2 = 0.946183$, indicating of high quality of the model.

The variance inflation factors for the regressors of the model (48) do not exceed 10 (no multicollinearity). Note that multicollinearity in the 0-1 MILPP cannot yet be controlled either.

Thus, the model (48) is quite interpretable.

The model (48) in the piecewise form is presented in the table.

The equations of model (48) for different ranges of variables

The NLR equation	Ranges of variables
$\tilde{y} = -24.527 + 0.0011x_{18} - 0.00013x_{20} - 0.0038x_{58} + 0.0254x_3$	$\frac{x_2}{x_{18}} \geq 0.000933, \frac{x_5}{x_{20}} \geq 0.00675, \frac{x_8}{x_{58}} \geq 0.117, \frac{x_3}{x_{22}} \geq 23.08$
$\tilde{y} = -24.527 + 0.0011x_{18} - 0.00013x_{20} - 0.0038x_{58} + 0.5857x_{22}$	$\frac{x_2}{x_{18}} \geq 0.000933, \frac{x_5}{x_{20}} \geq 0.00675, \frac{x_8}{x_{58}} \geq 0.117, \frac{x_3}{x_{22}} < 23.08$
$\tilde{y} = -24.527 + 0.0011x_{18} - 0.00013x_{20} - 0.0323x_8 + 0.0254x_3$	$\frac{x_2}{x_{18}} \geq 0.000933, \frac{x_5}{x_{20}} \geq 0.00675, \frac{x_8}{x_{58}} < 0.117, \frac{x_3}{x_{22}} \geq 23.08$
$\tilde{y} = -24.527 + 0.0011x_{18} - 0.00013x_{20} - 0.0323x_8 + 0.5857x_{22}$	$\frac{x_2}{x_{18}} \geq 0.000933, \frac{x_5}{x_{20}} \geq 0.00675, \frac{x_8}{x_{58}} < 0.117, \frac{x_3}{x_{22}} < 23.08$
$\tilde{y} = -24.527 + 0.0011x_{18} - 0.0196x_5 - 0.0038x_{58} + 0.0254x_3$	$\frac{x_2}{x_{18}} \geq 0.000933, \frac{x_5}{x_{20}} < 0.00675, \frac{x_8}{x_{58}} \geq 0.117, \frac{x_3}{x_{22}} \geq 23.08$
$\tilde{y} = -24.527 + 0.0011x_{18} - 0.0196x_5 - 0.0038x_{58} + 0.5857x_{22}$	$\frac{x_2}{x_{18}} \geq 0.000933, \frac{x_5}{x_{20}} < 0.00675, \frac{x_8}{x_{58}} \geq 0.117, \frac{x_3}{x_{22}} < 23.08$
$\tilde{y} = -24.527 + 0.0011x_{18} - 0.0196x_5 - 0.0323x_8 + 0.0254x_3$	$\frac{x_2}{x_{18}} \geq 0.000933, \frac{x_5}{x_{20}} < 0.00675, \frac{x_8}{x_{58}} < 0.117, \frac{x_3}{x_{22}} \geq 23.08$
$\tilde{y} = -24.527 + 0.0011x_{18} - 0.0196x_5 - 0.0323x_8 + 0.5857x_{22}$	$\frac{x_2}{x_{18}} \geq 0.000933, \frac{x_5}{x_{20}} < 0.00675, \frac{x_8}{x_{58}} < 0.117, \frac{x_3}{x_{22}} < 23.08$
$\tilde{y} = -24.527 + 1.1895x_2 - 0.00013x_{20} - 0.0038x_{58} + 0.0254x_3$	$\frac{x_2}{x_{18}} < 0.000933, \frac{x_5}{x_{20}} \geq 0.00675, \frac{x_8}{x_{58}} \geq 0.117, \frac{x_3}{x_{22}} \geq 23.08$
$\tilde{y} = -24.527 + 1.1895x_2 - 0.00013x_{20} - 0.0038x_{58} + 0.5857x_{22}$	$\frac{x_2}{x_{18}} < 0.000933, \frac{x_5}{x_{20}} \geq 0.00675, \frac{x_8}{x_{58}} \geq 0.117, \frac{x_3}{x_{22}} < 23.08$
$\tilde{y} = -24.527 + 1.1895x_2 - 0.00013x_{20} - 0.0323x_8 + 0.0254x_3$	$\frac{x_2}{x_{18}} < 0.000933, \frac{x_5}{x_{20}} \geq 0.00675, \frac{x_8}{x_{58}} < 0.117, \frac{x_3}{x_{22}} \geq 23.08$
$\tilde{y} = -24.527 + 1.1895x_2 - 0.00013x_{20} - 0.0323x_8 + 0.5857x_{22}$	$\frac{x_2}{x_{18}} < 0.000933, \frac{x_5}{x_{20}} \geq 0.00675, \frac{x_8}{x_{58}} < 0.117, \frac{x_3}{x_{22}} < 23.08$
$\tilde{y} = -24.527 + 1.1895x_2 - 0.0196x_5 - 0.0038x_{58} + 0.0254x_3$	$\frac{x_2}{x_{18}} < 0.000933, \frac{x_5}{x_{20}} < 0.00675, \frac{x_8}{x_{58}} \geq 0.117, \frac{x_3}{x_{22}} \geq 23.08$
$\tilde{y} = -24.527 + 1.1895x_2 - 0.0196x_5 - 0.0038x_{58} + 0.5857x_{22}$	$\frac{x_2}{x_{18}} < 0.000933, \frac{x_5}{x_{20}} < 0.00675, \frac{x_8}{x_{58}} \geq 0.117, \frac{x_3}{x_{22}} < 23.08$
$\tilde{y} = -24.527 + 1.1895x_2 - 0.0196x_5 - 0.0323x_8 + 0.0254x_3$	$\frac{x_2}{x_{18}} < 0.000933, \frac{x_5}{x_{20}} < 0.00675, \frac{x_8}{x_{58}} < 0.117, \frac{x_3}{x_{22}} \geq 23.08$
$\tilde{y} = -24.527 + 1.1895x_2 - 0.0196x_5 - 0.0323x_8 + 0.5857x_{22}$	$\frac{x_2}{x_{18}} < 0.000933, \frac{x_5}{x_{20}} < 0.00675, \frac{x_8}{x_{58}} < 0.117, \frac{x_3}{x_{22}} < 23.08$

According to the table, the composition of the variables affecting y changes depending on the conditions satisfied, and the parameter estimates $\lambda^-_{4,1} = 0.000933$, $\lambda^-_{16,2} = 0.00675$, $\lambda^-_{22,2} = 0.117$, and $\lambda^+_{12,3} = 23.08$ play the role of switching points for the following four automatically generated indicators:

– the ratio of the percentage of the working-age population (x_2) to the number of enterprises and organizations (x_{18});

– the ratio of the number of pensioners (x_5) to organizations' accounts payable (x_{20});

– the ratio of the number of private cars per 1000 people (x_8) to the rail freight tariffs (x_{58});

– the ratio of labor force (x_3) to electricity production (x_{22}).

Then the following interpretation is valid.

• If the indicator x_2/x_{18} is not smaller than 0.000933, the number of enterprises and organizations x_{18} will affect freight forward, whereas the percentage of the working-age population x_2 will have no effect. For example, increasing the number of enterprises and organizations x_{18} by 1 (under fixed values of the other variables) raises the freight forward y by 0.0011 million rubles on average. However, if the indicator x_2/x_{18} is less than 0.000933, the percentage of the working-age population x_2 will affect freight forward, whereas the number of enterprises and organizations x_{18} will have no effect. For example, increasing the percentage of the working-age population x_2 by 1% (under fixed values of the other variables) raises the freight forward y by 1.1895 million rubles on average.

• If the indicator x_5/x_{20} is not smaller than 0.00675, organizations' accounts payable x_{20} will affect freight forward, whereas the number of pensioners x_5 will have no effect. For example, increasing organizations' accounts payable x_{20} by 1 million rubles (under fixed values of the other variables) reduces the freight forward y by 0.00013 million rubles on average. However, if the indicator x_5/x_{20} is less than 0.00675, the number of pensioners x_5 will affect freight forward, whereas organizations' accounts payable x_{20} will have no effect. For example, increasing the number of pensioners x_5 by 1000 people (under fixed values of the other variables) reduces the freight forward y by 0.0196 million rubles on average.

• If the indicator x_8/x_{58} is not smaller than 0.117, the rail freight tariffs x_{58} will affect freight forward, whereas the number of private cars x_8 per 1000 people will have no effect. For example, increasing the rail freight tariffs x_{58} by 1 c.u. (under fixed values of the other variables) reduces the freight forward y by 0.0038 million rubles on average. However, if the indicator x_8/x_{58} is less than 0.117, the number of private cars x_8 per 1000 people will affect freight forward, whereas the rail freight tariffs x_{58} will have no effect. For example, increasing the number of private cars x_8 per 1000 people by 1 (under fixed values of the other variables) reduces the freight forward y by 0.0323 million rubles on average.

• If the indicator x_3/x_{22} is not smaller than 23.08, the labor force x_3 will affect freight forward, whereas the electricity production x_{22} will have no effect. For example, increasing the labor force x_3 by 1 thousand people (under fixed values of the other variables) raises the freight forward y by 0.0254 million rubles on average. However, if the indicator x_3/x_{22} is less than 23.08, the electricity production x_{22} will affect freight forward, whereas the labor force x_3 will have no effect. For example, increasing the electricity production x_{22} by 1 billion kWh (under fixed values of the other variables) raises the freight forward y by 0.5857 million rubles on average.

Thus, the interpretative characteristics of the NLR are richer and more diverse than those of the traditional linear regression model. Moreover, depending on the chosen construction strategy, the approximation characteristics of the NLR should in most cases exceed the same characteristics of linear regressions, which are only a particular case of the NLR. The proposed NLR are valuable: besides forecasting, they extract new interpretable mathematical laws to improve the efficiency of managerial decisions in various sectors of the economy.

Also, note that the NLR better suits modeling under multicollinearity conditions than the traditional linear regression. The more binary operations the NLR has, the higher the number of its degrees of freedom will be as compared to the linear regression. This means that the NLR can "accommodate" more variables with fewer regressors than the linear regression. For example, the NLR (48) contains only 4 regressors but 8 variables, so the chance of its multicollinearity is a priori lower compared to a linear regression with all 8 variables.

CONCLUSIONS

This paper has considered the NLR with the binary minimum and maximum operations. We have proposed an NLR construction method based on solving a 0-1 MILPP. The solution of this problem yields the structural specification of the NLR and its approximate OLS estimates. As shown, the structural specification of the NLR is regulated through constraints on the binary variables. The contradictory variables have been eliminated at the initial stage to reduce the solution time of the problem and make the NLR quite interpretable. The proposed method has been applied to model rail freight in Irkutsk oblast; the resulting NLR has revealed new rail freight regularities not available within classical linear regression analysis.

The method proposed above is universal and can be used to construct NLRs in any subject area based on statistical data with positive variables only. The parameter partitioning procedure forms a 0-1 MILPP; for a sufficiently large number of partitions, its optimal solution gives estimates slightly differing from the optimal OLS estimates of the NLR. Naturally, increasing the number of partitions requires more time to solve the problem. Nevertheless, as demonstrated in [20, 21] on the linear regression example, such a 0-1 MILPP is solved an order of magnitude faster compared to standard enumeration procedures. The speed of constructing NLRs for different-size samples using the proposed method will be tested in subsequent publications.

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UPPER BOUNDS ON TRAJECTORY DEVIATIONS FOR AN AFFINE FAMILY OF DISCRETE-TIME SYSTEMS UNDER EXOGENOUS DISTURBANCES

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Abstract. We propose a simple upper bound on trajectory deviations for an affine family of discrete-time systems under nonzero initial conditions subjected to bounded exogenous disturbances. It involves the design of a parametric quadratic Lyapunov function for the system. The apparatus of linear matrix inequalities and the method of invariant ellipsoids are used as technical tools. The original problem is reduced to a parametric semidefinite programming problem, which is easily solved numerically. Numerical simulation results demonstrate the relatively low conservatism of the upper bound. This paper continues the series of our previous publications on estimating trajectory deviations for linear continuous- and discrete-time systems with parametric uncertainty and exogenous disturbances. The results presented below can be extended to various robust formulations of the original problem and also the problem of minimizing trajectory deviations for an affine family of discrete-time control systems under exogenous disturbances via linear feedback.

Keywords: linear discrete-time system, trajectory deviations, parametric Lyapunov function, bounded exogenous disturbances, linear matrix inequalities, invariant ellipsoids.

INTRODUCTION

When investigating transients in linear systems, the behavior of the entire system trajectory is of great interest. In this case, the maximum deviation of the trajectory from zero is a crucial characteristic of transients.

There exist different methods for estimating trajectory deviations for a dynamic system; for example, see a survey in the paper [1]. In particular, a regular approach was proposed therein to estimate the maximum deviation for a linear continuous-time system; also, an approach was developed to minimize trajectory deviations via a static linear state feedback law based on linear matrix inequalities (LMIs). The latter approach was extended in [2] to discrete-time systems with structured matrix uncertainty.

Another important and promising line of research in this area concerns the localization method for invariant compact sets. Here, we mention the works of Russian researchers, A.P. Krishchenko, A.N. Kanatnikov, and S.K. Korovin; for example, see the papers [3–6].

The case where uncertain parameters are matrix elements is not common in practice: usually, the matrix coefficients have no direct physical meaning and depend on the parameters in a more sophisticated way. Affine uncertainty is the simplest model of such a *dependent* uncertainty structure; see the monograph [7] for details.

Discrete-time systems with parametric uncertainty were studied in [8–10]. From a technical point of view, the cited works involved the approach from [11]. This approach allows separating the system matrix and the Lyapunov function matrix in a matrix inequality expressing a sufficient condition for the stability of the family under consideration. At the same time, less conservative estimates are obtained by designing a parametric quadratic Lyapunov function.

Also, note the publications [12, 13] devoted to a close topic.

In this paper, we continue the described line of research to derive upper bounds on trajectory deviations for an affine family of discrete-time systems with nonzero initial conditions subjected to bounded exogenous disturbances. The main technical tool is the apparatus of LMIs [14, 15].

The following notations are used below: $\|\cdot\|$ means the spectral norm of a matrix and the Euclidean norm of a vector; $^{\mathrm{T}}$ is the transpose symbol; I denotes an identity matrix of appropriate dimensions. All matrix inequalities are understood in the sense of sign definiteness of corresponding matrices.

1. PROBLEM STATEMENT AND SOLUTION APPROACH

Consider a linear discrete-time dynamic system described by

$$x_{k+1} = A(\alpha)x_k + Dw_k \tag{1}$$

with the state vector $x_k \in \mathbb{R}^n$, a *nonzero* initial condition x_0, and an exogenous disturbance $w_k \in \mathbb{R}^m$ satisfying the constraint

$$\|w_k\| \le 1, \quad k = 1, 2, \ldots \tag{2}$$

Here $D \in \mathbb{R}^{n\times m}$, and Schur matrices $A(\alpha) \in \mathbb{R}^{n\times n}$ belong to the convex family

$$\mathbb{A} = \left\{A(\alpha) : A(\alpha) = \sum_{i=1}^{N} \alpha_i A_i,\ \sum_{i=1}^{N} \alpha_i = 1,\ \alpha_i \ge 0\right\}. \tag{3}$$

As is well known, a sufficient condition for the robust quadratic stability of a linear system consists in the existence of a *common* quadratic Lyapunov function

$$V(x) = x^{\mathrm{T}} P^{-1} x, \quad P > 0.$$

According to [8–10], the approach based on designing a *parametric* quadratic Lyapunov function

$$V(x) = x^{\mathrm{T}} P^{-1}(\alpha) x, \quad P(\alpha) > 0,$$

yields significantly less conservative estimates. In addition, the following assertion was established in [8] for system (1)–(3).

Theorem 1. *Assume that there exist matrices* $0 < P_i = P_i^{\mathrm{T}} \in \mathbb{R}^{n\times n}$ *and* $G \in \mathbb{R}^{n\times n}$ *such that*

$$\begin{pmatrix} P_i & A_i G & D \\ G^{\mathrm{T}} A_i^{\mathrm{T}} & \mu(G + G^{\mathrm{T}} - P_i) & 0 \\ D^{\mathrm{T}} & 0 & (1-\mu)I \end{pmatrix} \ge 0, \quad i = 1, \ldots, N,$$

for some $0 < \mu < 1$.

Then system (1)–(3) *has a parametric quadratic Lyapunov function with the matrix*

$$P(\alpha) = \sum_{i=1}^{N} \alpha_i P_i.$$

This paper mainly aims at estimating from above trajectory deviations for the family (1) under the exogenous disturbance (2).

For a discrete-time system, the maximum deviation of the trajectory from zero in transients is given by

$$\xi^* = \max_{k=1,2,\ldots} \max_{\|x_0\|=1} \|x_k\|.$$

Estimation of the value ξ^* is very difficult [1], but the method of invariant ellipsoids with the technique of LMIs yields simple upper bounds on this value.

Recall the following well-known result. A matrix $P > 0$ of a quadratic Lyapunov function for some dynamic system defines the so-called *invariant* ellipsoid

$$\mathcal{E} = \left\{x \in \mathbb{R}^n : \quad x^{\mathrm{T}} P^{-1} x \le 1\right\}, \quad P > 0.$$

In other words, a system trajectory starting at any point of the invariant ellipsoid will remain there. Hence, *for any* initial condition from the ball $\mathcal{B} = \{x \in \mathbb{R}^n : \|x\| \le 1\}$ contained in the ellipsoid, we have the upper bound

$$\|x_k\| \le \lambda_{\max}(P) = \sqrt{\|P\|}$$

for any time instant.

In view of this fact, our aim is to find a *minimum* invariant ellipsoid associated with the matrix $P = P(\alpha)$ of the parametric quadratic Lyapunov function for the family under consideration.

Since

$$\|P(\alpha)\| = \left\|\sum_{i=1}^{N} \alpha_i P_i\right\| \le \sum_{i=1}^{N} \alpha_i \|P_i\| \le \sum_{i=1}^{N} \alpha_i \max_i \|P_i\| \le \max_i \|P_i\|,$$

within the proposed approach we will minimize the upper bound on the major semiaxis of the invariant ellipsoid with the matrix $P(\alpha)$, i.e., the value

$$\max_i \|P_i\|.$$

Further, the condition $\mathcal{B} \subseteq \mathcal{E}$ is equivalent to the requirement

$$P(\alpha) \ge I$$

and is ensured by $P_i \ge I$, $i = 1, \ldots, N$. Indeed,

$$P(\alpha) = \sum_{i=1}^{N} \alpha_i P_i \ge \sum_{i=1}^{N} \alpha_i I = I.$$

According to Schur's complement lemma, the matrix inequality

$$\begin{pmatrix} P_i & A_i G & D \\ G^{\mathrm{T}} A_i^{\mathrm{T}} & \mu\left(G+G^{\mathrm{T}}-P_i\right) & 0 \\ D^{\mathrm{T}} & 0 & (1-\mu) I \end{pmatrix} \geq 0$$

can be equivalently written as

$$\begin{pmatrix} P_i-\frac{1}{1-\mu} D D^{\mathrm{T}} & A_i G \\ G^{\mathrm{T}} A_i^{\mathrm{T}} & \mu\left(G+G^{\mathrm{T}}-P_i\right) \end{pmatrix} \geq 0 .$$

Due to the upper bound

$$\left\|x_k\right\| \leq \sqrt{\|P(\alpha)\|} \leq \max_{i=1, \ldots, N} \sqrt{\left\|P_i\right\|},$$

we therefore arrive at the following assertion.

Theorem 2. *Let* P_i^*, i = 1, ..., N, *be the solution of the convex optimization problem*

$$\min \max_{i=1,\ldots,N} \left\|P_i\right\|$$

subject to the constraints

$$\begin{pmatrix} P_i-\frac{1}{1-\mu} D D^{\mathrm{T}} & A_i G \\ G^{\mathrm{T}} A_i^{\mathrm{T}} & \mu\left(G+G^{\mathrm{T}}-P_i\right) \end{pmatrix} \geq 0 ,$$

$$P_i \geq I,\ i = 1,\ldots,N,$$

with respect to the matrix variables $P_i = P_i^{\mathrm{T}} \in \mathbb{R}^{n\times n}$ and $G \in \mathbb{R}^{n\times n}$ *and the scalar parameter* $0 < \mu < 1$, *where the matrices* A_i, D *are given by*(1), (3).

Then the solutions of system (1) *under all admissible exogenous disturbances* (2) *have the upper bound*

$$\left\|x_k\right\| \leq \max_{i=1,\ldots,N} \sqrt{\left\|P_i^*\right\|} .$$

The optimization problem in Theorem 2 is a parametric semidefinite programming problem. It can be easily solved numerically through one-dimensional optimization by varying the parameter μ within the range (0, 1). In particular, the CVX package [16] can be effectively used in MATLAB.

2. AN EXAMPLE

Consider the system from [8] in a slightly modified form:

$$A_1=\begin{pmatrix} 0.0061 & -0.2630 & 0.2748 \\ 0.1266 & 0.1242 & -0.3029 \\ -0.5100 & 0.4678 & -0.9712 \end{pmatrix},$$

$$A_2=\begin{pmatrix} 0.1330 & 0.2009 & 0.1672 \\ 0.1224 & -0.5987 & 0.3100 \\ -0.5235 & 0.0297 & -0.4784 \end{pmatrix},$$

$$A_3=\begin{pmatrix} -0.2733 & -0.1868 & -0.0077 \\ -0.0253 & -0.2828 & 0.6112 \\ -0.2412 & -0.0844 & -0.8024 \end{pmatrix},$$

$$D=\begin{pmatrix} -0.4 \\ -0.5 \\ 0.2 \end{pmatrix}.$$

Solving the one-dimensional optimization problem in Theorem 2 yields (for μ = 0.873) the matrices

$$P_1^*=\begin{pmatrix} 4.0127 & 0.4418 & -2.1495 \\ 0.4418 & 4.3296 & 0.6922 \\ -2.1495 & 0.6922 & 2.8445 \end{pmatrix},$$

$$P_2^*=\begin{pmatrix} 2.7515 & 0.7640 & -1.2374 \\ 0.7640 & 4.8052 & -0.1782 \\ -1.2374 & -0.1782 & 2.2964 \end{pmatrix},$$

$$P_3^*=\begin{pmatrix} 2.2435 & 1.9108 & -0.7729 \\ 1.9108 & 3.9362 & -1.1877 \\ -0.7729 & -1.1877 & 1.4804 \end{pmatrix}$$

of the parametric Lyapunov function and the matrix

$$G^*=\begin{pmatrix} 2.7991 & -0.0704 & -1.4510 \\ 2.2860 & 5.1487 & -0.6981 \\ -1.0359 & 0.2141 & 1.9738 \end{pmatrix}.$$

Hence,

$$\sqrt{\left\|P_1^*\right\|}=2.3791,\ \sqrt{\left\|P_2^*\right\|}=2.2774,\ \sqrt{\left\|P_3^*\right\|}=2.3791,$$

and finally we have the upper bound

$$\left\|x_k\right\| \leq \max\left\{\sqrt{\left\|P_1^*\right\|}, \sqrt{\left\|P_2^*\right\|}, \sqrt{\left\|P_3^*\right\|}\right\}=\sqrt{\left\|P_1^*\right\|}=2.3791 .$$

For comparison, the common quadratic Lyapunov function for this system, found according to [5], has the matrix

$$P_{comm}^*=\begin{pmatrix} 27.1113 & -9.3697 & -23.8293 \\ -9.3697 & 76.0285 & 6.4098 \\ -23.8293 & 6.4098 & 47.9982 \end{pmatrix},$$

yielding more than triple the rough estimate:

$$\left\|x_k\right\| \leq \sqrt{\left\|P_{comm}^*\right\|}=9.0666 .$$

Figure 1 shows the projections of the ellipsoids with the matrices P_1^*, P_2^*, and P_3^* (the thin solid lines) and the invariant ellipsoid with the matrix P_{comm} (the thick dashed line) on the plane (x_2, x_3); the dotted line corresponds to the projection of a unit sphere.

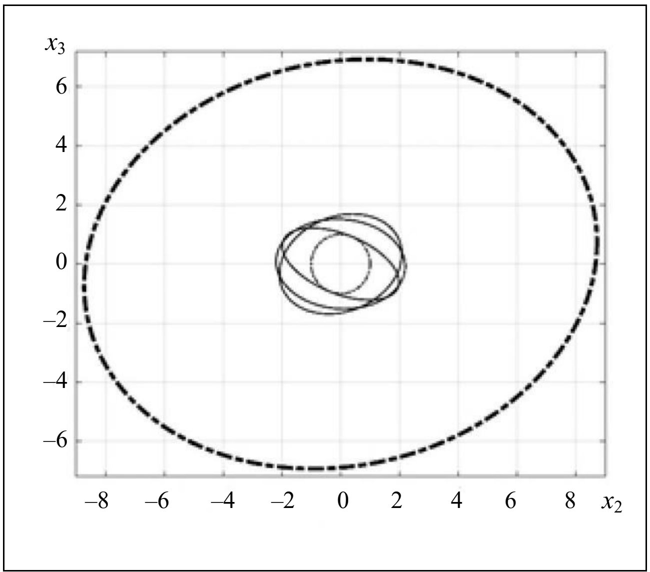

Fig. 1. The projection of ellipsoids on the plane (x_2, x_3).

Figure 2 shows the central part of Fig. 1 and the projection of the system trajectory under the initial condition

$$x_0 = \begin{pmatrix} 0 \\ 0.1391 \\ -0.9903 \end{pmatrix}, \ \|x_0\| = 1,$$

and the admissible exogenous disturbance

$$w_k = \mathrm{sign}(\sin(k/4)\cos(k/7)), \quad k = 1, 2, ...$$

(the dotted line).

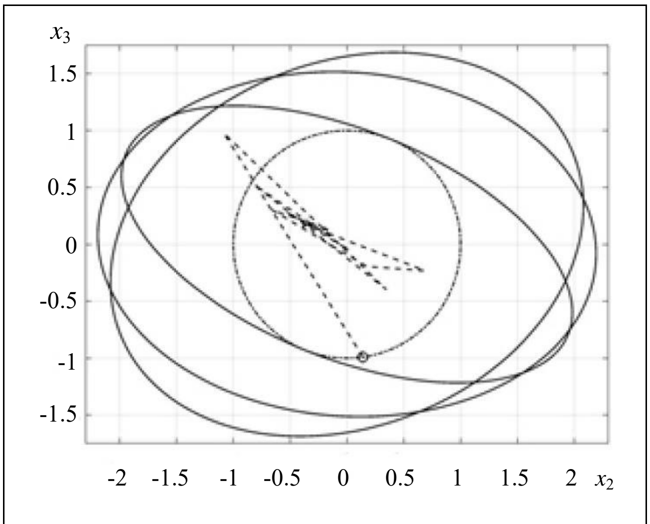

Fig. 2. The projection of ellipsoids and system trajectory on the plane (x_2, x_3).

Fig. 3 shows the dynamics of the value $\|x_k\|$ (the solid line) and its upper bound (the dashed line).

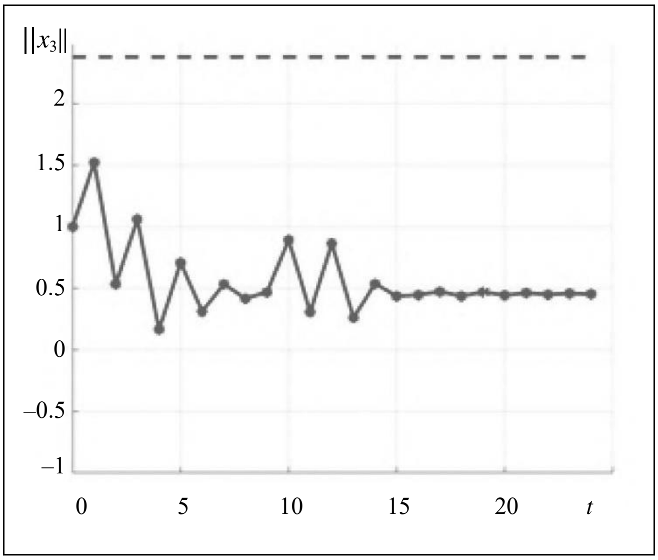

Fig. 3. Dynamics of $\|x_k\|$ and its upper bound.

3. CONCLUSIONS

This paper has presented a simple upper bound on a crucial characteristic of transients—the maximum trajectory deviation from zero—for an affine family of discrete-time systems with nonzero initial conditions subjected to bounded exogenous disturbances. Developing our previous research works, the estimation approach proposed above involves the design of a parametric quadratic Lyapunov function for the system under consideration. The apparatus of linear matrix inequalities and the method of invariant ellipsoids are used as technical tools. The original problem has been reduced to a parametric semidefinite programming problem, which is easily solved numerically, particularly in MATLAB using the CVX package.

We expect to extend these results to various robust formulations of the original problem and the problem of minimizing trajectory deviations for an affine family of discrete-time control systems under exogenous disturbances via linear feedback.

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A MATHEMATICAL MODEL OF MECHANICAL PENETRATION RATE WITH THREE CONTROL PARAMETERS TO OPTIMIZE OIL AND GAS WELL DRILLING

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Abstract. The types of rock destruction at the bottom hole under different loads on the drilling bit are considered, and well-known domestic and foreign models of the penetration rate are analyzed. As shown, they have no optima as power-type functions, being unsuitable for drilling optimization. In addition, they can be used for quick drilling control by adjusting only one parameter (the load on the bit). A mathematical model based on a sinusoid curve is constructed. This model allows the simultaneous control of three drilling mode parameters, namely, the axial load on the bit, its rotation frequency, and the mud flow rate for flushing the well. The adequacy of the model to the drilling process is verified, and its software implementation is performed. This model automatically recognizes the rock at the bottom hole during drilling, adapts to it, and calculates the optimal control parameters for destructing the traversed rock. The model is intended for an intelligent optimal adaptive control system for oil and gas well drilling.

Keywords: analysis of mathematical models of drilling rate, the optimum of a function, a model with three control parameters, optimal adaptive control, adequacy of the model.

INTRODUCTION

The main drilling process in well construction is the mechanical destruction of the rock with a bit at the bottom hole. This process is described by the equation of the mechanical penetration rate υ_m. Numerous factors affect the penetration rate; among them, note the load and torque on the bit, bit rotation frequency, mud flow rate and pressure, the rheological properties of the mud, and the lithological characteristics of the rock at the bottom hole.

On a large array of field and experimental drilling data, M.G. Bingham (the USA) studied in detail the function $\upsilon_m = f(\overline{G})$, where $\overline{G}$ is the specific axial load on the bit [1, 2] (the load reduced to the bottom hole area $S_{bot} = \pi D_{bit}^2/4$, where D_{bit} is bit diameter). As he concluded, this function is of power type, unimodal, and has the form of an S-shaped curve (Fig. 1). The qualitative relation of the function with physical and

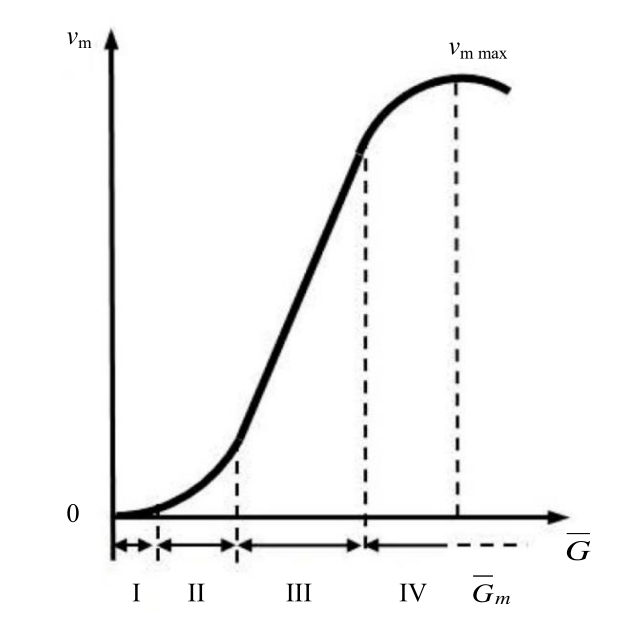


Fig. 1. Bingham's *S*-curve.

mechanical properties of the rock formation and parameters of flushing fluid was also established by Bingham. Domestic and foreign drilling practice confirms his conclusions; see [2–9] and other publications.

According to Bingham, the penetration rate function υ_{m} has several zones:

- zone I, where axial loads are low, the rock is destructed insignificantly (surface abrasion), and bit teeth pressure on the rock is smaller than its strength limit;
- zone II, where the contact pressure of bit teeth on the bottom hole increases and small pieces of the rock break off, causing a considerable increase in the penetration rate υ_{m} according to a nonlinear power-type law;
- zone III, where the load G exceeds the rock strength, causing the significant volumetric destruction of the rock according to an almost-linear law with a slope much greater than in zone I;
- zone IV, where the flushing fluid does not carry the drilled rock to the surface in due time; the cuttings are deposited on the bottom hole and are remilled. In addition, the penetration rate υ_{m} achieves maximum at the axial load G_m and then decreases.

The mathematical model of the drilling rate should reliably reflect these rock destruction zones and have an optimum for calculating the optimal values of the mode parameters. It is also important to determine the model's control parameters.

1. DRILLING MODELS

Many mathematical models of the penetration rate have been developed to describe the rock destruction process, both in Russia and abroad; see [1–3, 5–8, 10–12]. The basic (and typical) models and curves for the mechanical penetration rate are combined in Table 1.

Table 1

Mathematical models and graphs of the mechanical drilling rate

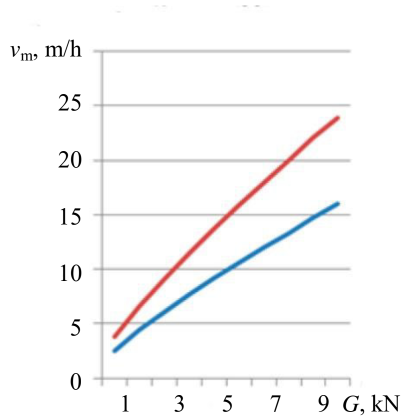
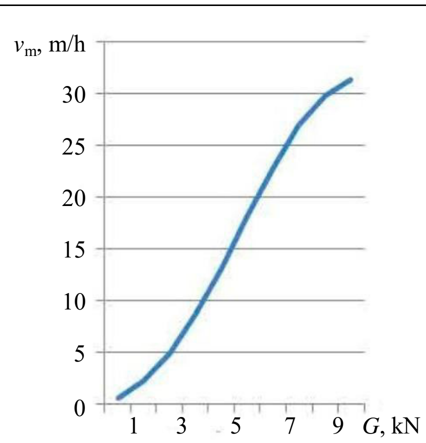
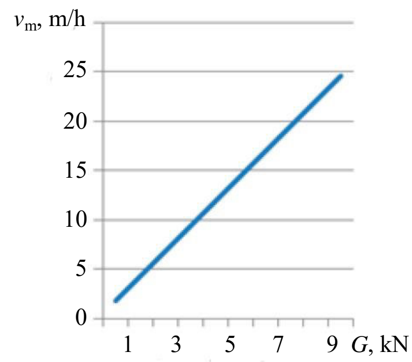
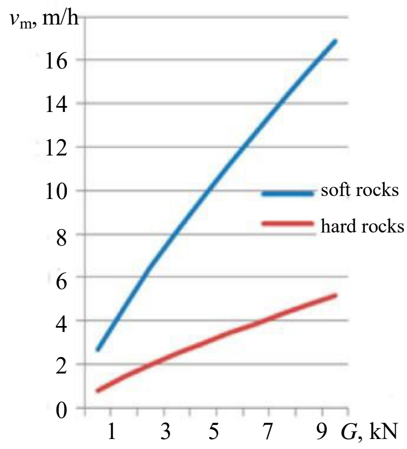
Models	Graphs
The VNIIBT model [6, 7]: $\upsilon_{\mathrm{m}} = k_{\mathrm{dril}}\, G^{\delta} n^{\alpha}$, where k_{dril} is the rock drillability coefficient, which depends on bit type and rock and mud properties; n is the bit rotation frequency, rpm; finally, δ and α are the slope parameters of the curve.	
The Pogarskii model [5–7]: $\upsilon_{\mathrm{m}} = \dfrac{a\, n^{\alpha} \overline{G}^{2}}{1 + b^{4} \overline{G}^{4}}$, where a is the drillability coefficient; α is an index affecting the curve steepness; finally, b is a coefficient depending on the mud flow rate.	

Table 1 (continued)

The Tenneco Oil Company model (the USA) [2, 7]: $$\upsilon_m = \frac{k_{dril}\,(G - G_0)\,n^{\alpha}}{f(h)},$$ where G_0 is the critical load on the bit for the teeth to penetrate into the rock; $f(h)$ is a function characterizing the bit condition.	
The Galle–Woods–Lubinski model (the USA) [2, 7]: $$\frac{dh}{dt} = k_{dril}\frac{\overline{G}^{\beta}\,r}{[a(D_3)]^b},$$ where β is the power index at the axial load; r is a function depending on the bit rotation frequency n, calculated separately for rocks of different hardness; $a(D_3)$ is the wear function of the bit armament; D_3 is the relative wear degree of the bit armament; finally, b is the power index for the function $a(D_3)$.	

According to analysis results, these models describe bit operation with different accuracy mainly within the linear zone III of Bingham's curve and have no maximum. Therefore, they are unsuitable for optimization. Moreover, in drilling practice, penetration rate control based on these models often adjusts the axial load G only: the parameters n and Q remain fixed during the trip. As a result, drilling modes are not optimal.

The contribution of the axial load G to the penetration rate reaches 43%; for the bit rotation frequency n and the mud flow rate Q, the corresponding figures are up to 14% and 7%, respectively [13]. Hence, they should be considered when calculating the optimal drilling parameters.

2. A DRILLING MODEL WITH THREE PARAMETERS

As a regression equation, Bingham's curve $\upsilon_m = f(G)$ can be represented as a fragment of a sinusoid shifted to quadrant I of the coordinate plane (Fig. 2):

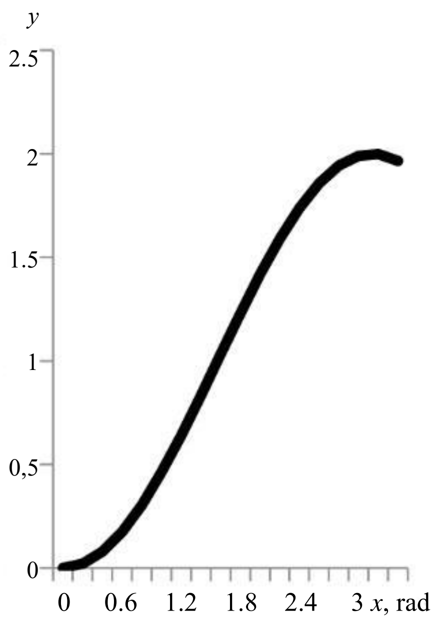


Fig. 2. A fragment of sin(x - π/2) + 1 shifted to quadrant I.

the graph should be raised by one on the y axis and shifted to the right by 1.57 rad on the x axis.

In drilling modes, depending on rig power and well depth, the axial load G on the bit varies from 0 to 40 N, and the drilling rate may reach 10–14 m/h and higher [4–8, 10–12, 14]. To match the argument x with the load G and the function y with the real drilling rate υ_m, we have to rescale them by introducing appropriate coefficients into the equation:

– the constant C_G to convert the radian measure of the argument x into the units of the load G, N;

– the proportionality factor k_υ to scale the function y vertically.

As a result, the dependence $\upsilon_m = f(G)$ takes the form

$$\upsilon_m = k_G \sin(C_G G - 1.57) + 1.$$

According to [3, 10–12] and others, the functions $\upsilon_m = f(n)$ with G = const and Q = const and $\upsilon_m = f(Q)$ with G = const and n = const are also unimodal. Their graphs are presented in Fig. 3 and can also be approximated by sinusoid fragments.

The curve $\upsilon_m = f(n)$ is described by the equation $\upsilon_m = k_n \sin(C_n n)$, and the curve $\upsilon_m = f(Q)$ by the equation $\upsilon_m = k_Q (\sin(C_Q Q - 0.7) + 0.645)$, where the values k_n, k_Q, C_n, and C_Q have the same meaning as k_G and C_G for the curve of the load on the bit. The operating ranges for wells are as follows: the bit rotation frequency, from 10 to 120 rpm; the mud flow rate, from 20 to 80 l/s [4–8, 10–12, 14].

The full mathematical model of the drilling rate $\upsilon_m = f(G, n, Q)$ as a function of the three parameters for optimal control of the drilling process has the form of their product:

$$\upsilon_m = k_\upsilon (\sin(C_G G - 1.57) + 1) \times \sin(C_n n) \times (\sin(C_Q Q - 0.7) + 0.645), \qquad (1)$$

where $k_\upsilon = k_G k_n k_Q$ is the total coefficient of the curve shape, equivalent to the drillability coefficient of the rock traversed by the bit at the bottom hole.

We verified the reliability of this model and its adequacy to real drilling conditions using drilling report data for completed wells in the Krasnodar region: Vostochno-Pribrezhnaya no. 9, Peschanaya no. 7, and Krupskaya no. 1 (wells nos. 1–3 in Fig. 4, respectively). The average deviations of the experimental data from the data based on model (1) were 12%, 13%, and 23%, respectively, which is a good outcome: the wells were drilled according to the drilling project documentation (not in optimal modes). The closest-to-optimal results were obtained for Vostochno-Pribrezhnaya no. 9 (well no. 1).

The graphs of the function (1) and its components and the drilling data for the three wells are shown in Fig. 4.

To plot the four-dimensional function $\upsilon_m = f(G, n, Q)$ on the two-dimensional coordinate plane, we represented the argument x in Fig. 4 in relative units, with $x = G$ for the function $\upsilon_m = f(G)$, $x = n/6$ for the function $\upsilon_m = f(n)$, and $x = Q/4$ for the function $\upsilon_m = f(Q)$.

As a result, the following conclusions can be made.

- The data obtained from the drilled wells confirm that the drilling model (1) accurately enough, with average errors of 12–23%, describes the mechanical destruction of rocks. Note that the wells were drilled on the parameter values recommended by the projects, which are compiled according to the results of the

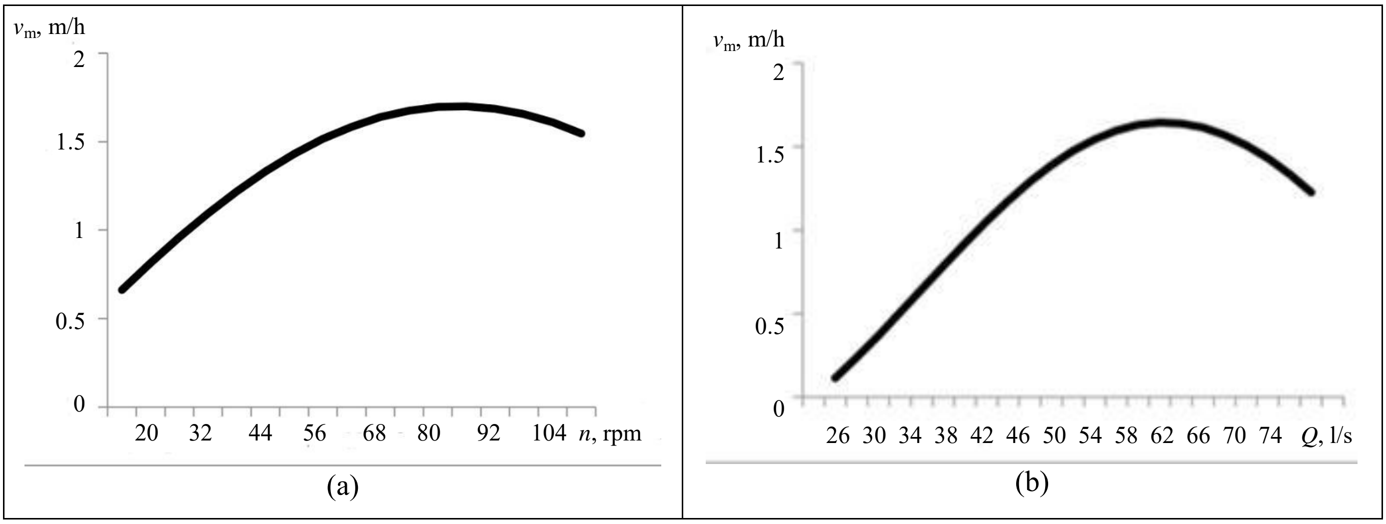


Fig. 3. The graphs of functions: (a) $\upsilon_m = f(n)$ and (b) $\upsilon_m = f(Q)$.

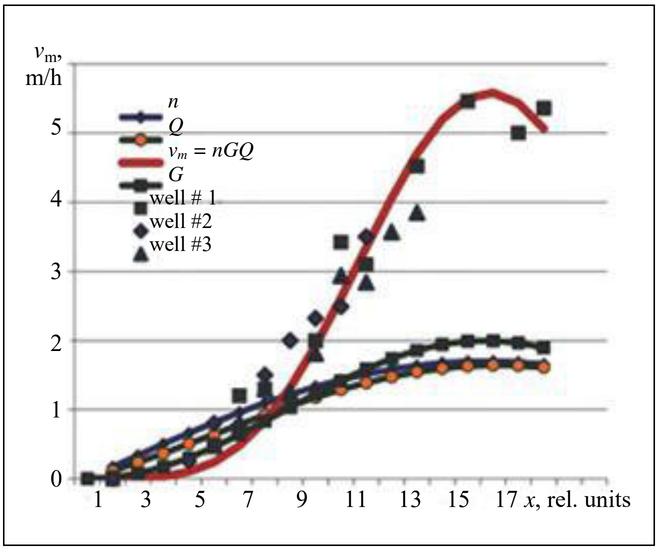


Fig. 4. The graphs of functions $\upsilon_m = f(G)$, $\upsilon_m = f(n)$, $\upsilon_m = f(Q)$, and $\upsilon_m = f(n, G, Q)$.

neighboring wells. For a new well, they are practically not optimal.

- During drilling, the optimal modes were achieved only at some depth intervals, mainly for well no. 1.
- The experimental and model-based data confirm that Bingham's curve is *S*-shaped.

3. ASSESSING THE ADEQUACY OF THE MODEL

As recommended in [15], generally accepted statistical criteria should be used for assessing the adequacy and quality of mathematical models and quickly estimating their main parameters. These recommendations were developed for transport networks. However, statistical criteria are universal and can be applied to models of any processes and objects.

Following the recommendations [15], we employed five criteria to assess the models:

– the absolute mean error δ_a,
– the relative mean error δ_p,
– the standard deviation ϑ_a,
– the relative standard deviation ϑ_p,
– the coefficient of correlation r.

At present, there are no precise values of these criteria under which a model is considered reliable. For applications, however, the relative criteria should not exceed 10%, and the coefficient of correlation should not be smaller than 0.9; for details, see [15].

The values of the adequacy criteria for model (1) are presented in Table 2.

According to the results, the model correlates well with real drilling processes and is suitable for optimal well control; the model's coefficient of correlation with drilling data is close to 1.

Table 2

Values of the adequacy criteria

Criterion	Well no. 1	Well no. 7	Well no. 9
δ_a	0.33	0.16	0.38
δ_p	10.01%	9.08%	18.40%
ϑ_a	2.02	0.93	1.21
ϑ_p	6.13%	5.59%	5.85%
r	0.98	0.74	0.89

4. PARAMETER OPTIMIZATION

The model was tested using the method of Bryansk partisans [16], i.e., an intelligent global optimization method for functions of several variables. This method includes two stages as follows. At the first stage (reconnaissance), the domain of the function is divided in half for each argument, and up to 30 agents are randomly initialized in each zone; then, the optimum of each zone is found, and the zone with the best optimum is selected. At the second stage (diversion), up to 500 agents are initialized in the selected zone, their optima are calculated, and the best optimum of the function is selected. We developed a Python program for optimum search and launched it with the following parameters: the number of partitions at the first stage, from 1 to 4; the number of reconnaissance agents, from 10 to 50; the number of diversion agents, from 200 to 500. The numerical results coincide; see Fig. 5 for one scenario of calculating the maximum drilling rate.

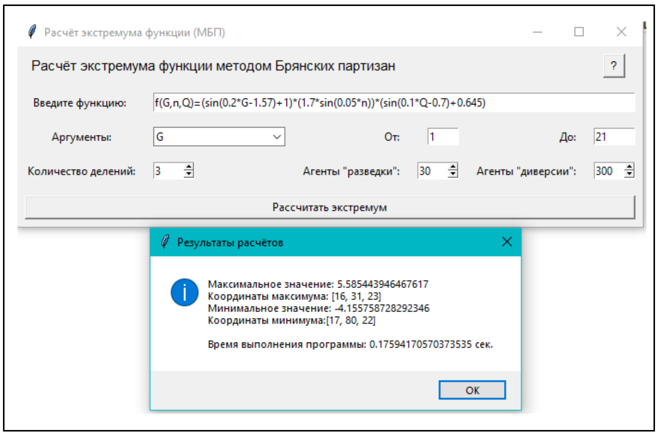


Fig. 5. The interface of the optimum search program.

The maximum mechanical penetration rate $\upsilon_{m\ max}$ = 5.58 m/h is achieved for G_{opt} = 16 N, n_{opt} = 31 rpm, and Q_{opt} = 23 l/s, which corresponds to the real parameters of drilling process control.

5. AN ADAPTIVE DRILLING METHOD

Model (1) is intended for the adaptive procedure of optimal drilling control.

The paper [17] described a methodology for adapting computer systems to exogenous impacts (intrusions) through their classification. It includes five modules (stages): input data (impact) processing, input data transformation (autocoding), searching for analogs in the database, classification, and feedback (developing the system response to the exogenous impacts). For the drilling process, this adaptation principle was modified as follows:

– With a chosen step of the penetration interval (e.g., every 0.3 m), the current values of the drilling parameters G, n, and Q and the resulting penetration rate υ_m are entered into the model.

– The model coefficients k and C are recalculated for the current values of G, n, Q, and υ_m. Therefore, the model is adapted to the rock at the bottom hole. The model automatically recognizes the type of rock traversed by the bit.

– The optimal values of the parameters G_{opt}, n_{opt}, and Q_{opt} are calculated on the adapted model. (The optimality criterion is $\upsilon_m = \max$.)

– The parameter values G_{opt}, n_{opt}, and Q_{opt} are set on the oil rig, and the next interval of 0.3 m is executed in the optimal mode.

This cycle (entering the new values of G, n, Q, and υ_m; recognizing the rock; adapting the model to it; calculating the optimal parameters; drilling in the optimal mode) is repeated until the well depth is reached, or the bit is worn. The described procedure has an obvious advantage: there is no need to identify the rock drilled at the bottom hole with the one in the lithological database of the well and classify it. (Note that the rock is not necessarily included in the database.)

CONCLUSIONS

As shown by the analysis, the widespread drilling models mainly involve the linear zone of Bingham's curve, adjust only one control parameter, have no optimum, and therefore are not suitable for optimization.

The new drilling model based on the sinusoidal curve allows the simultaneous optimal control of three drilling parameters (the load on the bit, the bit rotation frequency, and the mud flow rate) and has a common optimum for them. Moreover, the reliability of this model has been confirmed by the practical results obtained on the drilled wells: the model's coefficient of correlation with the drilling data is close to 1.

The optimal parameters calculated using the optimum search program have confirmed the suitability of the model for the optimal control of oil and gas well drilling.

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A LOGICAL-LINGUISTIC ROUTING METHOD FOR UNMANNED VEHICLES WITH THE MINIMUM PROBABILITY OF ACCIDENTS[1]

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Abstract. Forming optimal motion control laws for unmanned vehicles (UVs) by analyzing sensory data about the choice environment is an integral part of designing their situational control systems. The weakly predictable variability of the UV operating environment and the imperfection of measuring means reduce the possibility of obtaining comprehensive information about the environment state. Therefore, routing to minimize travel time and the probability of an accident is performed under uncertainty. An effective way to solve this problem is using logical-probabilistic and logical-linguistic models and algorithms. This paper is intended to develop new optimal routing methods for UVs with estimating the probability of an accident based on the logical-linguistic classification of route segments. For this purpose, the rows of parameters and characteristics of reference route segments are created and compared with the logical-probabilistic and logical-linguistic parameters and characteristics of classified route segments considering their significance for routing. After processing sensory and statistical data, the proposed logical-probabilistic and logical-linguistic methods are used to estimate the probabilities of accidents and minimize a performance criterion. As a consequence, the accuracy and speed of optimal routing for UVs are both increased. The results of this research can be used in the central nervous system of intelligent robots to classify route segments obtained by analyzing sensory and statistical data, which will improve the quality of motion control in an uncertain environment.

Keywords: optimization, control laws, the probability of an accident, sensory and statistical data, the attributes of reference route segments, logical-probabilistic and logical-linguistic analysis and classification.

INTRODUCTION

The development of unmanned vehicles (UVs), including unmanned aerial vehicles (UAVs), has recently become in high demand [1, 2]. The R&D works on UVs are determined by the following key problems:

1) extending the duration of autonomous operation;

2) improving navigation systems;

3) increasing payload;

4) raising the degree of autonomy based on artificial intelligence.

The fourth problem has recently been associated mainly with using neural network technologies [3–7].

Among their significant drawbacks, note the controversial problems of choosing a sufficient learning sample without overtraining the neural network and the problem of covering as many choice situations in decision-making as possible [8]. In addition, when forming control principles and algorithms, researchers and engineers consider information security problems for UVs [9] but often neglect motion safety issues of optimal routing [10, 11]. However, the prevention of accidents is the main operating principle of motion control systems for UVs and other robotic devices capable of moving in an automatic mode [12]. To implement this principle, it is necessary to develop algo-

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rithms for estimating the probability of an accident on a route and select the safest route under the existing constraints. Moreover, when solving the motion control problem, it is necessary to consider additional complexities due to the coordination of all motion participants: each participant must satisfy the corresponding kinematic equations and the existing state-space constraints, including dynamic constraints [13, 14] to minimize the probability of collision and related risks.

Risk assessments are predictive in nature since their uncertainty is associated with many factors that cannot be accurately estimated. The uncertainty of predictable risks causes situations reducing the probability of UV's accident-free motion along a route.

Qualitative and quantitative methods [15–17] are used to assess risks under uncertainty. The qualitative approach consists in determining all possible types of accident risks on a route and identifying their areas of occurrence and sources [18]. Further, this approach can serve for obtaining quantitative risk assessments. The quantitative approach allows calculating the value of individual risks on route segments and the entire route [19, 20]. Note that methods of probability theory and mathematical statistics are often used. In this case, it is necessary to study scenarios that simulate and analyze the simultaneous consistent change of all factors on route segments considering their interdependence. The conditions of implementing UV control algorithms are described by an expert through scenarios (e.g., pessimistic, optimistic, and most probable ones) or a system of constraints on the main parameters of the route and the corresponding indicators characterizing the probability of an accident.

This approach involves expert assessments obtained by complex procedures [21], starting with the selection of the number and qualification levels of experts. The results of the multi-step procedure are processed by statistics and qualitative analysis methods. Regression and correlation analysis tools are used for comprehensive risk analysis, and methods of the logical-probabilistic approach are employed for detailing and analyzing structurally complex routes [22].

In risk prediction under limited statistical data, it is reasonable to create a database of reference route segments that contains their qualitative attributes and quantitative expert assessments (the values of their membership functions and the values of their significance factors), as proposed in the logical-linguistic classification [23]. Within the scenario approach, which uses fuzzy set methods to calculate the values of the membership functions, it is then possible to rank the set of admissible routes by comparing a given route with the reference routes from the database [24]. In this case, the probability of an accident on reference route segments can be estimated by simulating UV's motion under uncertainty [25] and the available statistical data. Simulation modeling generates hundreds of possible accident combinations. After analyzing the simulation results and statistical data, it is possible to obtain distributions of the probabilities of accidents on reference route segments and give an integral assessment of the control efficiency and intelligence level of the UV [26] after optimal routing. In particular, this approach has been applied to determine the probabilities of accidents on reference route segments when forming the reference database in the proposed logical-linguistic method. The problem is to develop a method for an automatic control system (ACS) to select an optimal route of the vehicle that moves under uncertainty using logical-linguistic classification of route segments to certain reference models with the risk assessments or probabilities of accidents determined previously.

1. THE ROUTES RANKING PROBLEM

When searching for the best combinations of UV's motion control laws, the common problem is to find an optimal control minimizing the performance criterion

$$J_i = k_T T_i + k_R P_i,$$

where: $T_i = t_{if} - t_{i0}$ is the time to transfer the ith UV ($i=1,2,...$) located at the time instant t_0 in an initial point s_i of a bounded space $L^3 \subset E^3$ to a target point f_i of this space by the time instant t_f; E^3 denotes the three-dimensional Euclidean space; k_T is the significance factor of the goal achievement rate, adjusted by an expert or a group of experts; P_i is the estimated probability of an accident involving the ith UV while moving along the route during the time T_i; finally, k_P is the significance factor of the estimated probability of an accident, also adjusted by an expert or a group of experts.

In the proposed ACS, it is first necessary to determine UV's travel time on all possible routes. Under the existing logical-probabilistic, logical-linguistic, and other constraints, to calculate the product $J(R_v) = k_T T_i$ on each UV's route R_v, the ACS should evaluate the functional [3]

$$J\left(R_v\right) = k_T\left(\sum_{i,j} a\frac{L_{ij}}{V_{ij}} + \sum_{i,j} b\frac{\Psi_{ij}}{W_{ij}} + \sum_{i,j} c\tau_{i,j}\right), \quad (1)$$

where: a, b, and c are preference coefficients; V_{ij} and W_{ij} are the linear and angular velocities, respectively, which depend on the environment (e.g., air humidity and temperature); τ_{ij} is the delay time at an intersection depending on its type and load; Ψ_{ij} are the turning angles at an intersection; finally, L_{ij} are the lengths of segments between intersections.

As shown in [23], (i,j) is an element of the ordered set describing a given route from the starting point to the terminal point.

After evaluating the functional (1) for all possible UV's routes from point s_i to point f_i, the routes can be ranked by the time of arrival to point f_i. However, the fastest route may also turn out to be the most accident-prone. Therefore, the next step in optimal routing should be ranking of the routes R_v by the probability of an accident, $P_i(R_v)$.

2. THE DATABASE OF REFERENCE ROUTE SEGMENTS

Within the proposed method, when determining the probability of accidents on UV's routes and the product $k_R P_i$, we apply a logical-linguistic classification algorithm of route segments, which attributes a given route segment to a reference one. As shown in [23], this algorithm has high speed and efficiency. For implementing the algorithm, a database of reference route segments is created when developing the ACS for the UV. This database contains rows with the parameters (attributes) of reference route segments and the probabilities of an accident on such segments, determined in advance based on simulation modeling and statistical data. The presence of an attribute is indicated by one and its absence by zero.

Each route contains one or several segments at an intersection and one or several segments between intersections. Therefore, the database includes rows characterizing motion at an intersection and between intersections. Tables 1–8 show an example of reference rows from the database.

2.1. The Database of Reference Rows for Intersections

Table 1

Intersections

Database row	Type of intersection, direction of motion	Probability of accident
C_1 = /10000000000/	┤ with passage to the right	$P_{C1} = 0.12$
C_2 = /01000000000/	┤ with passage to the left	$P_{C2} = 0.15$
C_3 = /00100000000/	┬ with passage straight	$P_{C3} = 0.13$
C_4 = /00010000000/	┬ with passage to the right	$P_{C4} = 0.11$
C_5 = /00001000000/	┴ with passage straight	$P_{C5} = 0.14$
C_6 = /00000100000/	┴ with passage to the left	$P_{C6} = 0.17$
C_7 = /00000010000/	┼ with passage straight	$P_{C7} = 0.18$
C_8 = /00000001000/	┼ with passage to the right	$P_{C8} = 0.16$
C_9 = /00000000100/	┼ with passage to the left	$P_{C9} = 0.20$
C_{10} = /00000000010/	L with passage to the left	$P_{C10} = 0.10$
C_{11} = /00000000001/	Γ with passage to the right	$P_{C11} = 0.09$

Table 2

Turning angles

Database row	Angle and direction of turn	Probability of accident
Ψ_1 = /100000000/	-180° (left)	$P_{\Psi1} = 0.11$
Ψ_2 = /010000000/	-135° (left)	$P_{\Psi2} = 0.12$
Ψ_3 = /001000000/	-90° (left)	$P_{\Psi3} = 0.13$
Ψ_4 = /000100000/	-45° (left)	$P_{\Psi4} = 0.14$
Ψ_5 = /000010000/	0° (straight)	$P_{\Psi5} = 0.06$
Ψ_6 = /000001000/	$+45^\circ$ (right)	$P_{\Psi6} = 0.10$
Ψ_7 = /000000100/	$+90^\circ$ (right)	$P_{\Psi7} = 0.09$
Ψ_8 = /000000010/	$+135^\circ$ (right)	$P_{\Psi8} = 0.08$
Ψ_9 = /000000001/	$+180^\circ$ (right)	$P_{\Psi9} = 0.07$

Table 3

Angular velocities

Database row	Angular velocity, deg/s	Probability of accident
W_1 = /1000/	2	$P_{W1} = 0.10$
W_2 = /0100/	4	$P_{W2} = 0.11$
W_3 = /0010/	6	$P_{W3} = 0.12$
W_4 = /0001/	8	$P_{W4} = 0.13$

Table 4

Number of lanes

Database row	Number of lanes	Probability of accident
S_1 = /1000/	1	$P_{S1} = 0.10$
S_2 = /0100/	2	$P_{S2} = 0.12$
S_3 = /0010/	3	$P_{S3} = 0.13$
S_4 = /0001/	4	$P_{S4} = 0.14$

2.2 The Database of Reference Rows for Route Segments between Intersections

Table 5

Linear velocities

Database row	Linear velocity, m/s	Probability of accident
V_1 = /1000/	5	$P_{V1} = 0.10$
V_2 = /0100/	10	$P_{V2} = 0.11$
V_3 = /0010/	15	$P_{V3} = 0.12$
V_4 = /0001/	20	$P_{V4} = 0.13$

Table 6

Number of lanes

Database row	Number of lanes	Probability of accident
S_1 = /1000/	1	$P_{S1} = 0.10$
S_2 = /0100/	2	$P_{S2} = 0.12$
S_3 = /0010/	3	$P_{S3} = 0.13$
S_4 = /0001/	4	$P_{S4} = 0.14$

Table 7

Time of day

Database row	Time of day	Probability of accident
T_1 = /10000/	0 to 6 hours	$P_{T1} = 0.10$
T_2 = /01000/	6 to 10 hours	$P_{T2} = 0.13$
T_3 = /00100/	10 to 15 hours	$P_{T3} = 0.15$
T_4 = /00010/	15 to 20 hours	$P_{T4} = 0.14$
T_5 = /00001/	20 to 24 hours	$P_{T5} = 0.20$

Table 8

Route segment length

Database row	Route segment length	Probability of accident
L_1 = /10000/	very short, 200 m	$P_{L1} = 0.10$
L_2 = /01000/	short, 400 m	$P_{L2} = 0.12$
L_3 = /00100/	medium, 600 m	$P_{L3} = 0.13$
L_4 = /00010/	large, 800 m	$P_{L4} = 0.14$
L_5 = /00001/	very large, 1000 m	$P_{L5} = 0.15$

3. DETERMINING THE PROBABILITY OF AN ACCIDENT ON A ROUTE

To rank the routes R_v by the probability of an accident $P_i(R_v)$, the ACS of the UV first creates a list of intersections for each route. Next, for each list of intersections, the sensing system of the ACS determines the approximate values of their parameters corresponding to the attributes of the reference rows and fuzzifies these values to find the membership functions for the attributes of the corresponding reference rows. Then the ACS classifies the rows for intersections by comparing them with the reference rows from the database according to the algorithm described in [23]: it assigns values for the probabilities of accidents corresponding to the identified reference rows and calculates the probabilities of accidents at all intersections and the total probability of accidents at intersections along the entire route.

For example, a certain intersection is characterized by the following parameters (attributes): intersection $\perp$ with passage straight, 1 lane, turning angle 30º, and angular velocity 5.6 deg/s.

In this case, the row characterizing intersections has the form /00001000000/; classification using the logical-linguistic algorithm [23] attributes it to the reference row C_5 with the probability of an accident $P_{C5} = 0.14$. The row /1000/ characterizing the number of lanes is classified as the reference row S_1 with the probability of an accident $P_{S1} = 0.10$. After fuzzification, the row characterizing the turning angle takes the form /0 0 0 0 0 0 0.3 0.7 0 0 0 0/, being classified as the reference row Ψ_6 with the probability of an accident $P_{\Psi 6} = 0.10$. After fuzzification, the row characterizing the angular velocity takes the form /0 0 0.3 0.7 0/, being classified as the reference row W_3 with the probability of an accident $P_{W3} = 0.12$.

When passing an intersection, accidents are possible under one of the following events: C_i (i = 1, 2, ...), or Ψ_j (j = 1, 2,...), or W_q (q = 1, 2, ...), or S_g (g = 1, 2, ...). They correspond to the probabilities of an accident P_{Ci}, $P_{\Psi i}$, P_{Wq}, and P_{Sg}, respectively. According to the rules for calculating the probability of a logic function, the logic function $F_{1,2,\dots,n}$ in the Zhegalkin algebra [27] has the form

$$F_{1,2,\dots,n} \leftrightarrow f_1 ♀ f_2 ♀ f_3 ♀ \dots ♀ f_n,$$

where $f_1, f_2, f_3, \dots, f_n$ are logical functions or variables (events), ♀ denotes addition modulo 2, and ↔ denotes equivalence. According to the paper [24], the probability of an accident when passing such an intersection (n = 4) is given by

$$P = (-2)^0(P_{Ci} + P_{\Psi j} + P_{Wq} + P_{Sg}) + (-2)^1(P_{Ci}P_{\Psi j} + P_{Ci}P_{Wq} + P_{Ci}P_{Sg} + P_{\Psi j}P_{Wq} + P_{\Psi j}P_{Sg} + P_{Wq}P_{Sg}) + (-2)^2(P_{Ci}P_{\Psi j}P_{Wq} + P_{Ci}P_{\Psi j}P_{Sg} + P_{Ci}P_{Wq}P_{Sg} + P_{\Psi j}P_{Wq}P_{Sg}) + (-2)^3(P_{Ci}P_{\Psi j}P_{Wq}P_{Sg}). \quad (2)$$

For a large number of logical functions ($n > 8$), it is possible to calculate the probability approximately, being restricted to 8–10 row members; for details, see [24]. If there are N intersections on the route, fuzzification, classification, and formula (2) will be used to calculate the probabilities of an accident for each intersection; after that, formula (2) gives the probability of an accident P_N at all intersections of the route.

Then for each route, the ACS of the UV first creates a list of segments between intersections. Next, for each list of segments between intersections, the ACS determines the approximate values of their parameters corresponding to the attributes of the reference rows and fuzzifies these values to find the membership functions for the attributes of the corresponding reference rows. Then the ACS classifies the rows for segments between intersections by comparing them with the reference rows from the database according to the algorithm described in [24]: it assigns values for the probabilities of accidents corresponding to the identified reference rows and calculates the probabilities of accidents on segments between intersections and the total probability of accidents on all segments between intersections along the entire route.

For example, a certain segment between intersections is characterized by the following parameters (attributes): 1 lane, travel time 8 hours, linear velocity 12.7 m/s, length 500 m.

In this case, the row characterizing the number of lanes has the form /1000/, being classified as the reference row S_1 with the probability of an accident $P_{S1} = 0.10$. The row /01000/ characterizing travel time is classified as the reference row T_2 with the probability of an accident $P_{T2} = 0.13$. After fuzzification, the row characterizing the linear velocity takes the form /0 0.45 0.55 0/, being classified as the reference row V_3 with the probability of an accident $P_{V3} = 0.12$. After fuzzification, the row characterizing the segment length takes the form /0 0.5 0.5 0 /, being equally classified as the reference row L_2 with the probability of an accident $P_{L2} = 0.12$ or reference row L_3 with the probability of an accident $P_{L3} = 0.13$. Therefore, the probability of an accident due to the length of the segment between intersections can be estimated by the average value $(P_{L2} + P_{L3})/2 = 0.125$.

When passing a segment between intersections, accidents are possible under one of the following events: T_i ($i = 1, 2, ...$), or V_j ($j = 1, 2,...$), or L_q ($q = 1, 2, ...$), or S_g ($g = 1, 2, ...$). They correspond to the probabilities of an accident P_{Ti}, P_{Vi}, P_{Lq}, and P_{Sg}, respectively. According to [12], the probability of an accident on such a segment ($n = 4$) is given by

$$P = (-2)^0(P_{Ti} + P_{Vj} + P_{Lq} + P_{Sg}) + (-2)^1(P_{Ti}P_{Vj} + P_{Ti}P_{Lq} + P_{Ti}P_{Sg} + P_{Vj}P_{Lq} + P_{Vj}P_{Sg} + P_{Lq}P_{Sg}) + (-2)^2(P_{Ti}P_{Vj}P_{Lq} + P_{Ti}P_{Vj}P_{Sg} + P_{Ti}P_{Lq}P_{Sg} + P_{Vj}P_{Lq}P_{Sg}) + (-2)^3(P_{Ti}P_{Vj}P_{Lq}P_{Sg}). \quad (3)$$

If there are M segments between intersections on the route, fuzzification, classification, and formula (3) will be used to calculate the probabilities of an accident for each segment between intersections; after that, formula (3) gives the probability of an accident P_M on all segments between intersections of the route.

Finally, the probability of an accident on all routes R_v is calculated by the formula

$$P(R_v) = P_N(R_v) + P_M(R_v) - 2P_N(R_v)P_M(R_v).$$

4. RANKING AND OPTIMIZATION OF ROUTES

Due to the uncertain environment of the UV moving on a route, when calculating the performance criterion (1), it is necessary to consider the constraints in the form of logical and probabilistic modulo 2 equations [25]. As shown in [14], these constraints can be reduced to logical-interval ones. In this case, two values of the performance criterion (1), min $J(R_v)$ and max $J(R_v)$, are obtained for each route R_v. For the chosen values of the significance coefficient k_p, we calculate the two values below for each route to rank the routes R_v:

$$\min J_v = \{\min\{k_T J_T(R_v)\} + \min\{k_P P(R_v)\}\}; \quad (6)$$

$$\max J_v = \{\max\{k_T J_T(R_v)\} + \max\{k_P P(R_v)\}\}. \quad (7)$$

Usually, the values $\min\{k_P P(R_v)\}$ and $\max\{k_P P(R_v)\}$ coincide whereas $\min\{k_T J_T(R_v)\}$ and $\max\{k_P P(R_v)\}$ do not. Therefore, the ranking is performed by the minimum and maximum or the average value

$$J_v = 1/2(\max J_v + \min J_v).$$

The choice of an optimal route for the UV may depend on the opinion of an expert or a group of experts.

CONCLUSIONS

When selecting an optimal route for unmanned vehicles, it is necessary to minimize the probability of an accident. For this purpose, various algorithms are developed to assess accident risks at each route planning stage considering the “observed” area of the terrain.

Risk assessments are predictive in nature since their uncertainty is associated with many factors that cannot be accurately estimated. Therefore, when creating a database of reference route segments, the probabilities of an accident on such segments are determined at the ACS design stage based on simulation modeling and statistical data. Under limited statistical data, it is reasonable to predict accident risks using logical-linguistic and logical-probabilistic methods. For this purpose, databases of reference route segments are created, containing the qualitative attributes of segments and the probabilities of an accident obtained after modeling.

When the ACS of the UV determines the probability of an accident on a route, its sensory system obtains the quantitative values of attributes on route segments. After their fuzzification, the ACS finds the values of the membership functions for the specified attributes and creates rows similar to the reference rows of the database. For each route segment, the ACS identifies the closest reference row from the database and assigns to this segment the probability of an accident corresponding to the reference row. Using these probabilities of accidents on route segments, the ACS calculates the probability of accidents on the entire route using appropriate rules (calculating the probability of logical OR functions).

When selecting an optimal route, a trade-off between travel time and the probability of an accident must be observed by minimizing the following performance criterion: the sum of travel time and the probability of an accident, multiplied by given significance factors. These significance factors are adjusted by experts and entered into the ACS database at the formation stage. Usually, the performance criterion has an interval value, so the choice of an optimal route will depend on the expert's preferences.

Along with traditional approaches, the problems under consideration will require artificial intelligence technologies for determining the probabilities of accidents on reference segments. We emphasize that previously, optimal routing problems were considered without the probabilities of accidents.

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16TH INTERNATIONAL CONFERENCE ON STABILITY AND OSCILLATIONS OF NONLINEAR CONTROL SYSTEMS (PYATNITSKIY'S CONFERENCE)

The 16th International Conference on Stability and Oscillations of Nonlinear Control Systems (Pyatnitskiy's Conference) took place on June 1–3, 2022, at Trapeznikov Institute of Control Sciences, Russian Academy of Sciences (ICS RAS). The conference was organized by ICS RAS with the technical co-sponsorship of IEEE Russia Section. Chairman of the Organizing Committee was *V.N. Tkhai*, Chief Researcher of Laboratory No. 16 (Dynamics of Nonlinear Control Processes) named after E.S. Pyatnitskiy (ICS RAS).

The conference was devoted to presenting and discussing new results obtained by Russian and foreign researchers in the following areas:

- General problems of stability and stabilization;
- Nonlinear oscillations: general problems and methods;
- Lyapunov functions methods;
- Smooth and nonsmooth dynamics;
- Problems of controllability and observability;
- Robust control problems;
- Control in mechanical and electromechanical systems;
- Control in mechatronic systems and robotic control;
- Oscillations, stability, and stabilization in the network and coupled systems;
- Stability and control of hybrid and switched systems.

The event was held online using Russian video conferencing software. During the three days, there were 14 sessions, including 2 plenary ones. In total, 4 plenary talks and 137 section talks were delivered at the conference. The event was attended by researchers from Armenia, France, Germany, Kazakhstan, Kyrgyzstan, Russia, and Uzbekistan. Russian participants represented scientific organizations and universities from 19 cities.

At the first plenary session (June 1, 2022), two talks were presented. The first talk by *S. Dashkovskiy* (the Institute of Mathematics, the University of Wuerzburg, Germany) was devoted to the paradigm of input-to-state stability. Its origin as a natural extension of classical Lyapunov stability to systems with input was considered. Different applications of the paradigm were described, within which the theory of small-gain systems was developed. As noted by the author, the theory of input-to-state stability for finite-dimensional systems has now acquired a complete form. Recent results of this theory were surveyed, in particular, extensions to delayed systems, hybrid and switched systems, and infinite-dimensional systems. Finally, some open problems of the theory were outlined.

The second talk of that plenary session, entitled "Use of Feedback in Control Problems as an Optimization Problem," was presented by *B.T. Polyak* and *M.V. Khlebnikov* (ICS RAS and Moscow Institute of Physics and Technology). The talk dealt with an approach to linear control systems from an optimization point of view. In the classical linear-quadratic control problem, one can consider the linear feedback matrix as a variable and reduce the problem to minimizing a performance index by this variable. This approach goes back to the works of R. Kalman in the 1950s. In addition to the linear-quadratic control problem, the authors studied other problems from the same positions: the suppression of nonrandom bounded external disturbances by constructing a static linear output-feedback law and using a dynamic output-feedback law with an observer. For each of the three problems mentioned, a gradient method for finding the feedback law was described and justified. Several illustrative examples with simple and double pendulums were provided.

The second plenary session, held on June 2, 2022, also included two talks. The first one, entitled "Theory of Homogeneous Dynamical Systems and Their Application," was presented by *D. Efimov* from the National Institute for Research in Digital Science and Technology (INRIA, Lille, France). The author overviewed the theory of homogeneous dynamical systems and briefly described new results and applications for different classes of models, including those given by delayed or partial differential equations, and discrete-time systems. Also, connections of homogeneity with finite- or fixed-time convergence and input-to-state stability were explained.

In the second talk, "Attracting Cycle in a Controlled Mechanical System," *V.N. Tkhai* (ICS RAS) described the stabilization of oscillations of a controlled mechanical system by constructing an orbitally asymptotically stable cycle. He presented a general principle to stabilize arbitrary conservative systems through control actions based on self-oscillator signals. The control system is treated as a coupled system, and a Van der Pol oscillator is used as a generator. It acts on the mechanical system admitting a family of nondegenerate oscillations by one-way coupling control. The control system consists of electrical and mechanical parts with a mechatronic scheme to stabilize oscillations: the cycle is attracted in large.

The program of section sessions was made according to the scientific topics of the conference.

Sessions devoted to general problems of stability and stabilization concerned both theoretical problems and problems related to control, stability, and stabilization of particular objects, e.g., control problems in cerebral blood flow autoregulation modeling (*A.E. Golubev*, Ishlinsky Institute for Problems in Mechanics RAS), HIV infection models (*A.N. Kanatnikov*, Bauman Moscow State Technical University, and *O.S. Tkacheva*, ICS RAS), models of markets (*A.M. Kotyukov*, ICS RAS, and *N.G. Pavlova*, RUDN University), etc. Among theoretical problems, note the application of the theory of matrix inequalities to stability analysis (*V.A. Kamenetskiy*, ICS RAS), stability of periodic differential inclusions (*M.V. Morozov*, ICS RAS), stability and stabilization of systems with delay (*A.Yu. Aleksandrov*, St. Petersburg State University), state estimation of a continuous system by output discrete measurements (*A.I. Malikov*, Kazan National Research Technical University-Kazan Aviation Institute), and others.

Problems of control and stability of oscillations were considered at two section sessions, namely, (1) Nonlinear oscillations: general problems and (2) Oscillations, stability, and stabilization in the network and coupled systems. Among the objects of research were Hamiltonian systems at different resonances (*O.V. Kholostova*, Moscow Aviation Institute), self-oscillations of an aerodynamic pendulum (*D.V. Belyakov*, Moscow Aviation Institute), chaotic and periodic attractors (*I.M. Burkin*, Tula State University, and *N.V. Kuznetsov* and *T.N. Mokaev*, St. Petersburg State University), coupled conservative systems (*I.N. Barabanov* and *V.N. Tkhai*, ICS RAS), and multi-agent systems (*R.P. Agaev* and *D.K. Khomutov*, ICS RAS).

A large group of talks was devoted to control problems in mechanical, electromechanical, and mechatronic systems. The problems considered include control of string vibrations (*V.R. Barseghyan*, Yerevan State University, and *S.V. Solodusha*, Melentiev Energy Systems Institute SB RAS), stabilization of artificial Earth satellite rotations (*A.Yu. Aleksandrov* and *A.A. Tikhonov*, St. Petersburg State University), control of different manipulator robots (*Yu.F. Dolgui* and *I.A. Chupin*, Ural Federal University; *V.A.* Sobolev and *E.A. Shchepakina*, Federal Research Center "Computer Science and Control" RAS and Samara State University; *A.S. Andreev* and *O.A. Peregudova*, Ulyanovsk State University; and others). Also, optimal control problems for mechanical systems were considered, in particular, optimal damping of flexible rotor vibrations in electromagnetic bearings (*D.V. Balandin* and *R.S. Biryukov*, Lobachevsky State University of Nizhny Novgorod, and *M.M. Kogan*, Nizhny Novgorod State University of Architecture and Civil Engineering), the time-optimal movement of a platform with oscillators (*O.R. Kayumov*, Omsk State Pedagogical University), and others.

A separate session united talks on different control aspects for spacecraft and unmanned aerial vehicles. In particular, the following problems were discussed: control of a space robot-manipulator (*Ye.I. Somov*, *S.A. Butyrin*, and *S.E. Somov*, Samara State Technical University), attitude control of a geostationary satellite (*Ye.I. Somov*, *S.A. Butyrin*, and *T.E. Somova*, Samara State Technical University), control of space flight with a solar sail (*E.N. Polyakhova* and *V.S. Korolev*, St. Petersburg State University; *A.V. Rodnikov*, Moscow Aviation Institute; *D.V. Shimanchuk*, *A.S. Shmyrov*, and *V.A. Shmyrov*, St. Petersburg State University), and trajectory planning, stabilization, and attitude control for unmanned quadcopters (*V.A. Alexandrov* and *I.G. Rezkov*, ICS RAS; *A.I. Glushchenko* and *K.A. Lastochkin*, ICS RAS; *I.S. Trenev*, ICS RAS), etc.

The conference program and proceedings can be found on the conference website: https://stab22.ipu.ru/.

Note the high level of scientific discussion at the sessions as well as the high level of interest of the conference participants.

With the online format chosen by the Organizing Committee due to the epidemic situation, many organizational issues for the organizers and participants of the event were simplified. It was easier for researchers from far Russian cities (Blagoveshchensk, Khabarovsk, Irkutsk, Novosibirsk, etc.) and foreign countries to take part in the sessions. At the same time, in the final general discussion, it was declared that the improving epidemic situation would, hopefully, soon allow the traditional format of the conference to overcome the deficit of live communication between researchers.

The conference talks recommended by the Program Committee were published as extended papers in English; see Proceedings of 2022 16th International Conference on Stability and Oscillations of Nonlinear Control Systems (Pyatnitskiy's Conference) in the IEEE Xplore electronic library: https://ieeexplore.ieee.org/xpl/conhome/9807427/proceeding.

All conference proceedings were electronically published in Russian and are freely available on the conference website: https://stab22.ipu.ru/sites/default/files/news/Stab_2022_Rus%20%281%29.pdf.

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