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CONTENTS

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STATE ESTIMATION METHODS FOR FUZZY INTEGRAL MODELS. PART II: LEAST SQUARES METHOD AND DIRECT VARIATIONAL CALCULUS METHODS

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Abstract. This paper considers the least squares method (LSM) and its modifications for estimating the states of fuzzy integral models, namely, LSM with numerical integration, recurrent and nonlinear LSM, and fuzzy LSM, which is based on fuzzy rules for finding diagonal elements of the weight matrix in generalized LSM. Some examples of fuzzy systems of linear equations (FSLEs) arising in state estimation problems for fuzzy integral models are given and solved. The fuzzy Galerkin method is implemented for the approximate state estimation of a fuzzy integral model. This method leads to a fully FSLE. The emergence of "strong" and "weak" systems is explained using an illustrative example. Chebyshev quadrature methods and sinc functions for the approximate structural estimation of fuzzy integral models are considered. As noted in the paper, the same methodology can be applied to develop other algorithms for estimating fuzzy integral models based on the following methods: residuals, collocation, energy, Ritz, Courant, etc.

Keywords: fuzzy least squares method, fuzzy Galerkin method, fuzzy Chebyshev method, fuzzy sinc method.

INTRODUCTION

Part I of the survey—see [1]—mainly considered approximate state estimation methods for fuzzy integral models when an unknown fuzzy integrand is represented as a fuzzy polynomial with given crisp basis functions and fuzzy weight coefficients of the quadrature formula to be determined. This approach leads to solving fuzzy systems of linear equations (FSLEs) using Friedman's method of "embedding." However, other methods for solving FSLEs, equivalent to solving fuzzy integral equations, have become widespread in practice. Some of them [2–4] are described below.

In the paper [2], Friedman's method of embedding discussed in [1] was applied for a doubled FSLE when the desired fuzzy variables are calculated from a linear combination of the lower and upper unknown variables and the values yielded by embedding. This method was proposed by Ezzati [3].

Abbasbandy [4] suggested another method—a modification of the Ezzati method—often used for symmetric fuzzy membership functions of fuzzy variables defining the right-hand side of an FSLE.

The paper [5] introduced the fuzzy center method for solving a system of equations. This method is geometric and is used for symmetric and asymmetric triangular numbers on the right-hand side of an FSLE.

A characteristic feature of the methods mentioned above is that they apply to FSLEs of a relatively low dimension: as a rule, $\dim A \le 3$, where A denotes the crisp variables matrix of an FSLE, which corresponds to the dimension $\dim S = 2n \le 6$, where S is the extended matrix of the method of embedding and its modifications. Another feature of the methods is the appearance of "strong" and "weak" solutions; see the references [8] and [17] in part I of the survey [1]).

With an increase in the dimension $\dim S$ due to a higher estimation accuracy, traditional iterative methods [6, 7] for solving FSLEs with the matrix S are often adopted.

Most of these methods involve the QT factorization of the matrix S, where Q is a matrix of the diagonal elements of S and $T = Q - S$. The QT factorization leads to the methods of Richardson, Jacobi, Gauss–Seidel, relaxation, and their modifications. Another group of iterative methods is related to the HS_*S factorization of the matrix S, where H is an Hermitian matrix, S_* is the Skew matrix, and S is the splitting matrix. Here H is the arithmetic mean of the matrices S and S^{T}, whereas S_* is their averaged difference.

When estimating the states of traditional integral equations, an important place is occupied by direct variational calculus methods, such as the Galerkin method and quadrature formulas, associated with the numerical calculation of a proper integral. The same methods find application in the fuzzy case [8, 9].

The traditional Galerkin method is widely used in applied mathematics and modeling problems as one of the direct variational calculus methods [10], in the analysis of oscillatory processes in automatic optimization search systems [11], for the approximate solution of partial differential equations arising in the simulation of gyroscopes based on elastic waves in solids [12–14], for the solution of traditional integral equations [15], etc. At present, there are few publications on the modification of this method for fuzzy cases. As one example, we mention the paper [16], where the fuzzy Galerkin method was used to study fuzzy oscillatory processes in an automatic optimization search system. As it seems, the limited studies on using fuzzy methods, particularly for solving fuzzy integral equations, are due to relatively little research into developing fuzzy methods for applied problems. In what follows, we attempt to expand the capabilities of fuzzy methods to identifying fuzzy integral models.

When estimating the states of fuzzy integral models, the problem of solving fully FSLEs often arises. Some solutions of fully FSLEs will be considered in the section devoted to the fuzzy Galerkin method.

The unknown variables, together with the corresponding kernels, represent a certain integral. Therefore, it can be calculated using various numerical schemes (quadrature formulas) with fuzzy weight coefficients for crisp basis functions. The simplest fuzzy quadrature formulas are the fuzzy rules of rectangles, trapezoids, parabolas, etc. For the simplest quadrature formulas, partial approximation segments are usually used to reduce the error. The error also depends on the degree of the interpolation polynomial, the number and arrangement of the approximation segments, the use of various types of splines, and other factors. Further, some of these factors will be adopted for the approximate estimation of fuzzy integral models. These factors (methods) are a natural continuation of the ones presented in part I of the survey [1].

1. BASIC DEFINITIONS

By analogy with the book [10], we define a *fuzzy functional* J_{fuz} as a mapping $J_{\text{fuz}} : E \to R$, where E denotes the set of fuzzy functions.

For the set E, the definitions of fuzzy continuity and fuzzy differentiability at a fixed point or on a given interval are introduced. For this purpose, a fuzzy vector Banach space and its metric – the Hausdorff distance between its elements – are used.

A mapping $f_{\text{fuz}} : R \to E$ defines a fuzzy function $f_{\text{fuz}}(t),\ t \in R;\ E = \{r(t)\},\ r \in [0,\ 1] \subset R$, with the equivalent parametric representation $f_{\text{fuz}}(t) =$, $= f(t,\ r) = \left(\underline{f}(t,\ r),\ \overline{f}(t,\ r) \middle| r \in [0, 1]\right)$, where $r(\cdot)$ is a membership function.

A collection of fuzzy functions $\{f_{\text{fuz}\,i}(t)\}_{i=1}^{n}$ is *complete* if $f_{\text{fuz}\,n}(t) \to f_{\text{fuz}}(t)$, where the convergence $(\to)$ holds in the Hausdorff metric.

Fuzzy functions $x_{\text{fuz}\,i}(s) = x_i(s,\ r) = \left(\underline{x}_i(s,\ r),\right.$ $\left.\overline{x}_i(s,\ r) \middle| r \in [0, 1]\right)$ and $x_{\text{fuz}\,j}(s) = x_j(s,\ r) = \left(\underline{x}_j(s,\ r),\right.$ $\left.\overline{x}_j(s,\ r) \middle| r \in [0, 1]\right)$ are said to be *orthogonal* if

$$\int_a^b x_{\text{fuz}\,i}(s)\, x_{\text{fuz}\,j}(s)\, ds = \begin{cases} 0, & i \neq j, \\ 1, & i = j, \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \int_a^b \underline{x}_i(s,\ r)\, \underline{x}_j(s,\ r)\, ds, \\ \int_a^b \overline{x}_i(s,\ r)\, \overline{x}_j(s,\ r)\, ds, \end{cases} = \begin{cases} 0, & i \neq j, \\ 1, & i = j, \end{cases} \forall r \in [0, 1] \subset R.$$

Here the proper Riemann integral is interpreted in the fuzzy sense; see the definition in Section 1 of part I of the survey [1].

Other basic definitions employed below were introduced in the paper [6].

2. PROBLEM STATEMENT

There is a fuzzy integral model described by

$$x_{\text{fuz}}(s) = f_{\text{fuz}}(s) + \lambda\int_a^b K(s,\tau)x_{\text{fuz}}(\tau)d\tau. \quad (1)$$

It is required to consider numerical estimation schemes for its state using the least squares method (LSM) and its modifications, the fuzzy Galerkin method, and fuzzy quadrature formulas.

3. FUZZY ESTIMATION METHODS

3.1. Least squares method with numerical integration

This method was presented in the paper [17].

Consider the fuzzy integral model (1) in the parametric representation

$$\begin{cases} \underline{x}(s,r) = \underline{f}(s,r) + \lambda\int_a^b \underline{\mathrm{U}}(\tau,r)d\tau, \\ \overline{x}(s,r) = \overline{f}(s,r) + \lambda\int_a^b \overline{\mathrm{U}}(\tau,r)d\tau, \\ r\in[0,1]\subset R \text{ is a parameter,} \end{cases}$$

where

$$\underline{U}(\tau,r) = \begin{cases} K(s,\tau)\cdot\underline{x}(\tau,r), K(\cdot)\ge 0, \\ K(s,\tau)\cdot\overline{x}(\tau,r), K(\cdot)<0, \end{cases}$$

$$\overline{U}(\tau,r) = \begin{cases} K(s,\tau)\cdot\overline{x}(\tau,r), K(\cdot)\ge 0, \\ K(s,\tau)\cdot\underline{x}(\tau,r), K(\cdot)<0, \end{cases}$$

and $\lambda = 1$.

Assume that the kernel $K(s,\tau)$, where $\tau\in[a,b]$ is the interval of integration, satisfies the inequalities

$(i): K(s,\tau)\ge 0$ for $a\le\tau\le b$;

$(ii): K(s,\tau)\le 0$ for $a\le\tau\le b$;

$(iii): K(s,\tau)\ge 0$ for $a\le\tau\le c$,

and $K(s,\tau)<0$ for $c<\tau\le b$.

In case (i),

$$\underline{U}(\tau,r) = K(s,\tau)\underline{x}(\tau,r) \quad \text{and}$$

$$\overline{U}(\tau,r) = K(s,\tau)\overline{x}(\tau,r),\ \tau\in[a,b].$$

In case (ii),

$$\underline{U}(\tau,r) = K(s,\tau)\overline{x}(\tau,r) \quad \text{and}$$

$$\overline{U}(\tau,r) = K(s,\tau)\overline{x}(\tau,r),\ \tau\in[a,b].$$

In case (iii),

$$\underline{U}(\tau,s) = K(s,\tau)\underline{x}(\tau,r) \quad \text{for} \quad a\le\tau\le c, \quad \text{and}$$

$$\underline{U}(\tau,s) = K(s,\tau)\overline{x}(\tau,r) \quad \text{for} \quad c<\tau\le b,$$

$$\overline{U}(\tau,s) = K(s,\tau)\overline{x}(\tau,r) \quad \text{for} \quad a\le\tau\le c, \text{ and}$$

$$\overline{U}(\tau,s) = K(s,\tau)\underline{x}(\tau,r) \quad \text{for} \quad c<\tau\le b.$$

In case (i), we have:

$$x_{\text{fuz}}(s) = f_{\text{fuz}}(s) + \lambda\int_a^b K(s,\tau)x_{\text{fuz}}(\tau)d\tau \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \underline{x}(s,r) - \lambda\int_a^b K(s,\tau)\underline{x}(\tau,r)d\tau = \underline{f}(s,r), \\ \overline{x}(s,r) - \lambda\int_a^b K(s,\tau)\overline{x}(\tau,r)d\tau = \overline{f}(s,r), \\ r\in[0,1]\subset R. \end{cases}$$

The fuzzy solution $\underline{x}(s,r)$, $\overline{x}(s,r)$ is found in the form of the fuzzy approximate relation

$$\begin{aligned} \underline{x}(s,r) &\simeq \sum_{i=1}^n \underline{a}_i(r)\cdot h_i(s); \\ \overline{x}(s,r) &\simeq \sum_{i=1}^n \overline{a}_i(r)\cdot h_i(s), \end{aligned} \quad (2)$$

where $\{h_i(s)\}_{i=1}^n$ is a sequence of independent and complete functions; $\underline{a}_i(r)$ and $\overline{a}_i(r)$ are fuzzy variables to be determined.

Substituting formula (2) into the model (1), in case (i) we obtain

$$\begin{cases} \sum_{i=1}^n \underline{a}_i(r)h_i(s) - \lambda\int_a^b K(s,\tau)[\sum_{i=1}^n \underline{a}_i(r)h_i(\tau)]d\tau \simeq \underline{f}(s,r), \\ \sum_{i=1}^n \overline{a}_i(r)h_i(s) - \lambda\int_a^b K(s,\tau)[\sum_{i=1}^n \overline{a}_i(r)h_i(\tau)]d\tau \simeq \overline{f}(s,r). \end{cases}$$

Interchanging the symbols $\int$ and Σ yields

$$\begin{cases} \sum_{i=1}^n \underline{a}_i(r)h_i(s) - \sum_{i=1}^n \underline{a}_i(r)\times \\ \times(\lambda\int_a^b K(s,\tau)h_i(\tau)d\tau) \simeq \underline{f}(s,r), \\ \sum_{i=1}^n \overline{a}_i(r)h_i(s) - \sum_{i=1}^n \overline{a}_i(r)\times \\ \times\lambda\int_a^b K(s,\tau)h_i(\tau)d\tau) \simeq \overline{f}(s,r). \end{cases} \quad (3)$$

Using the notations

$$\begin{aligned} k_i(s) &= \lambda\int_a^b K(s,\tau)h_i(\tau)d\tau; \\ \ell_i(s) &= h_i(s) - k_i(s),\ i=\overline{1,n}, \end{aligned} \quad (4)$$

we write the system (3) as

$$\begin{cases} \sum_{i=1}^{n} \underline{a}_i(r)\cdot l_i(s) \simeq \underline{f}(s,r), \\ \sum_{i=1}^{n} \overline{a}_i(r)\cdot l_i(s) \simeq \overline{f}(s,r), \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \underline{f}(s,r) - \sum_{i=1}^{n} \underline{a}_i(r)\cdot l_i(s) = \underline{r}_n, \\ \overline{f}(s,r) - \sum_{i=1}^{n} \overline{a}_i(r)\cdot l_i(s) = \overline{r}_n, \end{cases} \quad (5)$$

where $\underline{r}_n$ and $\overline{r}_n$ are the approximation errors in the relation (2). The values $l_i(s)$, $i=\overline{1,n}$, in the expression (4) can be positive or negative. Multiplying the fuzzy variables $\underline{a}_i(r)$ and $\overline{a}_i(r)$ by the constant $l_i(s)\in R$, we arrive in

$$\underline{f}(s,r) - \sum_{i=1}^{n} b_i(r)\cdot l_i(s) = \underline{r}_n,$$

$$l_i(r) = \begin{cases} \underline{a}_i(r),\ l_i(s)\ge 0, \\ \overline{a}_i(r),\ l_i(s) < 0; \end{cases}$$

$$\overline{f}(s,r) - \sum_{i=1}^{n} c_i(r)\cdot l_i(s) = \overline{r}_n,$$

$$c_i(r) = \begin{cases} \overline{a}_i(r),\ l_i(s)\ge 0, \\ \underline{a}_i(r),\ l_i(s) < 0. \end{cases}$$

The unknown variables $b_i(r)$ and $c_i(r)$, $i=\overline{1,n}$, will be found using the traditional LSM. With the argument s written as $s=s_i$, $i=\overline{1,n}$, and the notations $l_i(s_j)=l_{ij}$, we obtain the following linear systems of equations for the variables $b_i(r)$ and $c_i(r)$:

$$\begin{cases} \underline{f}(s_j,r) - \sum_{i=1}^{n} b_i(r)\cdot l_{ij} = \underline{r}_j, \\ \overline{f}(s_j,r) - \sum_{i=1}^{n} c_i(r)\cdot l_{ij} = \overline{r}_j, \\ j=\overline{1,n}. \end{cases} \quad (6)$$

According to the LSM, the values of the variables $b_i(r)$ and $c_i(r)$, $i=\overline{1,n}$, are determined by minimizing the squared errors $\underline{r}_j^{\,2}$ and $\overline{r}_j^{\,2}$, respectively, on the interval of integration $[a,b]$ of the original equation (6):

$$\begin{cases} \min_{b_i} \int_a^b \underline{r}_j^{\,2}(s)ds, \\ \min_{c_i} \int_a^b \overline{r}_j^{\,2}(s)ds, \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \min_{b_i} \int_a^b [\underline{f}(s_j,r) - \sum_{i=1}^{n} b_i(r) l_{ij}]ds, \\ \min_{c_i} \int_a^b [\overline{f}(s_j,r) - \sum_{i=1}^{n} c_i(r) l_{ij}]ds, \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \frac{\partial}{\partial b_i}\left\{ \int_a^b [\underline{f}(s_j,r) - \sum_{i=1}^{n} b_i(r) l_{ij}]^2 ds \right\} = 0, \\ \frac{\partial}{\partial c_i}\left\{ \int_a^b [\overline{f}(s_j,r) - \sum_{i=1}^{n} c_i(r) l_{ij}] ds \right\} = 0, \end{cases} \Leftrightarrow \quad (7)$$

$$\Leftrightarrow \begin{cases} \sum_{i=1}^{n} b_i(l_i, l_j) = \sum_{i=1}^{n} (\underline{f}, l_j),\ j=\overline{1,n}, \\ \sum_{i=1}^{n} c_i(l_i, l_j) = \sum_{i=1}^{n} (\overline{f}, l_j),\ j=\overline{1,n}. \end{cases}$$

Here

$$(l_i,\ l_j) = \int_a^b l_i(s) l_j(s) ds;\ (\underline{f},\ l_j) = \int_a^b \underline{f}(s) l_j(s) ds;$$
$$(\overline{f}, l_j) = \int_a^b \overline{f}(s)\cdot l_j(s) ds,\ i,\ j=\overline{1,n}, \quad (8)$$

are scalar products.

The relation (7) has the matrix form

$$SA = Y,\ \det S \ne 0, \quad (9)$$

where

$$S = \begin{pmatrix} L & B \\ B & L \end{pmatrix},\quad L = (l_{ij});$$

$$A = (b_1,\dots,b_n \quad c_1,\dots,c_n)^{\mathrm{T}};$$

$$Y = (\underline{Y} \quad \overline{Y}),\quad \underline{Y} = (\underline{y}_1,\dots,\underline{y}_n)^{\mathrm{T}},$$

$$\overline{Y} = (\overline{y}_1,\dots,\overline{y}_n)^{\mathrm{T}},\ \underline{y}_i = (\overline{f}, l_i).$$

From the relation (9) we find the vector A^*:

$$A^* = S^{-1}Y,\ A^* = (b_1^*,\dots,b_n^* \quad c_1^*,\dots,c_n^*).$$

The approximate solution of the fuzzy equation (1) is given by (2) where the weight coefficients $\underline{a}_i$ and $\overline{a}_i$, $i=\overline{1,n}$, represent the components b_i^* and c_i^*, respectively, of the vector A^*:

$$\underline{x}_i^{\,*} \simeq \sum_{i=1}^{n} b_i^* h_i(s);\ \overline{x}_i^{\,*} \simeq \sum_{i=1}^{n} c_i^* h_i(s).$$

Note that in the system (7), we should first calculate proper integrals in scalar products and then use them to calculate the fuzzy vector A^*. For this purpose, let us apply a fuzzy quadrature formula – the fuzzy trapezoid rule. For the sake of definiteness, consider the following formulas in the system (7):

$$(\underline{f}, l_j) = \int_a^b \underline{f}(s) l_j(s) ds; \; (\overline{f}, l_j) = \int_a^b \overline{f}(s) l_j(s) ds. \quad (10)$$

In this case, the interval $[a,b]$ is partitioned into equal subintervals using the points s_i:

$$a = s_0 < s_1 < \ldots < s_{n-1} < s_n = b,$$

where

$$s_i = a + ih, \; s_i - s_{i-1} = (b-a)/h, \; i = \overline{1,n}.$$

Suppose that:

$$(\underline{f}, l_j) = \underline{\varphi_n}(r) = h[\underline{\varphi}(a, r) + \\ + \underline{\varphi}(b, r) + \sum_{i=1}^{n} \underline{\varphi}(s_i, r)],$$

$$(\overline{f}, l_j) = \overline{\varphi_n}(r) = h[\overline{\varphi}(a, r) + \\ + \overline{\varphi}(b, r) + \sum_{i=1}^{n} \overline{\varphi}(s_i, r)],$$

where $\underline{\varphi_n}(r)$ and $\overline{\varphi_n}(r)$ are the integrands in the expression (10). Then for any fixed $r \in [0, 1] \subset R$, we have

$$\lim_{n\to\infty} \underline{\varphi_n}(r) = \underline{F}(r) = \int_a^b \underline{\varphi}(s, r) ds \quad \text{and}$$

$$\lim_{n\to\infty} \overline{\varphi_n}(r) = \overline{F}(r) = \int_a^b \overline{\varphi}(s, r) ds.$$

The paper [17] proved a theorem on the uniform convergence of $\underline{\varphi_n}(s)$ and $\overline{\varphi_n}(s)$ to $\underline{F}(r)$ and $\overline{F}(r)$, respectively, in the Hausdorff metric. Case (*ii*) can be considered in a similar fashion. In case (*iii*), the interval of integration is partitioned into two corresponding subintervals.

Fuzzy spline interpolation [8] in the expressions (8) employs a similar fuzzy integration technique. For example, a fuzzy linear spline follows from the fuzzy variational problem

$$\min_{\varphi_{\text{fuz}}(s)} \int_{s_0=a}^{s_n=b} [\dot{\varphi}_{\text{fuz}}(s)]^2 ds, \; s_i \in [a, b], \; i = \overline{0, n},$$

with the boundary conditions $\varphi(s_0) = \varphi_{\text{fuz}\,0}$, $\varphi(s_n) = \varphi_{\text{fuz}\,n}$.

A fuzzy cubic spline follows from the fuzzy variational problem

$$\min_{\varphi_{\text{fuz}}(s)} \int_{s_0=a}^{s_n=b} [\ddot{\varphi}_{\text{fuz}}(s)]^2 ds$$

with given (fixed) fuzzy boundary conditions and the continuity of the function $\varphi_{\text{fuz}}(x)$ at the node points. Here $\dot{\varphi}_{\text{fuz}}(s)$ and $\ddot{\varphi}_{\text{fuz}}(s)$ are the Hukuhara derivatives [18, 19].

Example 1. Consider an integral equation of the form

$$x_{\text{fuz}}(s) = f_{\text{fuz}}(s) + \int_0^{2\pi} (0.1 \sin s \cdot \sin 0.5\tau) x_{\text{fuz}}(\tau) d\tau,$$

where

$$f_{\text{fuz}}(s) = f(s, r) = (\underline{f}(s, r), \\ \overline{f}(s, r) \mid r \in [0, 1]),$$

$$\underline{f}(s, r) = \sin(0.5s) \cdot [(13/15) \times \\ \times (r^2 + r) + (2/15) \cdot (4 - r^3 - r)],$$

$$\overline{f}(s, r) = \sin(0.5s) \cdot [(2/15) \times \\ \times (r^2 + r) + (13/15) \cdot (4 - r^3 - r)].$$

This equation describes the forced vibrations of a string of a given length $l = 2\pi$ under a fuzzy external disturbance $f_{\text{fuz}}(s)$. In the equation, $K(s, \tau) = \, = 0.1 \sin s \cdot \sin 0.5\tau$ is the string displacement at a point τ under the harmonic force applied at a point s. If $f_{\text{fuz}}(s) = 0$, this model represents the free vibrations of the string.

For solving this problem, let $h_1(s) = 1$ and $h_2(s) = s$.

From the expression (4) we find:

$$k_1(s) = \int_0^{2\pi} K(s, \tau) h_1(\tau) d\tau =$$

$$= \int_0^{2\pi} [(0.1 \sin s \cdot \sin 0.5\tau) \cdot 1] \, d\tau = 0.4 \sin s,$$

$$k_2(s) = \int_0^{2\pi} K(s, \tau) h_2(\tau) d\tau =$$

$$= \int_0^{2\pi} -[(0.1 \sin s \cdot \sin 0.5\tau) \cdot \tau] \, d\tau = 0.4\pi \sin s.$$

Calculations of the values $l_1(s)$ and $l_2(s)$ (see (4)) yield

$$l_1(s) = h_1 - k_1(s) = 1 - 0.4 \sin s \quad \text{and}$$

$$l_2(s) = h_2 - k_2(s) = s - 0.4\pi \sin s,$$

where

$$l_1(s) > 0 \text{ for } s \in [0, 2\pi],$$

and

$$l_2(s) = \begin{cases} l_2(s) = 0, \; s \in [0, \pm\pi/2], \\ l_2(s) > 0, \; s \in [\pm\pi/2, 2\pi]. \end{cases}$$

Using formulas (8), we determine the right-hand side of the system (7). Since $l_1(s) \ge 0, \forall s \in [0, 2\pi]$,

$$(\underline{f}, l_1) = \int_a^b (\underline{f}, l_1) ds = \int_0^{2\pi} \{(\sin 0.5s)[(13/15)(r^2 + r) + \\ + (2/15)(4 - r^3 - r)] \cdot [1 - 0.4 \sin s]\} ds =$$

$$= 4[(13/15)(r^2 + r) + (2/15)(4 - r^3 - r)],$$

$$(\overline{f}, l_1) = \int_a^b (\overline{f}, l_1) ds = 4[(2/15)(r^2 + r) +$$

$$+ (13/15)(4 - r^3 - r).$$

According to the fuzzy trapezoid rule,

$$(\underline{f}, l_2) = \int_a^b \underline{f} \cdot l_2 ds \simeq -0.3r^3 + 12.5r^2 + 12.5r + 0.1,$$

$$(\overline{f}, l_2) = \int_a^b \overline{f} \cdot l_2 ds \simeq -12.5r^3 + 0.03r^2 - 12.5r + 50.2.$$

The scalar products $(\ell_i, \ell_j),\ i,\ j = \overline{1, n}$, in the system (7) are given by (8):

$$(l_1, l_2) = \int_0^{2\pi} l_1(s) \cdot l_2(s) ds =$$

$$= \int_0^{2\pi} [(1 - 0.4\sin s) \cdot (1 - 0.4\sin s)] ds \simeq 6.8,$$

$$(l_1, l_2) = (l_2, l_1) = \int_0^{2\pi} l_1(s) \cdot l_2(s) ds =$$

$$= \int_0^{2\pi} [(1 - 0.4\sin s) \cdot (s - 0.4\pi\sin s)] ds \simeq 23.8,$$

$$(l_2, l_2) = \int_0^{2\pi} l_2(\mathrm{s}) \cdot l_2(s) ds =$$

$$= \int_0^{2\pi} [(s - 0.4\pi\sin s) \cdot (s - 0.4\pi\sin s)] ds \simeq 103.4.$$

As the result of these calculations, we arrive in the FSLE (9) with the unknown elements $b_i(r)$ and $c_i(r),\ i = 1, 2$:

$$\begin{pmatrix} \begin{matrix} 6.8 & 23.8 \\ 23.8 & 103.4 \end{matrix} & B \\ B & \begin{matrix} 6.8 & 23.8 \\ 23.8 & 103.4 \end{matrix} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} (\underline{f}, l_1) \\ (\underline{f}, l_2) \\ (\overline{f}, l_1) \\ (\overline{f}, l_2) \end{pmatrix}, \ |S| \neq 0.$$

The components of the vector $A^* = (\underline{a_1}^*, \underline{a_2}^*\ \ \overline{a_1}^*, \overline{a_2}^*)^{\mathrm{T}}$ are found using the system (6). Hence, the approximate solution of the fuzzy integral equation takes the form

$$\underline{x}^*(s, r) \simeq \underline{a_1}^* h_1(s) + \underline{a_2}^* h_2(s) = \underline{a_1}^* \cdot 1 + \underline{a_2}^* s,$$

$$\overline{x}^*(s, r) \simeq \overline{a_1}^* h_1(s) + \overline{a_2}^* h_2(s) = \overline{a_1}^* \cdot 1 + \overline{a_2}^* s.$$

For example, choosing $r = 0.5$ and $s = \pi$, we obtain the following solution: $\underline{x}^*(s, r) \simeq 1.1,\quad \overline{x}^*(s, r) \simeq 3.$ ♦

3.2. Recurrent and nonlinear least square methods

These methods were presented in [13, 14].

The method described in subsection 3.1 is applied for small data arrays: usually, $s_i = 1, 2.$ In the case $s_i,\ i > 2$, the recurrent LSM is preferable.

Let the relation (9) on k and $(k+1)$ measurements have the standard form:

$$S_k A_k = Y_k,\ S_{k+1} A_{k+1} = Y_{k+1} \Leftrightarrow L_k B_k = X_k{}^{\mathrm{T}} V_k, \quad (11)$$

where

$$L_k = X_k{}^{\mathrm{T}} X_k,\ X_k{}^{\mathrm{T}} V_k = Y_k,\ X_k = (l_i(s_j)),\ i,\ j = \overline{1, n}.$$

Then the recurrent LSM is to find the dependence $\Phi: B_{k+1} = \Phi(B_k)$.

The elements of the relation (11) have the block representations

$$V_{k+1} = \begin{pmatrix} u_1, ..., u_k & u_{k+1} \end{pmatrix}^{\mathrm{T}} = \begin{pmatrix} V_k & u_{k+1} \end{pmatrix}^{\mathrm{T}};$$

$$X_{k+1} = \begin{pmatrix} X_k \\ X_{k+1}^{\mathrm{T}} \end{pmatrix},\ X_k = (l_i(s_j)),\ i,\ j = \overline{1, k};$$

$$x_{k+1}^{\mathrm{T}} = (l_1(s_{k+1}), \ldots, l_{k+1}(s_{k+1})).$$

According to the multiplication rule of block matrices [20],

$$\begin{pmatrix} a & b \end{pmatrix} \cdot \begin{pmatrix} c \\ d \end{pmatrix} = ac + bd\ .$$

Hence, the right-hand side of (11) can be written as

$$\begin{gathered} V_{k+1}^{\mathrm{T}} \cdot V_{k+1} = \begin{pmatrix} X_k \\ x_{k+1}^{\mathrm{T}} \end{pmatrix} \cdot \begin{pmatrix} V_k \\ u_{k+1} \end{pmatrix} = \\ = \begin{pmatrix} X_k^{\mathrm{T}} & x_{k+1} \end{pmatrix} \cdot \begin{pmatrix} V_k \\ u_{k+1} \end{pmatrix} = \begin{pmatrix} X_k^{\mathrm{T}} V_k + x_{k+1} u_{k+1} \end{pmatrix}. \end{gathered} \quad (12)$$

Substituting the expression (12) into formula (11) yields

$$\begin{gathered} B_{k+1} = M_{k+1}(X_k^{\mathrm{T}} V_k + x_{k+1} u_{k+1}), \\ M_{k+1} = L_{k+1}^{-1} \Leftrightarrow M_{k+1}^{-1} = L_{k+1}. \end{gathered} \quad (13)$$

Performing trivial transformations for M_{k+1}^{-1}, we obtain:

$$M_{k+1}^{-1} = [(X_{k+1}^{\mathrm{T}}, X_{k+1})^{-1}]^{-1} = X_{k+1}^{\mathrm{T}},$$

$$X_{k+1} = \begin{pmatrix} X_k \\ x_{k+1}^T \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} X_k \\ x_{k+1}^T \end{pmatrix} =$$

$$= \begin{pmatrix} X_k^{\mathrm{T}} & x_{k+1} \end{pmatrix} \begin{pmatrix} X_k \\ x_{k+1}^{\mathrm{T}} \end{pmatrix} = X_k^{\mathrm{T}} X_k + x_{k+1} x_k^{\mathrm{T}} =$$

$$= M_k^{-1} + x_{k+1} I x_{k+1}^{\mathrm{T}},$$

where *I* denotes an identity matrix needed for auxiliary purposes.

As a result,

$$M_{k+1}^{-1} = M_k^{-1} + x_{k+1} I x_{k+1}^{\mathrm{T}} \Leftrightarrow$$
$$\Leftrightarrow L_{k+1} = (M_k^{-1} + x_{k+1} I x_{k+1}^{\mathrm{T}})^{-1}.$$

For inverting the matrix on the right-hand side of this expression, we use the well-known formula from matrix algebra [20]:

$$A^{-1} = (B + CDG)^{-1} =$$
$$= B^{-1} - B^{-1}C(D^{-1} + GB^{-1}C)^{-1}GB^{-1}.$$

Consequently,

$$L_{k+1} = M_{k+1}^{-1} = (M_k^{-1} + x_{k+1} I x_{k+1}^{\mathrm{T}})^{-1} =$$
$$= (M_k^{-1})^{-1} - (M_k^{-1})^{-1} x_{k+1} \cdot [I^{-1} + x_{k+1}^{\mathrm{T}} (M_k^{-1})^{-1}]^{-1} \times$$
$$\times x_{k+1}^{\mathrm{T}} (M_k^{-1})^{-1} =$$
$$= M_k - M_k x_{k+1} (I + x_{k+1}^{\mathrm{T}} M_k x_{k+1})^{-1} \cdot x_{k+1}^{\mathrm{T}} \cdot M_k.$$

The parenthesized element has the dimension $\dim(\cdot) = (1 \times 1)$. Therefore,

$$(I + x_{k+1}^{\mathrm{T}} M_k x_{k+1})^{-1} = \gamma_k \in R,$$

where

$$\gamma_k = (1 + \alpha_k)^{-1}.$$

Hence, the expression (13) takes the form

$$M_{k+1} = M_k - \gamma_k M_k x_{k+1} x_{k+1}^{\mathrm{T}} M_k. \qquad (14)$$

Substituting the expressions (14) and (12) into formula (11) and making some transformations, we get:

$$B_{k+1} = L_{k+1}^{-1} X_{k+1}^{\mathrm{T}} V_{k+1} \Leftrightarrow B_{k+1} = M_{k+1} X_{k+1}^{\mathrm{T}} V_{k+1} =$$
$$= (M_k - \gamma_k M_k x_{k+1} x_{k+1}^{\mathrm{T}} M_k)(X_k^{\mathrm{T}} V_k + x_{k+1} u_{k+1}) =$$
$$= M_k X_k^{\mathrm{T}} V_k + M_k x_{k+1} u_{k+1} -$$
$$- \gamma_k M_k x_{k+1} x_{k+1}^{\mathrm{T}} M_k X_k^{\mathrm{T}} V_{k+1} -$$
$$- \gamma_k M_k x_{k+1} x_{k+1}^{\mathrm{T}} M_k x_{k+1} u_{k+1} = B_k + \gamma_k M_k x_{k+1} \times$$
$$\times [\gamma_k^{-1} u_{k+1} - \alpha_k u_{k+1}] - \gamma_k M_k x_{k+1} x_{k+1}^{\mathrm{T}} B_k =$$
$$= B_k + \gamma_k M_k x_{k+1} (u_{k+1} - x_{k+1}^{\mathrm{T}} B_k).$$

Thus,

$$B_{k+1} = \Phi(B_k) = B_k + \gamma_k M_k x_{k+1} (u_{k+1} - x_{k+1}^{\mathrm{T}} B_k),$$

where $\gamma_k \in R$ is the adaptation coefficient.

The nonlinear LSM involves the recurrent form

$$B_{j+1} = B_j + \vartriangle B_j, \ j = \overline{0, k},$$

where $\vartriangle B_j = (X_j^{\mathrm{T}} X_j)^{-1} X_j^{\mathrm{T}} V_j$, $j = \overline{0, k}$, are given by (11) within the notations, and B_0 is an initial approximation vector.

3.3. Fuzzy least squares method

This method was presented in [21].

Consider initial data similar to those in subsection 3.1: a fuzzy linear mathematical model (6) in the vector form

$$\underline{f}(s, r) = (B(r), L(s)) + \underline{r_n}(s);$$
$$\overline{f}(s, r) = (C(r), L(s)) + \overline{r_n}(s), \qquad (15)$$

where $B(r) = (b_1(r), \ldots, b_n(r))$, $C(r) = (c_1(r), \ldots, {}_n(r))$ are the vectors of the unknown variables to be determined; $L(s) = (l_1(s), \ldots, l_n(s))$ is the vector of given basis functions; $\underline{r_n}(s)$ and $\overline{r_n}(s)$ are the model's errors; $(B(r), L(s))$ and $(C(r), L(s))$ are scalar products.

For the sake of simplicity, we will study the lower part of the model (15) indicated by the underline. Further considerations and transformations are easily repeated for its upper part (the overline).

Assume that for some values $s = s_i$, $i = \overline{1, m} (m \geq n)$, $s_1 < s_2 < \ldots < s_m$, the following data (measurements) were obtained on the left-hand side of the model (15):

$$\underline{f}(s_1, r), \ \underline{f}(s_2, r), \ldots, \ \underline{f}(s_m, r).$$

Substituting these data into the model (15) yields the FSLE

$$\underline{f}(s_j, r) = \sum_{i=1}^{n} b_i(r) \cdot l_i(s_j) + \underline{r}(s_j) \Leftrightarrow \underline{R} = \underline{Y} - XB, \qquad (16)$$

where and $X = (x_{ij})$, $x_{ij} = l_i(x_j)$, $i = \overline{1, n}$, $j = \overline{1, m}$ $\underline{R} = (\underline{r}(s_1), \ldots, \underline{r}(s_m))$, $\underline{Y} = (\underline{f}(s_1), \ldots, \underline{f}(s_m))^{\mathrm{T}}$, is a rectangular matrix, $\dim X = (m \times n)$, m is the number of measurements, and n is the number of the unknown parameters.

Let the weighted squared errors be given: $\underline{R}^{\mathrm{T}} \Lambda^{-1} \underline{R}$, where $\Lambda = \mathrm{diag}(\lambda_{ij})$ is a diagonal weight matrix.

The problem of calculating the unknown vector B in the FSLE (16) includes two stages.

At **Stage 1**, for each basis function $l_i(s = s_j) = l_{ij}$, $i = \overline{1, n}$, $j = \overline{1, m}$, the fuzzy IF-THEN rules are formed. A fuzzy rule base $R_j = \{R_{jp}\}_{p=1}^{p=2^n}$ is generated by specifying fuzzy identifiers r_{ij}^{ε}, $\varepsilon = \overline{1, f}$, for each fuzzy variable l_{ij} (the number of identifiers for the ith fuzzy variable for $s = s_j$). For example, for $s = s_1$, this base has the following form:

$$\begin{cases} R_{11}: \text{IF } l_{11} = r_{11}^{(1)} \text{ AND } l_{21} = r_{21}^{(1)} \\ \quad \text{AND} \ldots \text{AND } l_{n-1,1} = r_{n-1,1}^{(1)} \text{ AND } l_{n1} = r_{n1}^{(1)}, \\ \text{THEN } g_{11} = \min\left\{r_{11}^{(1)}, r_{21}^{(1)}, \ldots, r_{n-1,1}^{(1)}, r_{n1}^{(1)}\right\}, \\ \vdots \\ \text{OR} \\ R_{12}: \text{IF } l_{11} = r_{11}^{(1)} \text{ AND } l_{21} = r_{21}^{(1)} \\ \quad \text{AND} \ldots \text{AND } l_{n-1,1} = r_{n-1,1}^{(1)} \text{ AND } l_{n1} = r_{n1}^{(2)}, \end{cases}$$

$$\begin{cases} \text{THEN } g_{12} = \min\left\{r_{11}^{(1)}, r_{21}^{(1)}, \ldots, r_{n-1,1}^{(1)}, r_{n1}^{(2)}\right\}, \\ \text{OR} \\ \vdots \\ \text{OR} \\ R_{1p-1}: \text{IF } l_{11} = r_{11}^{(f)} \text{ AND } l_{21} = r_{21}^{(f)} \\ \quad \text{AND} \ldots \text{AND } l_{n-1,1} = r_{n-1,1}^{(f)} \text{ AND } l_{n1} = r_{n1}^{(f-1)}, \\ \text{THEN } g_{1p-1} = \min\left\{r_{11}^{(f)}, r_{21}^{(f)}, \ldots, r_{n-1,1}^{(f)}, r_{n1}^{(f-1)}\right\}, \\ \text{OR} \\ R_{1p}: \text{IF } l_{11} = r_{11}^{(f)} \text{ AND } l_{21} = r_{21}^{(f)} \\ \quad \text{AND} \ldots \text{AND } l_{n-1,1} = r_{n-1,1}^{(f)} \text{ AND } l_{n1} = r_{n1}^{(f)}, \\ \text{THEN } g_{1p} = \min\left\{r_{11}^{(f)}, r_{21}^{(f)}, \ldots, r_{n-1,1}^{(f)}, r_{n1}^{(f)}\right\}. \end{cases}$$

Here AND and OR are the Zadeh fuzzy logic functions:

$$r_1(x) \text{ AND } r_2(x) = \min(r_1(x), r_2(x));$$

$$r_1(x) \text{ OR } r_2(x) = \max(r_1(x), r_2(x)).$$

The rule base $R_1 = \left\{R_{1p}\right\}_{p=1}^{p=2^n}$ for $s = s_1$ contains $p = 2^n$ rules for the AND basis functions.

The resulting coefficients $g_{11}, g_{12}, \ldots, g_{1p}$ are normalized to $\lambda_{11} = g_{11} / \sum_{i=1}^{p} g_{1i}, \ldots, \lambda_{1p} = g_{1p} / \sum_{i=1}^{p} g_{1i}$.

For $s: s_2, s_3, \ldots, s_m$, the fuzzy rule bases $R_2, R_3, \ldots, R_m$, respectively, are generated by analogy. For each fuzzy rule base, the normalized weight coefficients are calculated and then used to construct the following table of dimensions $(m \times p)$ considering the rule base R_1:

$s = s_1$	$\lambda_{11}, \ldots, \lambda_{1p}$
$s = s_2$	$\lambda_{21}, \ldots, \lambda_{2p}$
$\vdots$	$\vdots$
$s = s_m$	$\lambda_{m1}, \ldots, \lambda_{mp}$

At **Stage 2**, the generalized LSM is applied to the collection of weight matrices $\left\{\Lambda_q\right\}_{q=1}^{p}$ to find the unknown parameters $b_i(r)$, $i = \overline{1, n}$, of the model (6). The parameters $c_i(r)$, $i = \overline{1, n}$, of the model (6) are calculated like $b_i(r)$.

Recall that in subsection 3.1, the fuzzy error model $r_n = (\underline{r_n}(s), \overline{r_n}(s))$ has been assigned the parameters $E\underline{r_n}(s) = E\overline{r_n}(s) = 0$ and $D\underline{r_n}(s) = D\overline{r_n}(s) = \sigma^2 I$, where E and D are the expectation and variance operators, respectively; I denotes an identity matrix of appropriate dimensions; σ^2 is the coefficient of proportionality. The resulting LSM estimates $\hat{c}_i(r)$ and $\hat{b}_i(r)$ are given by (9).

Now, with the weight matrices $\left\{\Lambda_q\right\}_{q=1}^{p}$ obtained for $s: s_1, \ldots, s_m (\dim \Lambda_q = (m \times m))$, the matrix error model $\underline{R}_q$ in (12) takes the form

$$E\underline{R}_q = 0, \; D\underline{R}_q = \sigma^2 \Lambda_q,$$

where $\Lambda_q \in \left\{\Lambda\right\}_{q=1}^{p}$ is a positive definite matrix.

In other words, for Λ_q there exists a linear transformation P_q such that:

$$P_q^{\mathrm{T}} \cdot P_q = P_q \cdot P_q = P_q^2 = \Lambda_q, \; P_q^{\mathrm{T}} = P_q.$$

Hence, multiplying the model (16) by P_q^{-1} on the left yields:

$$\begin{aligned} &\underline{R}_q = \underline{Y}_q - XB_q \Leftrightarrow \underline{Y}_q = XB_q + \underline{R}_q \Rightarrow \\ &\Rightarrow P_q^{-1}\underline{Y}_q = P_q^{-1}XB_q + P_q^{-1}\underline{R}_q \Rightarrow \\ &\Rightarrow \underline{Y}_q^{(1)} = X_q^{(1)}B_q + \underline{R}_q^{(1)}, \underline{Y}_q^{(1)} = \\ &= P_q^{-1}; X_q^{(1)} = P_q^{-1}X_q; \underline{R}_q^{(1)} = P_q^{-1}\underline{R}_q. \end{aligned} \tag{17}$$

To find the error model $\underline{R}_q^{(1)}$, we calculate the expectation $E\underline{R}_q^{(1)}$ and variance $D\underline{R}_q^{(1)}$:

$$E\underline{R}_q^{(1)} = E(P_q^{-1} \cdot \underline{R}_q) = P_q^{-1} \cdot (E\underline{R}_q) = 0 \text{ since}$$

$$\underline{R}_q^{(1)} = P_q^{-1} \cdot R_q \text{ and } E\underline{R}_q = 0;$$

$$\begin{aligned} D\underline{R}_q^{(1)} &= D(P_q^{-1} \cdot \underline{R}_q) = P_q^{-1} \cdot (D\underline{R}_q)(P_q^{-1})^{\mathrm{T}} = \\ &= \sigma^2 P_q^{-1}\Lambda_q(P_q^{-1})^{\mathrm{T}} = \sigma^2(P_q^{-1}P_q)(P_q(P_q^{-1})^{\mathrm{T}}) = \sigma^2 I \end{aligned}$$

since $\underline{R}_q^{(1)} = P_q^{-1} \cdot \underline{R}_q$, $D\underline{R}_q = \sigma^2\Lambda_q$, and $\Lambda_q = P_q \cdot P_q$.

Thus, the model has the form

$$\underline{Y}_q^{(1)} = X^{(1)}B_q + \underline{R}_q^{(1)} : E\underline{R}_q^{(1)} = 0; \; D\underline{R}_q^{(1)} = \sigma^2 I.$$

The uniform partition-based LSM (see subsection 3.1) can be therefore applied to this model. It yields equations of the form (9) for calculating the components of the vector B_q:

$$(X_q^{(1)\mathrm{T}} X_q^{(1)}) B_q = X_q^{(1)\mathrm{T}} \underline{Y}_q^{(1)}.$$

Substituting the original variables of formula (17) into this equation and using the properties of the transpose operation and the matrix P_q, we arrive in the following equation:

$$[(P_q^{-1} X_q)^{\mathrm{T}} \cdot (P_q^{-1} X_q)] B_q = (P_q^{-1} X_q)^{-1} (P_q^{-1} \underline{Y}_q) \Rightarrow$$
$$\Rightarrow [(X_q{}^{\mathrm{T}} P_q^{-1})(P_q^{-1} X_q)] B_q = X_q{}^{\mathrm{T}} P_q^{-1} \underline{Y}_q \Rightarrow$$
$$\Rightarrow (X_q{}^{\mathrm{T}} V_q^{-1} X_q) B_q = X_q{}^{\mathrm{T}} V_q^{-1} \underline{Y}_q. \quad (18)$$

Here $V_q, q = \overline{1, p}$, are the weight matrices found from the fuzzy rules of Stage 1.

The considerations for deriving equation (18) are repeated with respect to the vector C_q. As a result,

$$(X_q^{\mathrm{T}} V^{-1} X_q) C_q = X_q^{\mathrm{T}} V_q^{-1} \overline{Y}_q. \quad (19)$$

Combining the expressions (18) and (19) gives

$$S_q \cdot A_q = Y_q,\ |S_q| \neq 0,\ q = \overline{1, p}, \quad (20)$$

where

$$S_q = \begin{pmatrix} L_q & 0 \\ 0 & L_q \end{pmatrix}, L_q = (X_q^{\mathrm{T}} V_q^{-1} X_q);$$
$$A_q = \begin{pmatrix} B_q & C_q \end{pmatrix}^{\mathrm{T}}, B_q = (b_{q1}, \ldots, b_{qn} \quad c_{q1}, \ldots, c_{qn})^{\mathrm{T}};$$
$$Y_q = \begin{pmatrix} \underline{Y}_q{}^B & \overline{Y}_q{}^C \end{pmatrix},\ \underline{Y}_q{}^B = X_q^{\mathrm{T}} V^{-1} \underline{Y}_q,$$
$$\overline{Y}_q{}^C = X_q^T V^{-1} \overline{Y}_q.$$

The set of solutions obtained from (20) is

$$\{A_q^*\}_{q=1}^p = \{B_q^* \quad C_q^*\}_{q=1}^p.$$

It finally leads to the following set of fuzzy solutions of the integral equation (1):

$$\begin{cases} \underline{x}_q^*(s_j, r) = (B_q^*, H_j), \\ \overline{x}_q^*(s_j, r) = (C_q^*, H_j), \\ q = \overline{1, p};\ s_j = \overline{1, m}, \end{cases} \quad (21)$$

where $B_q^* = (b_{q1}^*, \ldots, b_{qn}^*),\ C_q^* = (c_{q1}, \ldots, c_{qn}),\ H_j = (h_1(s_j), \ldots, h_n(s_j)),$ and $(\cdot,\cdot)$ denotes scalar product.

The fuzzy solution of (1) is the generalized average solution obtained from the set (21):

$$\begin{cases} \underline{x}^*(s_j, r) = \sum_{q=1}^{p} \lambda_{1q} \underline{x}_q^*(s_j, r), \\ \overline{x}^*(s_j, r) = \sum_{q=1}^{p} \lambda_{1q} \overline{x}_q^*(s_j, r), \end{cases} \quad j = \overline{1, m}.$$

3.4. Fuzzy Galerkin method

Consider the fuzzy model

$$x_{\text{fuz}}(s) = f_{\text{fuz}}(s) + \lambda \int_a^b K(s, \tau) x_{\text{fuz}}(\tau) d\tau \Leftrightarrow$$
$$\Leftrightarrow \begin{cases} \underline{x}(s, r) + \lambda \int_a^b K(s, \tau) \underline{x}(\tau, r) d\tau = \underline{f}(s, r), \\ \overline{x}(s, r) + \lambda \int_a^b K(s, \tau) \overline{x}(\tau, r) d\tau = \overline{f}(s, r). \end{cases}$$

The problem is to estimate its state by a fuzzy approximation.

This problem can be solved using two possible approaches.

Approach 1:

$$\underline{x}(s, r) \simeq \sum_{i=1}^{n} \underline{a}_i(r) \underline{h}_i(s, r);$$
$$\overline{x}(s, r) \simeq \sum_{i=1}^{n} \overline{a}_i(r) \overline{h}_i(s, r),$$

where $\left\{h_{\text{fuz}\,i}(s) = h_i(s, r) = \left(\underline{h}_i(s, r), \overline{h}_i(s, r)\right)\right\}_{i=1}^n$ are fuzzy independent complete orthogonal functions; $a_{\text{fuz}\,i} = a_i(r) = \left(\underline{a}_i(r), \overline{a}_i(r)\right)$ are fuzzy coefficients to be determined.

Approach 2:

$$\underline{x}(s, r) \simeq \sum_{i=1}^{n} \underline{a}_i(r) h_i(s);\ \overline{x}(s, r) \simeq \sum_{i=1}^{n} \overline{a}_i(r) h_i(s),$$

where $\{h_i(s)\}_{i=1}^n$ are fuzzy independent complete orthogonal functions in the traditional sense; $\underline{a}_i(r)$ and $\overline{a}_i(r)$ are the coefficients in the parametric fuzzy representation.

Consider **approach 1**. After the transformations similar to deriving the system (5) from the expressions (1)–(4), we have:

$$\begin{cases} \sum_{i=1}^{n} \underline{a}_i(r) \underline{l}_i(s) \simeq \underline{f}(s, r), \\ \sum_{i=1}^{n} \overline{a}_i(r) \overline{l}_i(s) \simeq \overline{f}(s, r), \end{cases} \quad (22)$$

where

$$\underline{l}_i(s) = \underline{h}_i(s, r) - \underline{k}_i(s);$$

$$\underline{k}_i(s) = \int_a^b K(s, \tau)\underline{h}_i(\tau)d\tau,$$

$$\overline{l}_i(s) = \overline{h}_i(s, r) - \overline{k}_i(s);$$

$$\overline{k}_i(s) = \int_a^b K(s, \tau)\overline{h}_i(\tau)d\tau.$$

The values of the fuzzy variables $\underline{l}_i(s)$ and $\overline{l}_i(s)$ are positive or negative. Therefore, the system (22) can be represented as:

$$\begin{cases} \sum_{i=1}^{n} b_i(r)\underline{l}_i(s) \simeq \underline{f}(s, r), \\ b_i(r) = \begin{cases} \underline{a}_i(r), \underline{l}_i(s) \geq 0, \\ \overline{a}_i(r), \underline{l}_i(s) < 0, \end{cases} \\ \sum_{i=1}^{n} c_i(r)\overline{l}_i(s) \simeq \overline{f}(s, r), \\ c_i(r) = \begin{cases} \overline{a}_i(r), \overline{l}_i(s) \geq 0, \\ \underline{a}_i(r), \overline{l}_i(s) < 0. \end{cases} \end{cases} \quad (23)$$

The unknown variables $b_i(r),\ c_i(r),\ i = \overline{1, n}$, in (2) are obtained from the orthogonality of the functions $\underline{h}_i, \underline{h}_j$ and $\overline{h}_i, \overline{h}_j$. For this purpose, multiplying the system (23) successively by the functions $\underline{h}_i(s, r)$ and $\overline{h}_i(s, r)$, we arrive in the fully FSLE

$$(\mathrm{diag}\underline{H})B = \underline{F},\ (\mathrm{diag}\overline{H})C = \overline{F}, \quad (24)$$

where $B = (b_1, ..., b_n)^{\mathrm{T}}$; $C = (c_1, ..., c_n)^{\mathrm{T}}$; $\underline{F} = ((\underline{f}, \underline{l}_1), ..., (\underline{f}, \underline{l}_n))^{\mathrm{T}}$ and $\overline{F} = ((\overline{f}, \overline{l}_1), ..., (\overline{f}, \overline{l}_n))^{\mathrm{T}}$ are the vectors composed of the scalar products; $\underline{H} = ((\underline{l}_i, \underline{h}_j))$ and $\overline{H} = ((\overline{l}_i, \overline{h}_j)),\ i, j = \overline{1, n}$, are the matrices composed of the fuzzy scalar products; the parameters $\underline{l}_i$ and $\overline{l}_i$ are given by (8).

The system (24) is the fully FSLE defined in [22]. As is well known, it has two solutions: a) fuzzy and b) crisp.

Case a).

Presently, the theory of fully FSLEs is sufficiently developed. Some publications on the subject were discussed in [23–26].

We write the system (24) in the standard form

$$HD = F,$$

where $H = \mathrm{diag}\begin{pmatrix}\underline{H}\\ \overline{H}\end{pmatrix}$, $D = (C|D)^{\mathrm{T}}$, and $F = ((\underline{f}_i, \underline{l}_1), ..., (\underline{f}_n, \underline{l}_n)|(\overline{f}_i, \overline{l}_1), ..., (\overline{f}_n, \overline{l}_n))^{\mathrm{T}}$.

The matrix H and vector F contain fuzzy entries.

Note that the FSLEs considered above apply to fuzzy integral equations when the matrix H contains crisp elements and the vector F consists of fuzzy components. The term "fully FSLE" indicates that the matrix H and vector F have fuzzy entries. Some methods for solving fully and not fully FSLEs were surveyed in [27] and [28], respectively.

Case b).

The crisp method for solving a fully FSLE [22, 27] by changing the original variables leads to another system with the number of equations exceeding that of unknowns. Therefore, the traditional LSM is used to solve it.

The proposed technique is also applicable to nonlinear membership functions.

Approach 2. Within the fuzzy Galerkin method, the basis functions $h_i(s), i = \overline{1, n}$, are assumed to be traditional functions that satisfy completeness and orthogonality in the common sense. In this case, the fully FSLE (24) is transformed to the FSLE

$$(\mathrm{diag}\, H)\ C = F_{\mathrm{fuz}}, \quad (25)$$

where H is a matrix with crisp elements and $F_{\mathrm{fuz}} = (\underline{F} | \overline{F})$.

The system (25) is solved using a fuzzy method for FSLEs [27, 28].

3.5. Fuzzy Chebyshev quadrature methods

In the traditional case, this method is widespread in solving crisp integral equations: the equation's integral is replaced by a quadrature formula with nodes at the Chebyshev points [10]. The method can be also be used in the fuzzy case when the fuzzy integral model is represented in an equivalent parametric form. The lower and upper representations form a system of integral models solved by the traditional quadrature method under the assumption that the integrals are understood in a fuzzy sense.

Let a fuzzy integral model have the parametric representation, and let the kernels satisfy three types of inequalities $(i)-(iii)$, similarly to subsections 3.1 and 3.3 in part I of the survey [1].

In case (*i*) when $K(s, \tau) \geq 0,\ a \leq \tau \leq b$, we have:

$$x_{\mathrm{fuz}}(s) = f_{\mathrm{fuz}}(s) + \lambda \int_a^b K(s, \tau) x_{\mathrm{fuz}}(\tau) d\tau \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \underline{x}(s, r) = \underline{f}(s, r) + \lambda \int_a^b \underline{V}(\tau, r) d\tau, \\ \overline{x}(s, r) = \overline{f}(s, r) + \lambda \int_a^b \overline{V}(\tau, r) d\tau, \end{cases}$$

where $\underline{V}(\tau, r) = K(s, \tau)\cdot \underline{x}(\tau, r)$, $\bar{V}(\tau, r) = K(s, \tau)\times \times\bar{x}(\tau, r)$, $K(s, \tau)\ge 0$ on the interval of integration $[a, b]$, and $r\in[0,1]\subset R$ is a parameter.

Consider the equation for the variable's lower branch $\underline{x}(s, r)$, replacing the integral by the Chebyshev quadrature formula:

$$\int_a^b \Psi(x)dx = A\sum_{k=1}^{n}\Psi(x_k)dx+\rho,$$

$$x_k = 0{,}5(b-a)\left[1+x_k^{(n)}\right],\ A = n^{-1}(b-a),$$

where $x_k^{(n)}$ are the tabulated Chebyshev points, and ρ denotes the remainder term. Within ρ, we obtain

$$\underline{x}(s_i, r)-\lambda A\sum_{k=1}^{n}K(s_i, \tau_k)\times \times\bar{x}(s_i, r) = \bar{f}(s_i, r),\ i=\overline{1, n}.$$

In the matrix form, the system is written as

$$K\cdot X = \Phi,$$

where

$$X = \left(\underline{x}(s_1),..., \underline{x}(s_n)\middle|\bar{x}(s_1),..., \bar{x}(s_n)\right)^{\mathrm{T}};$$

$$\Phi = \left(\underline{f}(s_1),..., \underline{f}(s_n)\middle|\bar{f}(s_1),..., \bar{f}(s_n)\right)^{\mathrm{T}};$$

$K = \begin{pmatrix} R & 0\\ 0 & R\end{pmatrix}$, $R = \left(r_{ij} = \left(1-\lambda A\sum_{k=1}^{j}K(s_i, \tau_k)\right)\right)$ is a matrix.

The discrete fuzzy solution is given by

$$x_{\text{fuz}\,\partial}(s_i) \simeq x_\partial(s_i, r) =$$

$$\left(\underline{x}_g(s_i, r), \bar{x}_\partial(s_i, r)\middle| r\in[0, 1]\right),\ i=\overline{1, n}.$$

The fuzzy interpolation method [26] finally yields the approximate solution $x^*_{\text{fuz}} = x^*(s, r) = \left(\underline{x}^*(s, r), \bar{x}^*(s, r)\middle| r\in[0, 1]\right)$.

Cases (ii): $K(s, \tau)\le 0$ and (iii): $K(s, \tau)\ge 0$, $a\le\tau\le c$; $K(s, \tau)<0$, $c<\tau\le b$, $c\in[a, b]$, are considered by analogy with part I of the survey [1].

3.6. Fuzzy approximation method based on sinc function

This method was described in [29]. As before, the parametric representation is adopted for the original fuzzy integral model. Then it is transformed to a system of integral equations, and the sinc approach is applied to this system: the unknown fuzzy solution is approximated by basis functions with unknown fuzzy coefficients. The basis functions are described by a composition of sinc functions and a special function that defines the Kronecker function. This expansion of the solution finally yields an FSLE for the unknown coefficients. Its solution determines the required approximate state of the fuzzy integral model in parametric form.

The sinc wavelet is interpreted as a "small wave" function. Its main properties—see below—hold on the entire real axis [29]:

$$i_1: \operatorname{sinc}(t) = \begin{cases}\sin(\pi t)/\pi t,\ t\ne 0,\\ 1,\ t=0;\end{cases}$$

$i_2: S(j, h) = \operatorname{sinc}((t-jh)/h), j = 0, \pm1, \pm2,...;\ h>0$ is some step between nodes t;

$i_3: C(f, h)(t) = \sum_{j=-\infty}^{\infty} f(jh)\cdot\operatorname{sinc}((t-jh)/h)$ is the Whittaker cardinal expansion of the function $f(t)$ under the assumption that the series converges;

$i_4: C(f, h)(t) = \sum_{j=-n}^{n} f(jh)\cdot\operatorname{sinc}((t-jh)/h)$ is a finite Whittaker cardinal expansion of $f(t)$.

The approximation on an interval $[a, b]\subset R$ is constructed using a conformal mapping

$$\varphi(t) = \ln((t-a)/b-t)$$

that takes an eye-shaped domain D_E onto a strip D_d:

$$D_E = \{z = x+iy: |\arg(z-a/b-z)|<d\le\pi/2\},$$
$$D_d = \{\varsigma = \zeta + i\eta: |\eta|<d\le\pi/2\}, \tag{26}$$

where $h = (\pi d/\alpha n)^{0.5}$, $0<\alpha\le 1$, and n is an integer.

The basis functions on the interval $[a, b]$ have the form

$$S(j, h)\circ\varphi(t) = \operatorname{sinc}(\frac{\varphi(t)-jh}{h}), \tag{27}$$

where $\circ$ indicates the composition of functions.

The points on the interval $[a, b]$ are determined from the system (25) by resolving (27) with respect to t:

$$t_j = \varphi^{-1}(jh) = \left(a+be^{jh}\right)/\left(1+e^{jh}\right).$$

The basis functions (27) satisfy the Kronecker representation

$$\left[S(j, h)\circ\varphi(t)\right]\Big|_{t=t_i} = \delta_{ji}^{(0)} = \begin{cases}1,\ j=i,\\ 0,\ j\ne i.\end{cases}$$

These definitions and properties are used in the sinc method to estimate approximately the fuzzy integral model

$$x_{\text{fuz}}(s) = f_{\text{fuz}}(s)+\lambda\int_a^b K(s, \tau)x_{\text{fuz}}(\tau)d\tau \Leftrightarrow$$

$$\Leftrightarrow\begin{cases}\underline{x}(s, r) = \underline{f}(s, r)+\lambda\int_a^b K_1(s, \tau, \underline{x}(\tau, r), \bar{x}(\tau, r))d\tau, & (28)\\ \bar{x}(s, r) = \bar{f}(s, r)+\lambda\int_a^b K_2(s, \tau, \underline{x}(\tau, r), \bar{x}(\tau, r))d\tau,\end{cases}$$

where

$$K_1\left(s,\tau,\underline{x}(\tau,r),\overline{x}(\tau,r)\right)= =\begin{cases}K(\tau,r)\cdot\underline{x}(\tau,r), K(\cdot)\ge 0,\\ K(\tau,r)\cdot\overline{x}(\tau,r), K(\cdot)<0,\end{cases}$$

$$K_2\left(s,\tau,\underline{x}(\tau,r),\overline{x}(\tau,r)\right)= =\begin{cases}K(\tau,r)\cdot\overline{x}(\tau,r), K(\cdot)\ge 0,\\ K(\tau,r)\cdot\underline{x}(\tau,r), K(\cdot)<0,\end{cases}$$

for al $r\in[0,1]\subset R_1$ and $a\le\tau, s\le b$.

Let us approximate $\underline{x}(s,r)$ and $\overline{x}(s,r)$ with respect to the basis functions (27) with the corresponding coefficients α_j and β_j:

$$\begin{cases}\underline{x}(s,r)=\sum_{j=-n}^{n}\alpha_j\left[S(k,h)\circ\varphi(s)\right],\\ \overline{x}(s,r)=\sum_{j=-n}^{n}\beta_j\left[S(k,h)\circ\varphi(s)\right].\end{cases} \quad (29)$$

Using the property (27), we obtain:

$$\underline{x}(s,r)=\alpha_j,\ \overline{x}(s,r)=\beta_j,\ j=\overline{-n,n}. \quad (30)$$

Substituting the system (29) into formula (28) and considering the expression (30), we arrive in a system of traditional integral equations. This system is then transformed to $(4n+2)$ algebraic equations to find the desired coefficients $\{\alpha_j\}_{j=-n}^{n}$ and $\{\beta_j\}_{j=-n}^{n}$.

The algorithm of the sinc method includes the following steps.

Step 1. Apply the parametric form (28) to the fuzzy integral equation.

Step 2. Approximate the functions $\underline{x}(s,r)$ and $\overline{x}(s,r)$ using the system (29).

Step 3. Construct the system of algebraic equations for the desired coefficients $\{\alpha_j\}_{j=-n}^{n}$ and $\{\beta_j\}_{j=-n}^{n}$.

Example 2. Let: $a=0;\ b=2;\ \lambda=1;\ K(s,\tau)=(13)^{-1}\times$ $\times\left(s^2+\tau^2-2\right);\ 0\le\tau, s\le 2$; $\underline{f}(s,r)=r\cdot s\left[1-(2/13)^{-1}s\right]$; $\overline{f}(s,r)=(2-r)\cdot s\left[1-(2/13)s\right]$.

Solution. We choose $\alpha=0.5$ and $d=\pi/2$ for the relations (26). Then $h=\pi\cdot n$, and we obtain the parametric form

$$\begin{cases}\overline{x}(s,r)=(2-r)s\left[1-(2/13)^{-1}s\right]+\\ \underline{x}(s,r)=s\cdot r\left[1-(2/13)^{-1}s\right]+\\ +\int_0^2(13)^{-1}\left[s^2+\tau^2-2\right]\underline{x}(\tau,r)dr,\\ +\int_0^2(13)^{-1}\left[s^2+\tau^2-2\right]\overline{x}(\tau,r)dr.\end{cases}$$

The exact solution is given by the method of degenerate kernels (see subsection 3.4 in part I of the survey [1], case (*i*)):

$$\underline{x}_n(s,r)=r\cdot s;\ \overline{x}_n(s,r)=(2-r)s.$$

The approximate solution using the sinc method with $n=10$ yields

$$\begin{cases}\underline{x}(s,r)=rs+(3/5)-(3/26)r-(1/13)rs^2,\\ \overline{x}(s,r)=2s-rs+(3/26)r+\\ +(1/13)s^2r-(3/26)-(3/13)s^2.\ \blacklozenge\end{cases}$$

As shown in [29], the exact and approximate solutions well match one another.

4. FINDINGS

Generally speaking, the methods considered in this paper represent two classes, one combining exact methods and the other approximate ones. The latter methods are characterized by techniques for determining the fuzzy components of a given solution structure and techniques for choosing the solution structure. Exact methods constitute a relatively small group often associated with the kernel type of a fuzzy equation. If the kernel is of convolution type, then one of the exact methods based on the fuzzy Laplace transform is usually applied. If the kernel is degenerate, then the original fuzzy equation is transformed into an FSLE via an appropriate change of the kernel variables and then solved by the method of "embedding" or its modifications.

In approximate methods, the solution of an equation is described by an indefinite structure: the desired fuzzy solution is sought in the form of some expansion in a given crisp or fuzzy system of basis functions with indeterminate fuzzy coefficients. Substituting this expansion into the original fuzzy integral model gives a (not fully) FSLE or a fully FSLE, depending on the type of basis functions: if the basis functions are crisp, then a not fully FSLE arises; if the basic functions are fuzzy, a fully FSLE does. According to this technique, the paper has described the following methods: "embedding" (subsection 3.2 of the paper [1]); Taylor approximation (subsection 3.3 of the paper [1]); approximation (subsections 3.1–3.3 above); Galerkin (subsection 3.4 above).

The listed approximate methods can be supplemented by other methods [22]: residuals, collocations, energy, Ritz, Courant, difference-analytical, etc. The technique for solving fuzzy integral equations is implemented according to the same scheme as before: expanding the solution in terms of basis functions ⇒

reducing the original equation to an FSLE or a fully FSLE $\Rightarrow$ obtaining a fuzzy solution of this system by some method described in [25–28].

The following methods for choosing the structure of an approximate solution have been considered: Chebyshev quadrature methods and sinc functions (see Sections 3.5 and 3.6). According to the general approach to this choice, there are two main types of basis functions [15]: global (algebraic, trigonometric polynomials, special functions) and finite (B-splines, wavelets, automorphic, etc.). The finite methods have certain prospects when solving fuzzy integral equations due to their computational simplicity and acceptable accuracy.

CONCLUSIONS

Based on the definition of a fuzzy functional, this paper has considered the LSM for estimating the states of fuzzy integral models. The recurrent and nonlinear methods have been proposed as some modifications of the LSM.

To improve the accuracy, a fuzzy generalized LSM has been implemented. In this method, the diagonal elements of the weight matrix are obtained from fuzzy rules. Then the generalized LSM is employed to find the model's unknown parameters using a set of the weight matrices.

Based on the definition of the fuzzy orthogonality of fuzzy functions, the fuzzy Galerkin method has been developed for estimating approximately a fuzzy integral model. This method leads to a fully FSLE.

Structural identification methods – Chebyshev quadratures and sinc functions – have been considered to analyze approximate solution algorithms. The structural identification methods have certain prospects when obtaining fuzzy estimates for fuzzy integral models due to their computational simplicity and acceptable accuracy.

When implementing linear estimation methods, the given membership function does not change its form. However, for nonlinear methods, the given membership function is transformed in a nonlinear manner. It can be found, for example, by the method of sections.

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INFORMATION COMMUNITIES IN SOCIAL NETWORKS. PART II: NETWORKED MODELS OF FORMATION[1]

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Abstract. This survey deals with mathematical models for the formation of information communities under uncertainty. The models of opinion dynamics are considered in detail. Within these models, individuals change their opinions under the influence of other individuals in a social network of a nontrivial structure. Two classes of such models are presented: the models with rational (Bayesian) individuals and the models with naive (heuristic) individuals. For each of the classes, conditions for the formation of information communities in social networks are described. For various information communities to emerge in a society with rational agents, the rationality of individuals is often limited, and some assumptions on different awareness of individuals are introduced considering the network structure. For a society with naive individuals, different modifications of the opinion dynamics mechanism are often adopted.

Keywords: social networks, information community, formation of information communities, belief dynamics, naive individuals, rational individuals.

INTRODUCTION

As noted in part I of the survey (see [1]), identifying and studying information communities in social networks – the sets of individuals with similar and stable beliefs about a given issue – is an important problem in many subject areas. To solve this problem, we should understand the patterns of belief dynamics in a social network. Features of information processing by an individual are considered in cognitive science, psychology, and social psychology; for example, see [2, 3]. Formal microlevel models are developed to describe the belief dynamics in networks with these features; for example, see [4–8]. Models of belief dynamics and the formation of information communities in social networks based on microeconomic, cognitive, and socio-psychological foundations were discussed in part I of the survey [1]. Particularly, the concept of an information community was outlined, and a general conceptual model was introduced to describe information processing and decision-making by an individual in a social network. Within this model, agents seek to eliminate uncertainty about the environment's parameter, observing external signals and the actions of their neighbors in the social network. The factors affecting belief dynamics and the formation of information communities in social networks were considered. According to the analysis of the existing models, rational agents in a society of a degenerate structure often reach a true belief about the issue. For various information communities to emerge in such a society, the rationality of individuals and their awareness should be modified somehow; for example, see [9–11]. However, part I of the survey did not touch upon two key factors affecting the formation of information communities: the structure of a social network and agents with heuristic belief updating rules. These issues will be considered below.

Part II of the survey is organized as follows. Section 1 considers the formation of information communities in models with Bayesian agents interacting in the network. Section 2 considers the formation of information communities in a network of agents with heuristic belief updating rules.

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1. FORMATION OF INFORMATION COMMUNITIES IN SOCIAL NETWORKS WITH BAYESIAN AGENTS

In models with a network structure, a finite or countable set of individuals is specified. The main elements of networked models of belief dynamics are the awareness structure of agents, the set of their actions and payoff functions, and the observability of the actions of other agents. Let us describe them in detail.

The awareness structure of agents. The state of the world – the value of the parameter $\theta \in \Theta$ – is a realization of a random variable unobservable by individuals. Each individual i has private information: a private signal s_i representing a random variable whose distribution depends on the value θ. The signal value provides information about the true value of θ. Private signals are conditionally independent of the state of the world θ. The private belief is set initially and does not change over time. The agent's belief in some period t will depend on his observations in previous periods.

The actions and payoffs of agents. In a given period, each agent i can perform an action $x_i \in X$ once, gaining a payoff $u(x_i, \theta)$. When choosing his action, the agent is guided by the subjective probability θ and the expected payoff from performing the action, $U = E_i[u(x_i, \theta)]$, considering all available information. The informative value of the agent's action for observers depends on the set X.

In the case of binary actions $X = \{0, 1\}$ and the state space $\Theta = \{0, 1\}$, the payoff function is defined as $u(x, \theta) = \theta - c$, where $0 < c < 1$. Under uncertainty, the payoff is given by

$$u(x) = \left(E[\theta] - c\right)x.$$

The standard way to define continual choice is to assume that the agent chooses an action $x \in R^1$ by maximizing the expected value of a quadratic payoff function:

$$u(x, \theta) = -E\left[(x - \theta)^2\right].$$

The optimal action is $x = E[\theta]$, yielding the agent's expected payoff $U = -\mathrm{Var}(\theta)$.

Public information and action history. The order of agents' actions (interaction protocol) is defined in advance. Agent t ($t \geq 1$) chooses an action in period t. The action history by this period has the form

$$h_t = \{x_1, \ldots, x_{t-1}\}.$$

Agent t knows the action history h_t when choosing his action. At the beginning of period t (before making their decisions), the agents have the following common knowledge:

- the prior probability distribution of the state of the world θ,
- the distribution of private signals and the payoff functions of all agents,
- the action history h_t.

The payoffs of different agents are unobservable.

With the elements described above, *the belief updating process* of individuals is as follows. In a period $t \geq 1$, the probability distribution of the state of the world θ, which is based on the public information h_t only, is called *the public or social belief* $F(\theta \mid h_t)$. Agent t uses the public belief and private information (signal s_t) to form his belief about the state of the world, which has the distribution $F(\theta \mid h_t, s_t)$. Then he chooses an action maximizing the payoff $E[u(x_t, \theta)]$ depending on his belief. The other agents know the payoff function of agent t and his decision model. The observed action x_t is treated by them as the information available to agent t (the private signal s_t). According to this information, the agents update the public belief $F(\theta \mid h_{t+1})$.

Remark. Social learning is *effective* if the individual's action fully reveals his private information. This is possible if the set of admissible actions is large enough.

Let us consider basic models – the models of belief updating with continual and discrete actions – in which individuals observe the actions of *all* predecessors.

In *the model of belief dynamics with continual actions*, the state of the world is the realization of a random variable or vector with the Gaussian distribution $N(\bar{\theta}, 1/\rho_\theta)$ in the initial period. A countable number of individuals $i = 1, 2, ...$, is given. Each individual i receives a private signal s_i representing the sum of the true value and some noise $\epsilon_i \sim N(0, 1/\rho_\epsilon)$:

$$s_i = \theta + \epsilon_i.$$

The agent's payoff is $u(x, \theta) = -E\left[(x - \theta)^2\right]$. Individual t chooses an action $x_t \in R$. The public information at the beginning of period t consists of the prior distribution $N(\bar{\theta}, 1/\rho_\theta)$ and the action history $h_t = \{x_1, \ldots, x_{t-1}\}$.

Suppose that the public opinion about the value θ in period t obeys the Gaussian distribution $N(\mu_t, 1/\rho_t)$. Then the same assumption is valid in the period $t = 1$ for the parameters $\mu_1 = \theta$ and $\rho_1 = \rho_\theta$. As is easily demonstrated, it will hold for each subsequent period. In any period, the public belief is updated in three stages as follows.

- Calculating the belief of agent t. The public belief $N(\mu_t, 1/\rho_t)$ is updated according to Bayes' rule based on the private information $s_t = \theta + \epsilon$. The public belief is the distribution $N(\tilde{\mu}_t, 1/\tilde{\rho}_t)$ with the parameters

$$\tilde{\rho}_t = \rho_t + \rho_\epsilon,$$

$$\tilde{\mu}_t = \alpha_t s_t + (1-\alpha_t)\mu_t, \text{ where } \alpha_t = \rho_\epsilon / \tilde{\rho}_t.$$

• Choosing the action x_t of agent t. The agent seeks to maximize the payoff $-E\left[(x-\theta)^2\right]$. He chooses an action equal to the expectation of θ:

$$x_t = \alpha_t s_t + (1-\alpha_t)\mu_t.$$

• Social learning. Network agents observe the action x_t and update the public belief about θ during the next period. Recall that the decision rule of agent t and the values α_t and μ_t are known to all agents. Therefore, *the observed action x_t fully reveals the private signal s_t*. The public information at the end of period t is identical to the information of agent t: $\mu_{t+1} = \tilde{\mu}_t$ and $\rho_{t+1} = \tilde{\rho}_t$. Hence, in period $(t + 1)$, the belief still has the Gaussian distribution $N(\mu_{t+1}, 1/\rho_{t+1})$, and the learning process can be continued. Note that the action history $h_t = \{x_1, \ldots, x_{t-1}\}$ is equivalent by information content to the sequence of signals ($s_1, \ldots, s_{t-1}$).

The accuracy of public persuasion increases according to the law $\rho_t = \rho_\theta + (t-1)\rho_\epsilon$, i.e., the variance will converge to zero. Also, the significance of private signals tends to zero, and the agents will accordingly imitate each other's actions. Under "noisy" observations of the actions of other agents, the rate of social learning decreases [12]. There are modifications of the basic model of social learning [13] in which the agent *pays for the required accuracy p* of his private signal and then performs an action. Under minimum assumptions about the cost function $c(p)$, it can be proved that the agents will stop "buying" the signal after some time, and social learning will stop.

In *the model of belief dynamics with discrete actions*, the state of the world $\theta \in \Theta = \{0,1\}$ is specified randomly in the initial period, $\mu_1 = P(\theta = 1)$. A finite (N) or countable number of agents is indexed by integer t. Each agent receives a symmetric private signal q> ½: $P(s_t = \theta|\theta) = q$. Agent t chooses an action $x_t \in \{0,1\}$ in period t (and only in this period). The agent's payoff is given by the state of the world:

$$u(x, \theta) = \begin{cases} 0, \; x = 0, \\ \theta - c, \; x = 1, \end{cases}$$

where $0 < c < 1$. (In the classical BHW-model, proposed by Bikhchandani, Hirshleifer, and Welch [144], the state of the world θ is the agent's payoff from action 1 (accept), and the parameter c describes the costs incurred by action 1.) Since $x \in \{0,1\}$, the payoff can be written as $u(x,\theta) = (\theta - c)x$. Under uncertainty, the agent considers the payoff to be the expectation of $u(x, \theta)$ given the available information.

As before, the information available to agent t is his private signal and the action history h_t. The public belief at the beginning of period t is the probability of state 1 given the public history h_t:

$$\mu_t = P(\theta = 1|h_t).$$

As was shown in [14], *an information cascade* can quickly arise in such models: the agents in sequence ignore their private signals and act in the same way as their predecessors, thereby not providing their followers with new information. In other words, the entire society ineffectively aggregates the available information and may come to wrong beliefs. In a possible modification [15] of this model, the agents acquire information to change their actions. If the information helps break the current consensus and is reasonably "inexpensive," the agents will reach the right beliefs and actions.

Thus, in the models of belief updating with a canonical structure of a social network (in which each agent observes the actions of all predecessors), social interactions will yield one information community with a true or false belief about the issue of interest. (The conditions of true or false belief have been discussed above.) Let us now consider the models of belief updating for Bayesian agents with a more complex topology of social networks.

Formation of information communities in networks with a nontrivial structure

Let us start with the exemplary model of sequential social learning with a nontrivial structure. Consider a countable set of agents (individuals) indexed by $n \in \mathbb{N}$ [16]. Agents make decisions sequentially and once. The payoff of agent n depends on his action and the initial state of the world θ. For simplicity, the state of the world and the actions of agents are assumed binary: for agent n, the action is $x_n \in \{0,1\}$, and the state of the world is $\theta \in \{0,1\}$. The payoff of agent n is given by

$$u_n(x_n, \theta) = \begin{cases} 1, \; x_n = \theta, \\ 0, \; x_n \neq \theta. \end{cases}$$

Also, the values of the state of the world are supposed equally probable, i.e., $P(\theta = 0) = P(\theta = 1) = 1/2$. The agents do not know the state θ. Each agent forms a belief about the state of the world by observing a private signal $s_n \in \bar{S}$ ($\bar{S}$ is a metric space) and the actions of other agents. The signals are conditionally independently generated according to the probability measure F_θ. The pair (F_0, F_1) is called *the signal structure*. The measures F_0 and F_1 are absolutely continuous relative to each other: a signal that would completely reveal the state of the world is impossible.

Each agent n observes the actions of his neighbors in the social network only, i.e., the agents from the set $B(n) \subseteq \{1, 2,\ldots, n-1\}$. Each network neighborhood $B(n)$ is generated according to some probability distribution Q_n defined on the set of all subsets {1, 2,…, n – 1}. Each distribution Q_n in the sequence $\{Q_n\}_{n \in N}$ does not depend on other distributions and the realizations of private signals. The sequence $\{Q_n\}_{n \in N}$ forms the social network topology, which is common knowledge, unlike the realized neighborhood $B(n)$ and the private signal s_n. The canonical topology considered in the literature, when each agent observes all previous actions, is realized if for any $n \in \mathbb{N}$ the probability of the neighborhood {1, 2,…, n – 1} is equal to 1 in the distribution Q_n. Other options are also possible, for example, the realization of a random graph model.

The information set of agent n is defined as

$$I_n = \{s_n,\ B(n),\ x_k \text{ for all } k \in B(n)\}.$$

Let $\boldsymbol{I}_n$ denote the set of all admissible information sets of agent n. The strategy of agent n, σ_n: $\boldsymbol{I}_n \to \{0,1\}$, is a mapping of the set of all admissible information sets into the action set. The strategy profile is a sequence of strategies $\sigma = \{\sigma_n\}_{n \in N}$. For a given strategy profile σ, the sequence of actions of network agents is a random process $\{x_n\}_{n \in N}$. This process generates a measure P_σ. A strategy profile σ^* is a *perfect Bayesian equilibrium* in the class of pure strategies in the social learning game if, for any $n \in N$, the strategy σ^*_n maximizes the expected payoff of agent n under the opponents' strategy profile σ^*_{-n}.

For a given strategy profile σ, the expected payoff of agent n from the action $x_n = \sigma_n(I_n)$ is $P_\sigma(x_n = \theta \mid I_n)$. Hence, for any equilibrium σ^*,

$$\sigma^*_n(I_n) \in \underset{y \in \{0,1\}}{\mathrm{Argmax}}\, P_{(y,\sigma^*_{-n})}(y = \theta | I_n).$$

This social learning game has a perfect Bayesian equilibrium in the class of pure strategies. The set of all such equilibria is denoted by Σ^*.

The following question is of interest: Will the equilibrium behavior guarantee asymptotic learning? Formally speaking, *asymptotic learning* arises in an equilibrium σ if the value x_n converges in probability to θ:

$$\lim_{n\to\infty} P_\sigma(x_n = \theta) = 1.$$

The agents' actions can be characterized as a function of the sum of two posterior beliefs: the agent's private belief and the *social belief*. In an equilibrium $\sigma \in \Sigma^*$, the decision of agent n, $x_n = \sigma_n(I_n)$, has the form

$$x_n = \begin{cases} 1, & p_n + q_n > 1, \\ 0, & p_n + q_n < 1, \end{cases}$$

and $x_n \in \{0,1\}$ otherwise. Here $p_n = P_\sigma(\theta = 1 \mid s_n)$ is *the private belief*, and $q_n = P_\sigma(\theta = 1 \mid B(n),\ x_k,\ k \in B(n))$ is *the social belief*.

The private belief of agent n does not depend on the strategy profile. Using Bayes' rule, it can be written as

$$p_n = \left(1 + \frac{dF_0}{dF_1}(s_n)\right)^{-1}.$$

The support of the private beliefs is the range $[\underline{\beta}, \overline{\beta}]$, where $\underline{\beta} = \inf\{r \in [0, 1] | P(p_1 \le r) \rangle 0\}$ and $\overline{\beta} = \sup\{r \in [0,1] | P(p_1 \le r) < 1\}$. The signal structure has *bounded private beliefs* if $\underline{\beta} > 0$ and $\overline{\beta} < 1$, and *unbounded* if $\underline{\beta} = 1 - \overline{\beta} = 1$. In the latter case, the agents can receive an arbitrarily strong signal in favor a certain state.

Consider some properties of network topologies and signal structures to present further results on asymptotic social learning. A network topology has *expanding observations* if, for all $K \in N$,

$$\lim_{n\to\infty} Q_n\left(\max_{b \in B(n)} b < K\right) = 0.$$

The following theorem was proved: if the network topology $\{Q_n\}_{n \in N}$ does not have expanding observations, then *there exists no equilibrium σ in which asymptotic learning will be achieved*. If the network topology does not have expanding observations, there is a finite set of agents whose actions will be observed with a positive probability by an infinite number of agents. As a result, they will be unable to aggregate information dispersed through the network. (Such a finite set of agents is called *excessively influential* [16].)

If the network topology $\{Q_n\}_{n \in N}$ has expanding observations, and the signal structure (F_0, F_1) implies the unboundedness of private beliefs, then the theorem on asymptotic learning in each equilibrium $\sigma \in \Sigma^*$ holds. In particular, this theorem guarantees learning in the case of moderately influential agents (i.e., their actions are visible to the entire society): they are not the only sources of information in the network. As an example, consider a network in which all other agents observe the actions of the first K agents, but each agent also observes its immediate neighbor, i.e., $B(n) = \{1, 2, ..., K, n - 1\}$. This network topology has expanding observations and, therefore, leads to learning under unbounded private beliefs. This conclusion contradicts the results for non-Bayesian learning models (see [17, 18]), in which the new beliefs of agents are the weighted average of the private beliefs and the beliefs of the agents they observe: if the first K agents are influential (other agents observe their actions), there will be no asymptotic learning.

Now consider the signal structure (F_0, F_1) in which the private beliefs are bounded, and the network topology $\{Q_n\}_{n \in N}$ satisfies one of the following conditions:

– $B(n) = \{1, \ldots, n-1\}$ for all n (see the paper [199]).

– $|B(n)| \le 1$ for all n.

– There exists a constant M such that $|B(n)| \le M$ for all n and

$$\lim_{n\to\infty} \max_{b \in B(n)} b = \infty \text{ a.s.}$$

Then asymptotic learning will not be achieved in any equilibrium $\sigma \in \Sigma^*$. Particularly, there is no asymptotic learning in a network where each agent n chooses $M \ge 1$ neighbors from the set $\{1, \ldots, n-1\}$ uniformly and independently.

Note that in this model, the agents perform their actions once, subsequently gaining their payoffs. In some situations, *action can be postponed:* agents may exchange messages—their information—without considerable costs (except for time) to obtain additional information. An example of the agent's payoff function [20] is

$$u_i(x_i, \theta) = \begin{cases} \delta^\tau \pi \text{ if } x_{i,\tau} = \theta \text{ and } x_{i,t} = \text{wait for } t < \tau, \\ x \text{ otherwise,} \end{cases}$$

where $x_i = [x_{i,t}]_{t=0,1\ldots}$ is the action sequence of agent i ($x_i \in \{\text{wait}, 0, 1\}$); $\pi > 0$ specifies the agent's payoff; $\delta \in (0, 1)$ denotes the discount factor. At the qualitative level, an analog of this two-stage model in the case of agents with heuristic belief updating rules is the model [21], in which agents first form their opinions and then simultaneously perform their actions in accordance with the payoff functions.

The conditions for achieving learning in a social network [16] are rather mild. The typical outcome of Bayesian social learning models is a long-term consensus. In order to obtain information communities with different beliefs, it is necessary to relax the rationality requirement for social network individuals.

In particular, the concept of *quasi-Bayesian updating* of agents is widespread [22–24]: each agent in the network believes that the actions of other agents are caused exclusively by their private signals. (This concept is associated with *cognitive constraints*—the limited depth of agent's inference.) The paper [24] considered sequential social learning in a social network (observation network) under the following assumptions. The world can be in one of two equiprobable states $w \in \{0, 1\}$. There is a countable set of agents indexed by $i = 1, 2, 3, \ldots$, which act once and in turn (sequentially). At his move, agent i receives a private signal about the state of the world, $s_i \sim N(1, \sigma^2)$ if $w = 1$, or $s_i \sim N(-1, \sigma^2)$ if $w = 0$. In addition, agent i observes the actions of his predecessors in a directed observation network $N_i \subseteq \{1, 2, ..., i - 1\}$. Based on this information, the agent forms his belief about the state of the world w and chooses an action $a_i \in [0, 1]$ maximizing his utility function $u_i(a_i, w) = -(a_i - w)^2$, more precisely, the expected utility $\mathbb{E}\left[-(a_i - w)^2\right]$ (Thus, the action chosen by him corresponds to his belief about the probability of the event $\{w = 1\}$.) The agents in the model are Bayesian, but the cited authors made *a rather strong assumption about the naivety of network participants*: agent i mistakenly believes that the action of his predecessor j in the observation network is conditioned by the private signal of agent j only (he has no predecessors). In other words, agent i supposes that $a_j = \mathrm{P}[w = 1 \mid s_j]$, underestimating the correlation of the actions of his predecessors. Interestingly, the agent's optimal action can be derived by a rule similar to the updating rule in DeGroot's opinion dynamics model. As was established in [24], the society (all agents) of denser observation networks more often comes in the long run to a false estimate of the state of the world compared to that of sparse networks. *Erroneous learning* takes place: either $\lim_{n\to\infty} a_n = 0$ if $w = 1$, or $\lim_{n\to\infty} a_n = 1$ if $w = 0$. This effect can be explained as follows: in sparse networks, "early" agents do not strongly affect each other, and the consensus reached "includes" more independent sources of information and is likely to be correct. Also, it was demonstrated that the agents will almost surely come to a consensus in the case of continuous actions: their disagreements will disappear. However, if the agents' actions are binary (for any agent i, $a_i \in \{0, 1\}$), and the set of agents is divided into two groups with even and odd numbers so that, in accordance with the stochastic block model, the probability of an observation connection from agent i to agent j is equal to q_s if they belong to the same group, and equal to q_d otherwise ($q_s > q_d > 0$), then there is a positive probability that all odd (even) agents will choose action 0 (action 1, respectively). Thus, *information communities with opposite beliefs about the state of the world* will be formed in the connected network.

In the paper [25], agents were assumed *locally Bayesian*: they process information as Bayesian agents, but each considers his ego (local) network to be the entire initial global network, undirected and connected. (In contrast to the models discussed above, the agent does not suppose that his neighbors are guided only by private signals.) In each period, agents form their beliefs about the state of the world based on the private signal received in the previous period and messages from their neighbors (the complete history of messages from neighbors since agents have perfect memory) and then exchange their beliefs. A simple belief updating rule was proposed: agents attribute unexpected changes in the beliefs of their neighbors to the new private signals they receive. (Agents suppose that there are no other agents outside their local network.) As was shown, agents' beliefs fluctuate without stabilizing in some networks.

Constraints on the rationality of network agents are also imposed in models with repeated actions, in which agents repeatedly revise their beliefs and actions (repeated Bayesian updating). This class includes models of repeated actions (1) with locally optimal agents, (2) with heuristic inclusion of information from neighbors, and (3) with rational expectations of agents.

Models of repeated actions with locally optimal agents. In each period, agents choose the best response based on their current beliefs (formed rationally), neglecting the influence of their actions on other agents and the possibility of obtaining additional information in the future. If the agents' actions are continuous and their prior beliefs coincide, a consensus is reached in any connected network with discrete states of the world [23] and Gaussian states of the world [26]. In the case of discrete actions of agents ($x_n(t) \in \{0, 1\}$), a consensus can also be reached in a connected network; see [27, 28]. In the model [28], each agent performs a locally optimal action in each period, taking into account his current beliefs.

Models of repeated actions with heuristic inclusion of information from neighbors. A striking example is the model [29, 30], partly resembling DeGroot's approach. Agents have prior beliefs about the state of the world $\theta \in \{0, 1\}$. At the beginning of each period, each agent receives a private signal and observes the beliefs of his neighbors. In period t, agent n has the belief $p_n(t) = P(\theta = 1)$. First, he updates the belief according to Bayes' rule, taking into account the received signal $s_n(t)$:

$$p'_n(t) \equiv P(\theta = 1 | s_n(t)) =$$
$$= \frac{P(s_n(t)|\theta = 1)\, p_n(t)}{P(s_n(t)|\theta = 1)\, p_n(t) + P(s_n(t)|\theta = 0)(1 - p_n(t))}.$$

Then he averages the resulting belief based on the beliefs of his neighbors using DeGroot's rule:

$$p_n(t+1) = a_{nn} p'_n(t) + \sum_m a_{nm} p_m(t),$$

where the matrix A specifies the weights of his neighbors. If the signals received by the agents are not informative, then their beliefs are formed according to DeGroot's rule; see Section 2. If the signals are informative, the network graph is strongly connected, and each agent "trusts" himself, then the agents' beliefs will almost surely converge to the true estimate of the state of the world.

Models of repeated actions with rational expectations of agents. In the paper [31], the states of the world are from the set $\Theta = \{0, 1\}$. In the initial period, each agent n receives an informative signal s_n. In each period, agent n observes the action of each neighbor $m \in B(n)$ and chooses the action $x_n(t)$, obtaining the payoff

$$u(x_n(t), h_n(t), s_n) = P(\theta = x_n(t) | h_n(t), s_n),$$

where $h_n(t)$ is the history of neighbors' actions by the beginning of period t. Agents discount their future payoffs with a factor $\lambda \in (0, 1)$ and play a repeated game with incomplete information. If the network graph is L-locally connected and there is an upper bound d on the number of observed neighbors, all agents in an infinite (large) network will almost surely (with a high probability) reach the true estimate of the state of the world. A graph G is L-locally connected if, for each edge (n, m), the length of a path from m to n does not exceed L. The property of L-connectedness and the existence of the bound d can be interpreted as the absence of excessively influential agents in the network.

2. FORMATION OF INFORMATION COMMUNITIES IN SOCIAL NETWORKS WITH HEURISTIC AGENTS

Bayesian models initially do not consider the psychological components of personality. As is known from psychology and social psychology, individuals have cognitive limitations and are subject to various socio-psychological factors (including a predisposition to their point of view, the social impact of some individuals on others, conformism, etc.). Various theories and models are being developed to explain the emerging effects. Since the 1950s, mathematical models with simple empirical rules for updating agents' beliefs have been developed and improved to demonstrate the effects observed in practice. The fundamental works in the field of *opinion dynamics modeling* investigate and describe, first of all, the phenomenon of coordinating the opinions of agents (reaching a consensus) when the interaction between network members gradually decreases the disagreement of their opinions. This phenomenon is explained in social psychology by several reasons, particularly by conformism, the acceptance of evidence (persuasion), incomplete information, uncertainty in one's own decisions, etc.

In classical formal models of opinion dynamics (see [4, 7, 32–35]), the sequential averaging of continuous (*continual*) opinions of agents in discrete time was considered. There are some variations of this kind of models with continuous-time averaging [36, 37].

Here is a slightly modified example of the classical DeGroot's model of reaching a consensus in a network structure. In this structure, each agent from the set $N = \{1, \ldots, n\}$ forms his current opinion at each step as the weighted sum of the opinions of all other network agents and his opinion at the previous step:

$$x_i^{(t+1)} = \sum_{j \in N} a_{ij} x_j^{(t)},\ t \geq 0,$$

where $x_i^{(0)}$ denotes the initial opinion of agent i. The parameter $a_{ij} \in [0,1]$ reflects the degree of influence of agent j on agent i ($\sum_j a_{ij} = 1$).

In matrix form, the opinion dynamics can be written as

$$x^{(t+1)} = Ax^{(t)},$$

where A is a row-stochastic influence matrix.

Note that DeGroot's model has microeconomic foundations and relation to Bayesian models. In particular, if the initial beliefs of individuals are noisy, then the DeGroot updating rule is optimal at the first step [17]: the new opinion of an individual is the weighted sum of the opinions of his neighbors, and the weight of the neighbor's opinion is the accuracy of his information. In subsequent periods, the individual must tune the weights of his neighbors since the incoming information can be repeated. This procedure is not easy, so DeGroot's rule with constant weights can be treated as a behavioral heuristic. Another microeconomic foundation is the representation of agents as players participating in a simple coordination game [38]. In this game, the locally optimal best response dynamics (coinciding with the dynamics in DeGroot's model) yield a Nash equilibrium.

The opinion dynamics in DeGroot's model allow reaching a consensus in a strongly connected social network. The agents' opinions gradually coincide since each agent has a direct or indirect impact on any other agent in the network, and the deviations in their opinions finally vanish.

The structure of the interaction network restricts the possibility of reaching a consensus. For example, in a disconnected network, consensus can only be reached in special cases. The disagreement of opinions can also be observed in strongly connected networks if, e.g., agents have initial beliefs somewhat "insensitive" to any influence [39]. In such models, the agent's opinion at each step is the weighted sum of the opinions at the previous step and his initial opinion:

$$x^{(t+1)} = \Lambda A x^{(t)} + [I_n - \Lambda] x^0,$$

where $\Lambda = I_n - \mathrm{diag}(A)$.

The initial opinions of agents can be interpreted as individual preferences or ingrained beliefs remaining in force during opinions exchange.

The opinion dynamics similar to the considered ones can be obtained using the model with compound nodes [40] in which each node consists of two agents––external and internal—interacting with each other. Each node exchanges information with other nodes through its external agent, and the internal agent (a trusted person of the external one, his friend or consultant) interacts only with the corresponding external agent.

A multidimensional generalization of the model with "insensitive" agents is the model [41] where each agent has an opinion on several interrelated issues (m different topics). The opinion of agent i ($i \in N$) on m different topics is given by the vector $x_i^{(t)} = \left(x_i^{(t)}(1), \ldots, x_i^{(t)}(m)\right)$. The opinion dynamics of agent i in period t are described by

$$x_i^{(t)} = \lambda_{ii} \sum_{j \in N} a_{ij} y_j^{(t-1)} + (1 - \lambda_{ii}) x_i^{(0)},.$$

$$y_j^{(t-1)} = C x_j^{(t-1)},$$

where C denotes the mutual influence matrix of the topics under discussion, and $y_j^{(t-1)}$ are convex combinations of agent j on several topics. The dynamics can be written in matrix form:

$$x^{(t)} = \left[(\Lambda A) \otimes C\right] x^{(t-1)} + \left[(I_n - \Lambda) \otimes I_m\right] x^{(0)},$$

where $\otimes$ indicates the Kronecker product, and $\Lambda = I_n$ or $\Lambda = I_n - \mathrm{diag} A$ (depending on the model).

Despite the additional factors of these models (the presence of biases and mutually influencing topics) that preserve some mismatch of opinions, the mutual influence of agents gradually decrease their disagreements. In particular, the averaging assumption implies that opinions will never go beyond the range of initial opinions.

Numerous theoretical results were obtained for opinion dynamics models, often associated with reaching a consensus in networks. Such models are studied using the theory of stochastic matrices and the theory of homogeneous and inhomogeous Markov chains. As is known, opinion dynamics can be modeled by Markov chains. In a homogeneous Markov chain, reaching a consensus is determined by the convergence of the powers of its stochastic matrix. Some sufficient conditions for the convergence of the powers of a stochastic matrix were given in [4, 42]. For the class of stochastic matrices without guaranteed consensus, the necessary conditions for reaching a consensus were presented in [42]. The minimum changes in the initial beliefs of agents leading to a consensus were found in [43]. Also, some particular results were established. For example, in [38], random (stochastically block) networks were considered, and the dependence of the rate of convergence of agents' beliefs (updated by the simple averaging rule) on the value of homophily was determined.

The analysis above deals with long-term consensus. In practice, it is often interesting to know the possibility of reaching disagreements in a finite time. How does the network structure affect *medium-term disagreements and the formation of medium-term information communities*? If the initial beliefs of the agents are the same, a consensus is reached immediately and does not depend on the network structure. In general, consensus depends on the initial beliefs and the network structure. The worst case [44] can be examined: What is the rate of convergence of beliefs (the number of steps required for making the disagreements sufficiently small) under any initial beliefs? For simplicity, irreducible and primitive influence matrices A are considered. For a typical matrix of this kind, the following relation holds almost everywhere:

$$A^t = \sum_{l=1}^{n} \lambda_l^t P_l.$$

Also, this matrix has the following properties:

– The values $\lambda_1 = 1$, λ_2, λ_3, …, and λ_n are different eigenvalues of the matrix A sorted in the non-ascending order of the magnitude.

– The matrix P_l is a projection operator corresponding to a nontrivial one-dimensional subspace associated with the eigenvalue λ_l.

– $P_1 = A^\infty$ and $P_1 x^{(0)} = x^{(\infty)}$.

– $P_l 1 = 0$ for all $l > 0$.

The matrix P_1 corresponds to the resulting influence matrix A^∞ and determines a stable state of the system (the resulting beliefs of the agents). The other matrices $P_{l>1}$ reflect the deviation from this resulting matrix in period t. The domination of the matrix P_1 depends on λ_2: the smaller this value is, the faster the stable state will arise in the network [455]. As it turned out, the agents' beliefs in period t satisfy the inequality

$$\frac{1}{2}|\lambda_2|^t - (n-2)|\lambda_3|^t \le \sup_{x^{(0)} \in [0,1]^n} \left\| x^{(t)} - x^{(\infty)} \right\|_\infty \le (n-1)|\lambda_2|^t.$$

Thus, the value $|\lambda_2|^t$ determines the maximum deviation of the agent's belief from the resulting belief in the network. The matrix P_2 mainly determines the deviation of beliefs from the resulting belief (consensus) and corresponds to *the metastable state of the network in which most of the disagreements disappear, but some stable part remains valid*.

Since $P_2 = \sigma\rho^{\mathrm{T}}$, where ρ^{T} and σ are the left and right eigenvectors, respectively, of the matrix A corresponding to the eigenvalue λ_2, for sufficiently large t the deviation $x^{(t)} - x^{(\infty)}$ will be equal to $\lambda_2^t \sigma(\rho^{\mathrm{T}} x^{(0)})$. If the eigenvalue λ_2 is a positive real number, then the deviation of agent i from the consensus is proportional to the component σ_i, irrespective of the initial beliefs $x^{(0)}$. The order of the medium-term beliefs of the agents is determined by one network-dependent number. An interesting interpretation was given in [17]: the opinions of individuals about different issues can be approximated well by a line; the individual's position on this line (in the left-right spectrum) determines his opinion about all issues. (For example, the opinions of many people about a wide range of fundamentally unrelated issues can be characterized by a measure of their conservatism/liberality.)

The following question arises immediately: *How does the network structure (influence matrix) affect the preservation of different information communities in the network?* The effect of splitting (dividing) the network into groups can be estimated, e.g., using the Cheeger isoperimetric constant (an indicator of graph bottlenecks) [45]:

$$\Phi_*(A) = \min_{S: S \subseteq N, \sum_{i \in S} \pi_i \le \frac{1}{2}} \frac{\sum_{i \in S, j \notin S} \pi_i A_{ij}}{\sum_{i \in S} \pi_i},$$

where π denotes the left eigenvector of the matrix A corresponding to the eigenvalue λ_1 (the influence vector of agents). This constant is small if there exists some set of agents with at most 50% of social influence and a relatively small external influence. If the influence matrix corresponds to a reversible Markov chain, then $\Phi_*^2(A) \le 1 - \lambda_2 \le 2\Phi_*(A)$ [45]. In other words, the Cheeger isoperimetric constant provides lower and upper bounds on λ_2: the smaller this constant is, the greater the eigenvalue λ_2 will be, and the longer the disagreements will hold in the network.

The influence (trust) matrix in the belief dynamics models considered above remains invariable. There are studies in which the influence matrix changes over time. The belief dynamics model was generalized in [46] as follows: the influence matrix evolves at each step, and the iterative process is given by the product of matrices:

$$x^{(t)} = A^{(t)} x^{(t-1)}.$$

In this problem statement, the opinions are coordinated by studying the convergence of inhomogeneous Markov chains. Note that the class of opinion formation models is closely related to extensive research into consensus in multi-agent systems (see the surveys [47, 48]). The theoretical results obtained in this area can be transferred to social networks. (Of course, in the case of simple belief updating rules, which often neglect the specifics of the subject area.)

Consensus formation is given much attention in the publications generalizing the classical DeGroot's model, unlike the issues of social learning (the formation of a society with true beliefs).

Forming true beliefs in DeGroot's model. If the agents' beliefs in the network reflect their estimates of the state of the world μ∈ [0, 1], the agents come to a true or false consensus. Is *social learning* possible in a network where the agents update their beliefs by DeGroot's rule? In [18], the initial beliefs (private signals) $x_i^{(0)}$ were supposed independent random variables on [0, 1] with a mean μ and a positive variance. As is known, under sufficiently weak conditions, the agents' beliefs converge to the same final opinion. To assess convergence to the true resulting estimate in a growing society, the sequence of influence matrices $\left(A(n)\right)_{n=1}^{\infty}$ indexed by n (the number of agents in each network) was considered. Each network (the corresponding influence matrix) was assumed *convergent*: any initial beliefs of agents have some limit with respect to it. A sequence $\left(A(n)\right)_{n=1}^{\infty}$ is said to be *wise* if $\operatorname{plim}_{n\to\infty} \max_{i \le n} \left| x_i^{(\infty)}(n) - \mu \right| = 0$.

One of the society's "wisdom" conditions is associated with the influence of agents. Without loss of generality, agents are rearranged in the descending order of their influence so that $s_i(n) \ge s_{i+1}(n) \ge 0$ for any i and n, where $s_i(n)$ is the weight of the initial opinion of agent i in the final opinion of network n. As was established, the sequence of converging stochastic matrices . is wise if and only if the corresponding influence vectors have the property $s_1(n) \to 0$: the influence of the most influential agent tends to 0 as the society grows infinitely.

Another obstacle to wisdom is the so-called *prominent* groups, which receive a disproportionate share of social attention and lead it astray. For a fixed network with n agents, a *group B* is a subset of the agents' set $\{1,\ldots, n\}$. A group B is *t-prominent* with respect to A (or prominent within t steps) if any agent $i \notin B$ is indirectly influenced by it: $A_{i,B}^t > 0$. The minimum weight of such an influence is called *the t-step prominence of B*: $\pi_B(A;t) = \min_{i \notin B} A_{i,B}^t$. A *family* is a sequence of groups (B_n), where $B_n \subset \{1,\ldots,n\}$ for each n. A family (B_n) is *uniformly prominent* with respect to $\left(A(n)\right)_{n=1}^{\infty}$ if there is a constant $a > 0$ such that, for each n, $\pi_{B_n}(A(n);\, t) \ge a$ at some step t. A family (B_n) is *finite* if it eventually stops growing, i.e., there exists a number q such that $\sup_n |B_n| \le q$. As was established, if there is a finite uniformly prominent family with respect to ($A(n)$), then the sequence is not wise. Thus, large societies with a "small" group affecting everyone in the network will never reach true beliefs about the state of the world.

The opinion dynamics models considered above describe the phenomenon of consensus (disagreements in the opinions of interacting agents decrease over time) and the accompanying phenomenon of social learning. In addition, the effect of medium-term disagreements is possible due to the network structure. In many cases, social and psychological phenomena are observed in social networks [49] as the result of long-term disagreements and stable information communities: the persistence of disagreements, group polarization (during a group discussion, any initially dominant point of view will strengthen), opinion polarization (disagreements between two opposition groups will increase), etc. The classical opinion dynamics models in the long term do not explain well the persistence of disagreements or even the strengthening of radical opinions in strongly connected networks. For these phenomena, new formal mathematical models are being developed [5, 6, 37, 50–53]. In particular, the formation of sets of agents with different beliefs is described by *bounded confidence* models [5, 6, 51] in which only sufficiently close agents can influence each other. (This rule of interaction is usually motivated by the phenomena of homophily and social identification.)

In the model [54], the opinion dynamics were described by a vector $x = (x_1, \ldots, x_n) \in R^n$. Agent i perceives the opinions of other agents only if they are sufficiently close to his opinion. In other words, the set of influence agents of agent i has the form $I(i, x) = \{1 \le j \le n \,|\, |x_i - x_j| \le \epsilon_i\}$, where $\epsilon_i > 0$ is the degree of confidence. (As a rule, $\epsilon_i = \epsilon$.) Then the opinion dynamics are given by

$$x_i(t+1) = |I(i, x(t))|^{-1} \sum_{j \in I(i, x(t))} x_j(t).$$

These dynamics match DeGroot's model with an influence matrix depending on the agents' opinions: $a_{ij}(x) = 1/|I(i, x)|$ if $j \in I(i, x)$, and $a_{ij}(x) = 0$ otherwise. The necessary and sufficient conditions for reaching a consensus were presented. In order to reach a consensus, it is necessary that the opinion vector at any step be an ϵ-profile. (A vector is an ϵ-profile if after sorting its elements in the ascending order, the distance between two neighbor elements will not exceed ϵ.) Otherwise, consensus is impossible: *the trust network splits into connected components, and groups of individuals with the same beliefs (information communities) appear in it*. In any case, the agents will come to equilibrium in a finite number of steps [55].

In [56], "cautious" agents were considered: their trust to received messages depends on the content of messages (opinions expressed in them). In order to reflect this dependence, a trust function $G(x, u)$ was introduced, where x and u denote the agent's opinion and message received by him, respectively. A series of assumptions were accepted regarding the properties of the trust function: different combinations of assumptions lead to different formalizations of the trust function corresponding to: an agent trusting to the received messages regardless of the content, a conservative agent, an innovator agent, a moderate conservative agent, and a moderate innovator agent. For example, the trust function of a moderate conservative agent has the form

$$G(x, u) = \beta\, [1 - (1 - \exp \times \\ \times(-\gamma\, |x - u|))\, \exp(-\gamma\, |x - u|)],$$

where β and γ are constants. In a practical interpretation, the agent selects and perceives information coinciding with his opinion until the disagreement becomes significant enough. Under very large deviations, the probability that he will notice such information increases. In the general case, the controlled opinion dynamics in a social network are described by

$$x_i^k = a_{ii} x_i^{k-1} + G_i\left(x_i^{k-1}, u^{k-1}\right) u^{k-1} + \\ + \sum_{j \in N_i} a_{ij} G_i\left(x_i^{k-1}, x_j^{k-1}\right) x_j^{k-1},\ k = 1, 2, \ldots,$$

where u denotes an external control (e.g., the media), and the individual trust functions $\{G_i(\cdot)\}_{i \in N}$ are such that the normalization condition holds. Within this model, the matrix A reflects the trust of agents to *information sources*; the trust functions, the trust of agents to *the content of information*. In a particular case of a homogeneous and regular network, an optimal informational control problem was stated: find a sequence of controls maximizing an efficiency criterion. This problem can be solved by standard methods. The dependence of the agent's degree of trust to the messages of other agents on their content led to the development of another model [8] of two interrelated processes: the propagation of actions through the network and the formation of agents' opinions. According to numerical simulations, various information communities can be formed in the network as a result of exchanging the beliefs.

We have discussed opinion dynamics models of "heuristic" individuals in a social network with continual beliefs. Generally speaking, these kinds of models—with gradually changing opinions—seem to be most natural. This fact was confirmed by studies in social psychology and behavioral economics. Particularly, as noted in [57], society (Indian villages) is divided into two types of agents: Bayesian agents and those acting by DeGroot's rule. Nevertheless, there are numerous publications where the opinions have ordinal or even nominal scales, etc. Models *with discrete beliefs* (*opinions*) also include voter models [58], majority models, and threshold models [59]. Many of these models, considering the network structure of interactions, are also known as models of the propagation of activity (information) in the network; for example, see the paper [60]. Here are some examples of voter models with discrete beliefs of individuals that illustrate the effect of disagreements in the network.

The paper [61] considered a set of N agents on an $L \times L$ regular lattice ($L^2 = N$). Agent $i \in \{1, 2, \ldots, N\}$ chooses an action $a^{(i)} \in A = \{a_1, \ldots, a_g, \ldots, a_G\}$, guided by his opinion according to a rule $r^{(i)} \in R = \{r_1, \ldots, r_k, \ldots, r_K\}$. The peculiarity of this model is the possibility of specifying multiple relations between opinions-rules and actions: rules are exclusive (one action is mandatory, the rest are prohibited) and inclusive (one of several actions may be performed with equal probability). The agents know the sets A and R and the relation matrix S of dimensions

$K \times G$. They can also observe the actions of neighbors in the network but not their opinions. At the initial step, each agent i randomly chooses a rule $r^{(i)} \in R$ with equal probability and acts according to it. At each next step τ, a randomly chosen agent i updates his beliefs about the rule followed by his neighbor $j \in M_i$ based on the observed action $a^{(j)}(\tau)$:

$$P^{(i)}\left(r^{(j)}(\tau)=r_k|a^{(j)}(\tau)\right)=\frac{P^{(i)}\left(a^{(j)}(\tau)|r_k\right)}{\sum_{k=1}^{K}P^{(i)}\left(a^{(j)}(\tau)|r_k\right)}.$$

Then—at the same step—agent i accepts an action rule in accordance with the probability of its use by the network neighbors.

As was established, *various information communities exist in the network if the agents apply inclusive rules*. If the agents apply only exclusive rules, the model reduces to the classical voter model.

A voting-based belief formation model richer by practical interpretations was proposed in [62]. As noted, long-term disagreements are rare in traditional economic models of social learning, although disagreements arise more often in reality than consensus. Many economic models of social learning rest on the assumption that each new piece of information received by an agent is true: the agent observes an element of the partitioned state space, containing the true state. This assumption (in some cases implicit) leads the agents to a consensus. The concept of information processing by agents was also introduced: agents can receive false information, making disagreements in such models a common outcome of network interactions. Two classes of axioms were presented for the belief updating rules of agents: the axioms of willingness-to-learn (the updating rule allows the agent to learn and reach the true estimate of the state of the world) and the axiom of non-manipulability (the updating rule leads the agent to the same belief when receiving the same information regardless of its form). Different updating rules are possible depending on the selected combinations of axioms. The author discussed two of them in detail:

– the agents adhering to the stubborn updating rule never give up their beliefs;

– the agents follow the stubborn updating rule, but they can completely change their beliefs if the new information completely eliminates the uncertainty about the state of the world.

The agents' interaction protocol is as follows: at each discrete time instant (step), an edge ij of the social network graph is selected randomly; agent i reports a randomly chosen statement P included in his belief B_j to agent j; then agent j updates his belief according to a rule $U_j(B_j, P)$, where the statement P is a subset of the states of the world Ω, i.e., $P \subseteq \Omega$. The network agents will reach a consensus if, at some step, there is a statement P^* such that $P(B_i) = P^*$ for each agent i, where $P(B_i)$ is the intersection of all statements from the beliefs of agent i. As it turned out, to reach a consensus in the network, rather strong assumptions about the truth of the initial beliefs and the number of stubborn agents have to be accepted. As was established, *the formation of different information communities in large networks is possible and even inevitable.*

CONCLUSIONS

Part II of the survey has considered the formation of information communities in societies with a nontrivial network structure. Individuals—the members of society—interact with each other within this structure. Observing the actions of his neighbors in the network, an individual (agent) can obtain additional information about the issue of interest.

Rational agents in such networks reach a consensus in the long run (come to a true or false agreement, depending on the conditions imposed on the topology and/or their initial beliefs). It is necessary to relax the agents' rationality requirement to obtain different information communities. In the first class of the models discussed, the agents are conditionally Bayesian: for example, they have "naive" beliefs about the composition and structure of the network or naively take into account the signals of neighbors. In the second class of the models, the agents are "simple" and forming their beliefs based on heuristic rules. In the case of individuals unconditionally trusting to the actions of their network neighbors, a consensus is a common outcome of network interactions: a single information community will appear. However, it is possible to specify the conditions for forming various medium-term (metastable) information communities. As has been demonstrated, stable information communities differing from each other may appear when: (a) in addition to social influence, individuals are affected by other socio-psychological factors (homophily, the inclination to confirm one's own point of view, etc.) and (b) along with the true information, false information propagates through the network.

Part III of the survey will consider empirical studies related to the existence of information communities in real social networks and their features.

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A SELF-ORGANIZATION MODEL FOR AUTONOMOUS AGENTS IN A DECENTRALIZED ENVIRONMENT[1]

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Abstract. A self-organization model for autonomous agents operating in a transparent decentralized environment is developed and investigated. Transparency means that all information about the environment and the agents' community is open. Each agent informs the entire community about his current resources and intentions. The environment consists of cells, and during operation, each cell can generate a new resource using the resources received from agents. Each agent is also aware of the efficiency and resources of the cells. The agent-based approach is adopted to consider the efficient allocation of agents' resources in cells and analyze different resource allocations. Each agent acts rationally based on his goals. An iterative resource allocation method is proposed, in which the agents exchange information to make their decisions. Computer simulations are carried out for several modes of operation: 1) without learning but with iterations, 2) with learning and iterations, 3) without learning and iterations, and 4) with learning but without iterations. As indicated by the simulation results, the total resource of the agents' community is significantly higher in the model with learning and iterations; due to self-organization and learning, the agents are distributed so that their number in each cell is small. According to the experimental evidence, learning works only in combination with iterations.

Keywords: multi-agent systems, self-organization, decentralization, transparent environment.

INTRODUCTION

In research into artificial intelligence, the theory of *multi-agent systems* has become widespread in recent decades. The multi-agent approach is used in optimization and control, collective behavior and market modeling, and investment allocation. Unlike dynamic systems, discrete-event modeling, and system dynamics, the individual characteristics of agents and their local interaction are important in this approach: the model can be constructed bottom-to-top. Therefore, it is possible to observe how the interaction of agents affects the overall behavior of the entire system. Currently, there exist many models and lines of research in this area [1–4]. One line is *self-organizing* multi-agent systems [5, 6]. The importance of *self-organization* was emphasized by W. Ashby [7]. Multi-agent systems allow studying self-organization processes and describing complex systems and have high flexibility. Also, note some papers on multi-agent systems and related ones on robot communities [8, 9].

The theory of multi-agent systems often operates two closely related concepts: distributed artificial intelligence (DAI) and decentralized artificial intelligence (DzAI). Let us explain their difference. Speaking about the problems solved within these approaches, we indicate that DAI is intended for the joint global control of a distributed group of agents. The solution of such problems is joint because mutual information exchange helps perform one common task. Unlike DAI, DzAI focuses on the activities of an autonomous agent in a multi-agent world. This approach employs the concept of "agent" widely, referring to a subject who acts rationally based on his goals. Moreover, an autonomous agent may exist regardless of other autonomous agents. Autonomous agents can cooperate and exchange information in a common world for performing personal or global tasks. Thus,

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in DAI, a certain global task is initially defined, and it is required to design distributed objects to perform it. In DzAI, decentralized autonomous subjects are first defined, and the main problem is to study their behavior to understand what tasks they can perform [10]. In this paper, we will follow the decentralized approach.

The phenomena of "competition" and "cooperation" are of particular interest in multi-agent modeling. In the paper [11], R. Axelrod experimentally proved the profitability of cooperation for two competing players based on game theory and computer simulation. He constructed and studied a multi-agent model consisting of a community of competing agents in which each agent has independent decision-making. Due to cooperation (information exchange and transparent environment) and learning, more efficient operation of the entire system is possible.

This paper develops the authors' previous research into collective behavior processes in a transparent environment [12–14]. Note that "transparent environment" is not a strict term accepted in the scientific community. However, in recent years, it has been frequently encountered in socio-economic studies. We will assign it an interpretation close to the term "transparent market" in economics [15]. This paper aims at showing that in the course of competition and cooperation, autonomous agents can be distributed in cells—singly or in small groups—using learning and iterative information exchange about the environment's state.

1. MODEL DESCRIPTION

Let us give a formal description of the model. Consider some environment consisting of numbered cells. The number of cells is fixed and equal to M. Each cell is characterized by its efficiency k_i and resource R_{i0}, which does not change over time. Assume that during operation, each cell can generate a new resource using the available resource. In this paper, the resource is similar to the capital of investors and producers and the cells to producers [12–14]. Unlike the models from [12–14], the cell's resource does not change. The parameter k_i characterizes how efficiently cell i can process the available resource.

A community of N agents operates in the environment. Each agent j is also characterized by his resource K_j. The agent allocates a share of his resource to some cell, receiving back a share of the generated resource. Moreover, agent j receives a share of the generated resource from cell i proportionally to his contribution to this cell. Agents operate during T periods in the transparent environment in the same way as described in [12–14]: they have complete information about the efficiency of cells and the total resource of each cell after receiving resources from other agents.

Note that time is divided into periods, and each period has a sufficient number of iterations. The periods are numbered by $T = 1, 2, \ldots, N_T$ and the iterations by $t = 1,2, \ldots, t_{\max}$. One period T includes several stages:

– stage 1, representing an iterative process in which the agents decide on a resource allocated by a particular agent to a certain cell,

– stage 2, allocating agents' resources to cells,

– stage 3, receiving newly generated resources from the cells,

– stage 4, agents' learning.

At the beginning of period T, agents choose shares of their resources to be allocated to particular cells. This decision is made during an *iterative process* (step 1), considered in detail below. Each agent can select any number of cells for resource allocation.

Let us first describe stages 2 and 3 in period T. Assume that after receiving the resources from all agents (stage 1), cell i has the total resource

$$R_i = R_{i0} + \sum_{j=1}^{N} r_{ij} , \tag{1}$$

where r_{ij} is the resource allocated by agent j to cell i. The resource that all agents in aggregate can receive from cell i is equal to

$$E_i(R_i) = \exp(-k_s s_i) k_i \varphi_i(R_i) , \tag{2}$$

where k_i denotes the efficiency of cell i ($0 < k_i \le 1$); k_s is the resource spent by the cell on one agent (for example, the resource consumed for interacting with the agent); s_i is the number of agents selecting cell i; φ_i stands for a cell performance function given by

$$\varphi_i(x) = \alpha_1 \left[1 - \exp(-\alpha_2 x)\right], \tag{3}$$

where α_1 ($\alpha_1 \in \mathbf{R}$) and α_2 ($0 < \alpha_2 \le 1$) are some parameters. Figure 1 shows the graph of the performance function (3). In contrast to the paper [13], the performance function is nonlinear and more flexible for adjustment. Note also that formula (2) for calculating the new resource generated by the cell differs from the one considered previously [12–14] by the factor $\exp(-k_s s_i)$. This factor in (2) reduces the generated resource: the more agents interact with the cell, the greater the decrease will be. In the papers [12–14], such consumption was not taken into account for producers. As shown by computer simulations below, the agents with this consumption can be learned and distributed in cells in small groups.

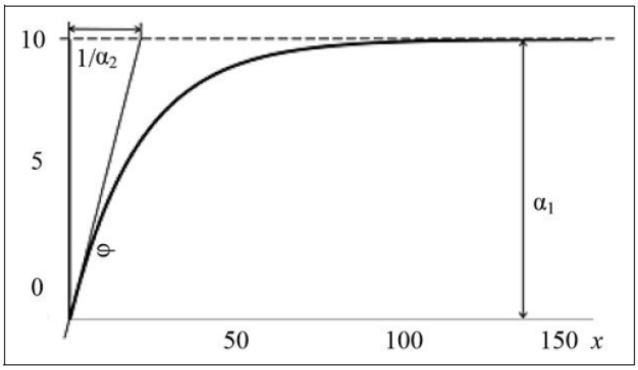


Fig. 1. Graph of performance function with $\alpha_1 = 10$ and $\alpha_2 = 0.05$.

At stage 3 of period *T*, the resource received by each agent from cell *i* is calculated by the formula

$$P_{ij} = E_i(R_i)\frac{r_{ij}}{\sum_{l=1}^{N} r_{il}},$$

Since the environment is transparent, each agent has information about the efficiencies of all cells and the intentions of all other agents.

The total resource received by agent *j* in period *T* is the sum

$$SP_j = \sum_{i=1}^{M} P_{ij}.$$

Next, the resource of agent *j* is increased by the value SP_j:

$$K_j(T) = K_j(T-1) + SP_j.$$

The total resource of the entire community at the end of period *T* is the sum

$$SK(T) = \sum_{j=1}^{N} K_j(T).$$

The agents choose the shares of the resources to be allocated to a particular cell during the following iterative process (stage 1). At the first iteration, the agents consider the efficiencies of all cells and the resource of a given cell for estimating the effect obtained from it. The estimates are calculated by the formula

$$A_{ij} = d_{ij}k_i\varphi_i(R_{i0}), \tag{4}$$

where d_{ij} specifies the current degree of belief of agent *j* to cell *i*. At the beginning of operation, the degrees of belief are all equal to 0.1. During the iterative process, the degrees of belief remain the same: they will change with learning at stage 4 of period *T* (see the description below).

At the second and subsequent iterations, the intentions of other agents are considered, and the estimates are calculated by the formula

$$A_{ij} = d_{ij}P_{ij} = d_{ij}\exp(-k_s s_i)k_i\varphi(R_i')\frac{r_{ij}}{\sum_{l=1}^{N} r_{il}}, \tag{5}$$

where R_i' is the expected resource of cell *i* after the contributions of all agents; r_{il} denotes the resource to be allocated by agent *l* to cell *i* (see the previous iteration). The resource R_i' is calculated by formula (1). Thus, the value R_i' is iteratively recalculated considering the intentions of all agents. The resource received by the cell is determined at the last iteration. In computer simulations, the model's behavior was studied for the modes with iterations and without iterations. In the latter mode, the estimates are once calculated by the agents using formula (4). (Formula (5) is not involved here.)

After receiving the estimates, the agents choose shares of their resources for allocating to each cell. The resource allocated by agent *j* to cell *i* is calculated by

$$r_{ij} = K_j\frac{A_{ij}}{\sum_{l=1}^{M} A_{lj}},$$

where K_j specifies the resource of agent *j*. A sufficient number of iterations are performed; at the last iteration, each agent chooses a share of his resource for allocating to a particular cell. This share is equal to the resource r_{ij} obtained at this iteration.

At the end of each period *T*, the agents are learned (stage 4). Learning takes place without a teacher by changing *the degrees of belief* to the cells. As soon as the agent knows the resource received from the cell (stage 3), he recalculates the current degrees of belief according to the rule

$$d_{ij}(T+1) = d_{ij}(T) + \beta Q(P_{ij})\left[1 - d_{ij}(T)\right] - \gamma d_{ij}(T),$$

where β ($0 < \beta \le 1$) is the learning rate; $Q(x) = x / (1 + x)$; P_{ij} represents the resource received by agent *j* from cell *i*; γ ($0 < \gamma < 1$) denotes the "forgetfulness" parameter. Thus, the values d_{ij} are reestimated depending on the resource received by the agent from a particular cell. The greater the resource received is, the higher the agent's belief to this cell will be. Note that if the profit increase is insignificant, the degree of belief will be reduced. The last term characterizes a decrease in the degree of belief due to skills "forgetting." In computer simulations, the modes with learning and without learning were compared. In the mode without learning, the degrees of belief remain the same between periods: each period has no stage 4.

This learning algorithm allows adjusting the degrees of belief so that the agents are distributed in different cells. Different distribution patterns were analyzed using computer simulations.

2. SIMULATION RESULTS

For the model under consideration, we developed a computer program and carried out numerical experiments. The following values of the basic parameters were used: the number of periods $N_T = 100$; the number of iterations in each period, $t_{max} = 150$; the number of agents $N = 5$, 10, or 20; the number of cells $M = 10$ or 30; the performance function parameters $\alpha_1 = 10.0$ and $\alpha_2 = 0.05$; the learning rate $\beta = 1.0$; the "forgetfulness" parameter $\gamma = 0.8$; the resource spent by the agent in each cell, $k_s = 0.3$. The initial resources of all agents and the efficiencies of all cells were random and uniformly distributed on the range [0, 1]. Computer simulations were carried out for several operating modes of the model:

– mode 1 (without learning but with iterations),
– mode 2 (with learning and iterations),
– mode 3 (without learning and iterations),
– mode 4 (with learning but without iterations).

2.1. Convergence of iterative process

Figure 2*a*,*b* shows the final total resource of the agents' community depending on the number of iterations in the last period (modes 1 and 2). The data were averaged over one hundred different calculations. Clearly, the iterative process in mode 1 converges quickly. In mode 2, the iterative process is convergent as well.

Figure 3 presents the difference between the total resource obtained at iteration $(t + 1)$ and during t iterations in mode 2. According to the simulation results, the total resource of the agents' community in this node is much higher than in mode 1; for details, see subsection 2.2. Due to this effect, the number of iterations t_{max} was set equal to 150.

2.2. Simulation results

Now we demonstrate that in mode 2, the agents' community accumulates a larger resource than in modes 1, 3, and 4. Figure 4 shows the total resource dynamics for the agents' community in these modes. Clearly, the total resource of the agents' community in mode 2 is much higher. At the same time, the total resource almost coincides in modes 3 and 4 (without iterations); in mode 1, the total resource slightly differs from that in modes 3 and 4. In modes 3 and 4 (without iterations), the agent's resource grows much slower than in those with iterations. Hence, we will analyze the modes with iterations only (modes 1 and 2), comparing them with one another.

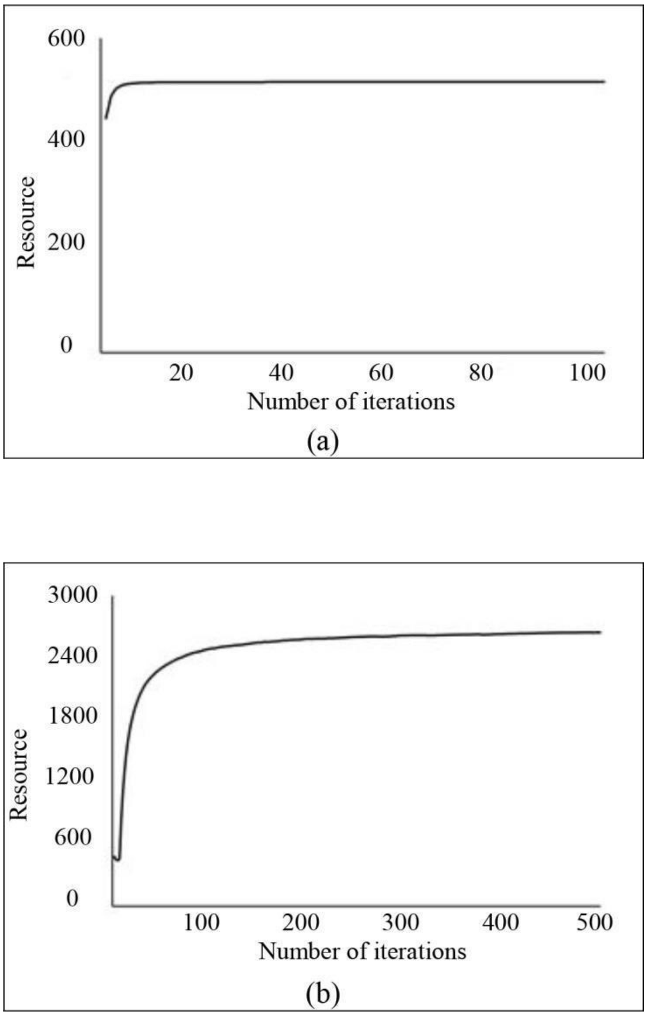


Fig. 2. Total resource of agents' community depending on the number of iterations ($N = 5$, $M = 10$, $T = 100$):
(a) mode 1 and (b) mode 2.

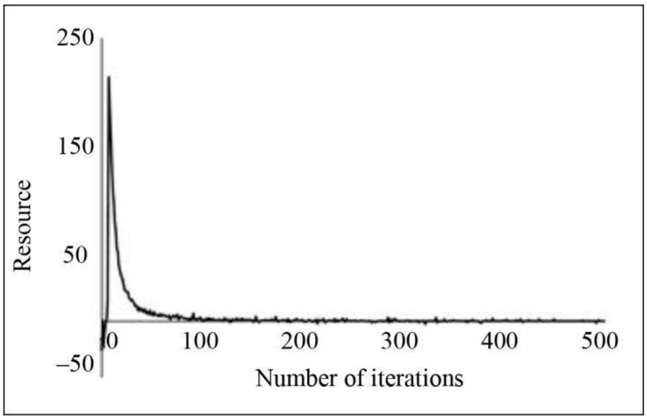


Fig. 3. Difference between the total resource obtained by agents' community at iteration $(t + 1)$ and during t iterations in mode 2 with $T = 100$.

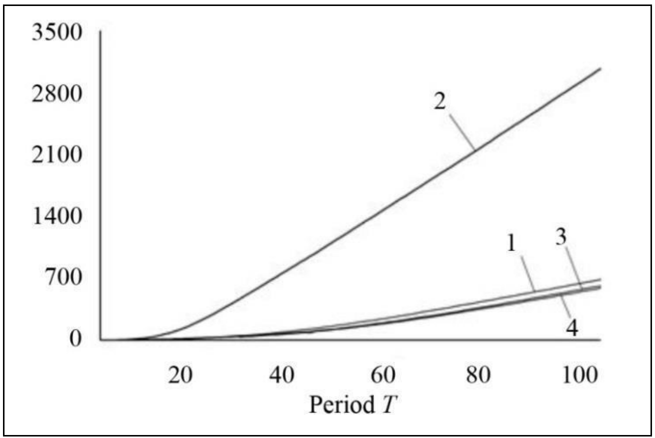


Fig. 4. The role of learning and iterations. Dynamics of community's total resource (N = 5, M = 10, t_{max} = 150) in different modes:
(*1*) without learning but with iterations, (*2*) with learning and iterations, (*3*) without learning and iterations, and (*4*) with learning but without iterations.

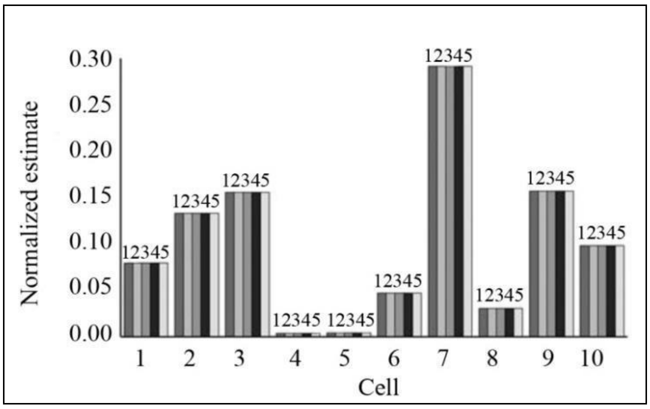


Fig. 5. Normalized estimates of cells at last iteration in mode 1: t_{max} = 150 in period T = 2, N = 5, and M = 10. Agents are indicated on the diagram columns.

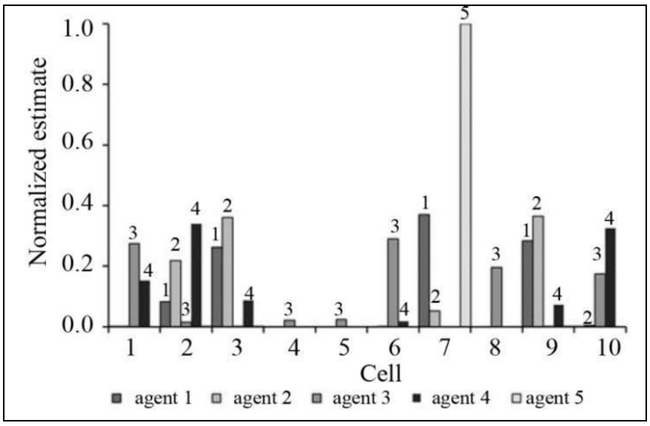


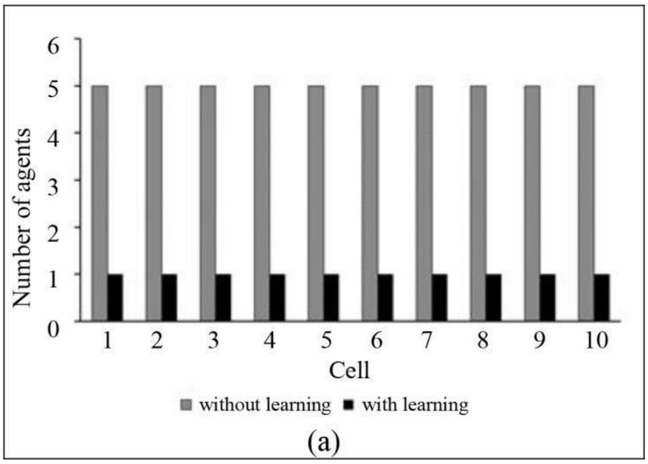
Fig. 6. Normalized estimates of cells at last iteration in mode 2: t_{max} = 150 in period T = 2, N = 5, and M = 10. Agents are indicated on the diagram columns.

Thus, learning and iterations are effective only together. To explain this result, consider how agents rank cells in different modes according to formula (5). Let the most efficient cell be the one with the largest product $k_i R_{i0}$. First, we analyze the cell ranking procedure in mode 1 (without learning but with iterations). Figure 5 shows the normalized estimates calculated by the agents for each cell at the last iteration t_{max} = 150 of period T = 2. Clearly, the agents rank the cells in the same way: cell 7, most efficient, is followed by cells 3, 9, 2, 1, 6, 8, 5, and 4. Therefore, *cell ranking* is effective in mode 1. In this case, the agents do not compete with each other but *cooperate*: between iterations, the contribution to the more efficient cell increases synchronously for all agents.

Now let us include learning in the model and consider mode 2. Figure 6 shows the corresponding simulation results: the cell rankings according to formula (5) are different for different agents. When learning is included, the cell ranking procedure is supplemented by *the competition of agents*. The agents having a greater contribution to a given cell displace other agents from this cell gradually, changing the degree of belief between periods. For example, agent 5 allocates his entire resource to cell 7 (the most efficient one). Agents 1 and 2 also allocate some shares of their resources to this cell, but the picture will change over time (see Table 1): only agent 5 will stay in cell 7, whereas agent 1 will pass to cell 9, displacing the other agents from it.

Next, we examine the agents' distribution in cells in modes 1 and 2. First of all, note that each agent can select an arbitrary number of cells. In addition, if several agents select the same cell, then each agent receives less resource from the cell (see formula (2)). As it seems, the agents benefit from being distributed in cells singly or in small groups. Figure 7*a*,*b* shows how many agents select each cell in two modes with iterations – modes 1 (without learning) and 2 (with learning) – in the last period T = 100. The agents' distribution depends on their number and the number of cells as well.

We begin with the case when there are twice as many cells as agents (Fig. 7*a*). Clearly, during the operation of the entire community in the mode *with learning*, exactly one agent selects each cell. As emphasized above, one agent can allocate his resource to several cells. Despite this opportunity, in the mode with learning, the agents are singly distributed in cells. This effect is achieved through *iterations and learning*. Here we observe self-organization in the community of agents. In the mode *without learning*, each agent allocates his resource to all available cells according to the estimates A_{ij}. Clearly, each cell is selected by all ten agents. In this case, the resource allocated by an agent will depend on the cell's efficiency and resource.



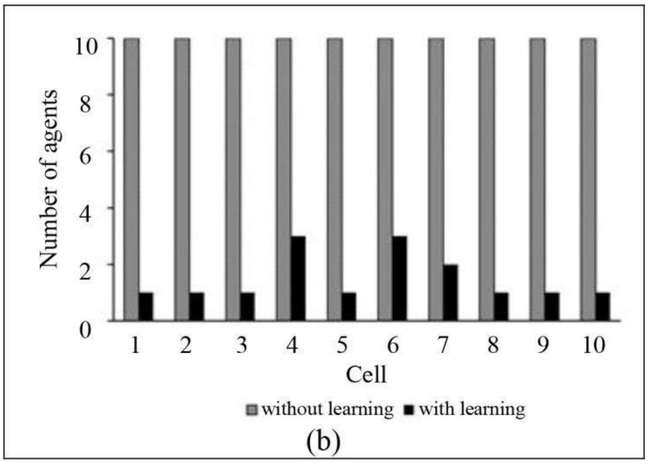


Fig. 7. Distribution of agents in cells:
(a) $N = 5$ agents and $M = 10$ cells, (b) $N = 10$ agents and $M = 10$ cells.

Now consider the case when the numbers of cells and agents coincide (Fig. 7*b*). In this case, the agents in the mode with learning are distributed in cells in small groups: one, two, or three agents per cell. In the mode without learning, the agents allocate their resources to the cells similarly to the previous case.

Increasing the number of agents and cells, we obtain results similar to Fig. 7; see Fig. 8. The agents' distribution is considered in the last period $T = 100$. Clearly, due to self-organization and learning, the agents can be distributed in small groups of one, two, or three agents per cell. In contrast, in the mode without learning, each agent allocates a share of his resource to each cell available. Moreover, more agents select more efficient cells. (In the experiment under consideration, these are cells 1, 2, and 20.)

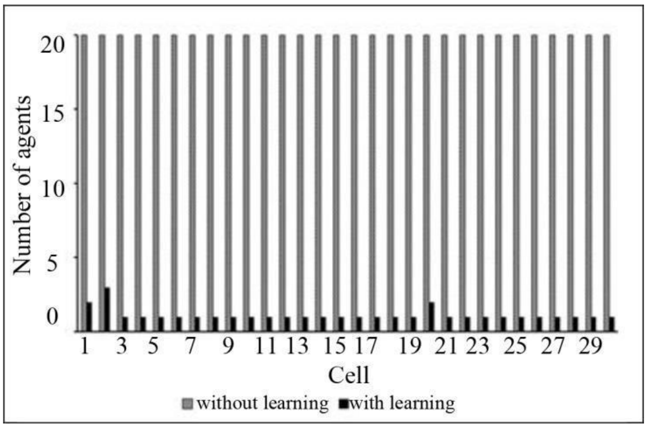


Fig. 8. Distribution of agents in cells: $N = 20$ agents and $M = 30$ cells.

Consider mode 2 (with learning) to investigate the agents' distribution in cells when there are more cells than agents ($M = 10$ and $N = 5$). The resulting distribution is presented in Table 1. The rows are sorted in the descending order of the initial resource of the agents.

Due to self-organization, the agents select disjoint cells; see the last column. In addition, two agents (2 and 3) allocate their resources to several cells. Note that agent 3, having the smallest initial resource, chooses less efficient cells with a smaller value of the product $k_i R_{i0}$. According to Table 1, all other agents select cells with a greater value of this product. At the same time, the agent with the maximum initial resource has a better chance of capturing a more efficient cell in the competition. The agent with the least resource (in this experiment, agent 3) obtains less efficient cells. Despite this fact, as the result of the operation of the entire community, the total resource of agent 3 becomes greater compared to agent 5 choosing the most efficient cell. This phenomenon can be explained as follows. Agent 3 allocates resources to a larger number of cells, and since none

Table 1

Distribution of agents in cells

Agent	Agent's initial resource	Efficiency of cells selected by agent	Cell resource	Cells selected by agent
5	0.94	0.97	0.72	7
1	0.54	0.57	0.99	9
2	0.48	0.66. 0.64	0.67. 0.83	2, 3
4	0.33	0.91	0.27	10
3	0.25	0.86. 0.08. 0.02. 0.23. 0.15	0.24. 0.21. 0.73. 0.92. 0.96	1, 4, 5, 6, 8

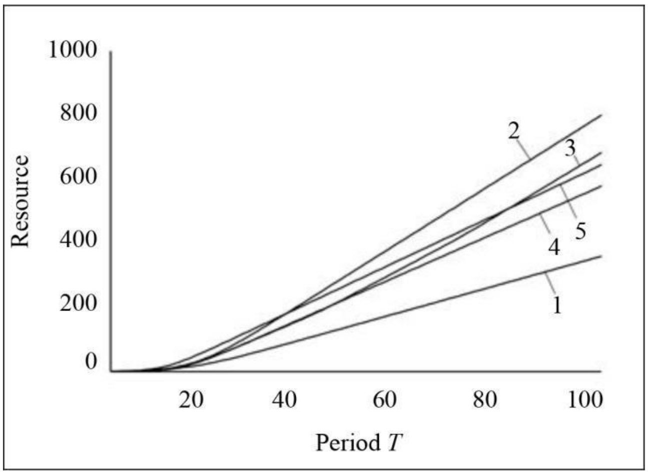


Fig. 9. Dynamics of agents' resources in mode 2, N = 5 and M = 10.

else chooses these (less efficient) cells, the entire resource generated by these cells is taken by agent 3. The results are shown in Fig. 9, where curves 1–5 indicate the resources of the corresponding agents from Table 1.

Thus, *learning and iterative information exchange* reduce the number of agents in a separate cell due to their competition. The agents with a greater initial resource choose more efficient cells, i.e., the ones with higher estimates in the iterative process. Since the new resource generated by a cell is distributed among all agents proportionally to their contributions to it, the agent with the greatest contribution will have the maximum return. Accordingly, his belief to this cell will increase more compared to the other agents allocating their resources to it. The situation described above will be repeated in the next period: it will be beneficial for other agents to choose less efficient displaces all other agents from a more efficient cell. The rest of them are distributed in the remaining cells, considering their estimates during the iterative information exchange. This case applies to a situation when there are significantly more cells than agents. If the numbers of cells and agents coincide, one or more agents may stay in more efficient cells.

Also, an interesting case is when all cells have the same efficiency and the same resource. For example, let these values be equal to 0.9 for all agents, N = 5, and M = 10. The observations were carried out for the period T = 100. According to the simulation results, the agents cannot be distributed in cells. Each agent allocates a share of his resource to each of the available cells. This effect occurs because all estimates and degrees of belief are the same, and each agent allocates his resource uniformly to all possible cells. This problem can be solved by randomizing the initial degrees of belief d_{ij}. Then the simulation results show that the agents are singly distributed in each cell. In this case, the competition is specified through the degrees of belief: it does not matter which cell an agent chooses. For clarity, Table 2 presents the belief matrix of the agents obtained in the last period T = 100. Clearly, each cell contains one agent (see the last column of Table 2). However, some agents select several cells, e.g., agents 1, 4, and 5.

Figure 10 shows the total resource of the agents' community depending on the period T in the two cases described. Clearly, when the agents are distributed singly in each cell, the total resource of the entire community is higher.

Table 2

Belief matrix of agents

Cell	Agent					Number of agents
	1	2	3	4	5	
1	0.00	0.00	0.00	0.00	**0.52**	**1**
2	0.00	0.00	0.00	0.00	**0.52**	**1**
3	0.00	**0.52**	0.00	0.00	0.00	**1**
4	0.00	0.00	0.00	**0.52**	0.00	**1**
5	0.00	0.00	**0.52**	0.00	0.00	**1**
6	0.00	0.00	0.00	0.00	**0.52**	**1**
7	0.00	0.00	0.00	**0.52**	0.00	**1**
8	**0.52**	0.00	0.00	0.00	0.00	**1**
9	0.00	0.00	0.00	0.00	**0.52**	**1**
10	**0.52**	0.00	0.00	0.00	0.00	**1**
Number of cells	2	1	1	2	4	10

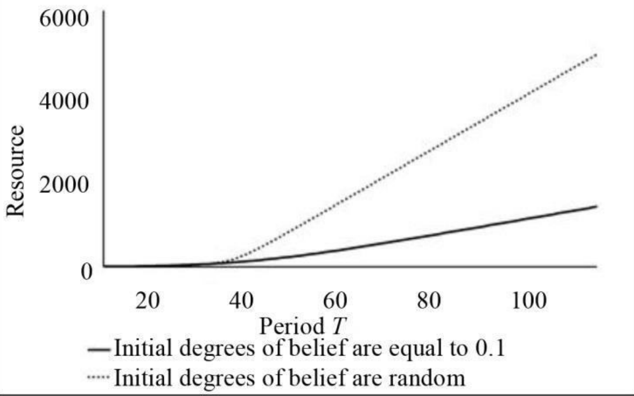


Fig. 10. Dynamics of agents' resources in mode 2 with $N = 5$, $M = 10$, and $k_i = 0.9$.

We also examined the effect of significant parameters (the learning rate β and the "forgetting" parameter γ) on the model's behavior.

Figure 11 shows the community's total resource depending on the "forgetfulness" parameter γ for β = 1.0 in the period T = 100. The best results were observed for β = 1.0 and γ = 0.8.

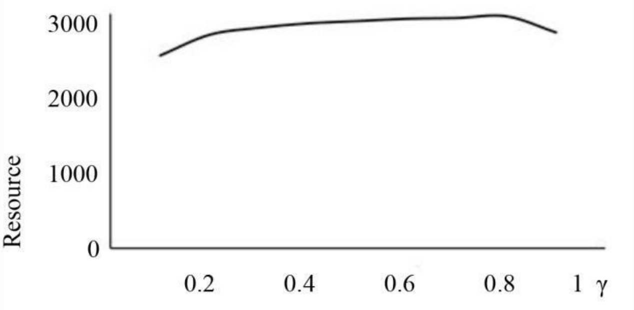


Fig. 11. Community's total resource in mode 2 depending on parameter γ for β =1.0.

CONCLUSIONS

The previous research into collective behavior processes [12–14] was focused on the interaction of different agents (investors and producers) in a transparent environment. Simultaneously, a simple attempt was made to learn investors by adjusting their degrees of belief to producers. However, this attempt was unsuccessful: learning did not significantly increase the resource of the economic community. In this paper, the previous versions of the models have been developed. The rule of interaction between the model's elements (particularly between cells and agents) has been modified by introducing an additional consumption of the agent's resource for interaction with other agents. As a result, the agent's learning has become effective.

According to the computer experiments presented above, this version of the model is operational. An important result of this paper is that under learning and iterative information exchange, the agents are distributed so that their number in each cell is small, and the total resource accumulated by the entire community is higher than in the model without learning and iterations.

The proposed algorithm can be applied to collective behavior problems. The developed model can also be used to study competition and cooperation in economic and social sciences, in which these categories play an important role.

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KEY AREAS FOR IMPLEMENTING MANAGERIAL INNOVATIONS WITHIN DOMESTIC AND MULTINATIONAL COMPANIES OPERATING IN RUSSIA

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Abstract. The highly complex, ambiguous, and turbulent business environment forces the leading multinational companies to search for new strategic capabilities, and managerial innovations are treated as an imperative for this development. However, top-management of the domestic companies operating in the Russian market is not focused sufficiently on managerial innovations. This paper considers the process of managerial innovations and key areas of their implementation within domestic and multinational companies operating in Russia. The empirical study described below involves 1025 employees from 791 companies operating in Moscow and Moscow region. According to the collected data, the companies operating in the Russian market primarily focus on employee motivation and building an effective communication process as the priority areas for implementing managerial innovations. Moreover, the type of economic activity, business size, and the company's degree of internationalization are taken into account in the empirical study. Several peculiarities of the implementation areas of managerial innovations for domestic and multinational companies operating in Russia are identified.

Keywords: managerial innovations, process of managerial innovations, implementation of managerial innovations, implementation areas of managerial innovations, domestic companies, multinational companies.

INTRODUCTION

Under the conditions of volatility, uncertainty, complexity, and ambiguity (VUCA), managerial innovations (MIs) are an integral part of forming the competitive advantages of most multinational companies. As noted in [1], managerial innovations are more important for creating a competitive advantage than R&D results. The global scientific community also pays great attention to studying the formation and implementation of managerial innovations in managing multinational companies; for example, see the papers [2–4] and other publications.

The term "managerial innovations" was first used by J. Kimberly. He defined it as "...programs and techniques related to strategy, structure, and processes, representing the first, not previously implemented, transition from the current state of management, affecting the essence, quality, and quantity of information available in the decision-making process..." [5]. Later, G. Hamel gave a broader definition of this concept, noting that managerial innovation is something that "changes the content of manager's work" [6]. Due to the keen interest in managerial innovations, such multinational companies as DuPont, GE, Procter, Visa, Linux, Toyota, and Whole Foods achieved outstanding success [6]. The Oslo Manual, the main methodological document in the field of innovations in the Organization for Economic Cooperation and Development (OECD), defines "managerial innovation" as an innovation representing "a new or improved business process for one or more business functions that differs significantly from the firm's previous business processes and that has been brought into use in the firm" [7].

Analysis of Russian and foreign researches in the field of managerial innovations for more than fifty years allows distinguishing two main processes of managerial innovations:

– the formation of a managerial innovation [2, 8–16],

– the implementation of a managerial innovation [2, 10–12, 14–17].

This paper considers the second process—the implementation of managerial innovations—and presents the authors' empirical study of the implementation areas of managerial innovations within domestic and multinational companies operating in the Russian market, based on the data obtained within the HSE research project "Study of managerial practices and innovations of Russian and global companies operating in Russia" in 2019–2020.

This paper consists of five sections as follows. Section 1 considers the theoretical aspects of the implementation of managerial innovations. In Section 2, the methodology and empirical base of the authors' study are described. Section 3 examines the key implementation areas of managerial innovations within Russian and multinational companies operating in the Russian market, highlighting the key features in the implementation areas of managerial innovations depending on the types of economic activity and business size. Section 4 presents an analysis of the implemented managerial innovations within domestic and multinational companies operating in the Russian market for 2016–2019. In Section 5, the limitations of this study and some lines for further research are discussed. In the Conclusions, the main outcomes on the implementation areas of managerial innovations within domestic and multinational companies operating in the Russian market are summarized.

1. IMPLEMENTATION OF MANAGERIAL INNOVATIONS

The implementation of managerial innovations was used in many English-language sources to describe the process of adopting new managerial practices, approaches, processes, and techniques [18–21]. Webster's Dictionary1 defines the term "implementation" as "the process of making something active or effective." In this paper, the implementation of managerial innovations is understood as the process of deciding by an organization to start using new managerial practices, approaches, processes, and techniques and using them as well. This interpretation reflects the organization's execution of the approaches and processes embedded in the essence of managerial innovation.

In domestic and foreign theoretical sources, various approaches to describing the implementation of managerial innovations were presented. A common feature of most approaches is the description of two subprocesses that are integral parts of implementing a managerial innovation:

– the decision to implement an MI (see the papers [13, 14, 16, 21–24]),

– the direct implementation of an MI (see the papers [13, 14, 16, 18, 23, 24]).

The authors [13] defined the decision to implement an innovation as the starting point of this process when company leaders decide to develop an idea and allocate resources. Simultaneously, it was noted that the managers' consent is a distinctive feature of this stage since direct implementation requires the consent and commitment of ordinary employees. The difference in the need to involve different levels of employees at different implementation stages of managerial innovations is a development of the idea expressed in the paper [25]. The cited authors drew attention to the fact that company leaders are not active innovators themselves but act as "arbitrators."

As we believe, the most interesting study of the decision process on implementing managerial innovations is the paper [26], which divided the decision process under consideration into internal and external validation. Thus, the authors expanded the concept of "decision-making on the implementation of managerial innovations" by company leaders: all participants of the process must agree with such a decision, and only then can the innovation be successfully implemented. For internal validation, the need for using internal examples of the new idea's efficiency and the concept of "small wins" was emphasized. For external validation, it was proposed to involve four main subjects: academic business schools, consultants, media, and professional associations. Combining internal and external validation allows overcoming barriers to innovation among employees and launches the immediate implementation and popularization of innovation.

The paper [18] defined "the implementation of innovation" as a transition period within which organization employees acquire the skills to use innovation and accept it as a new approach to work. "The transition period," according to the authors, is a critical gateway from deciding to introduce an innovation to its sustainable use and routinization. The fundamental problem in the innovation process is to guarantee the use of an innovation by the organization's employees to which it is directed. In other words, the problem is to change the day-to-day behavior of the employees.

As underlined in [20], a managerial innovation is implemented not at the moment of a corresponding decision within an organization but when this innovation begins to be actively used in it. According to the descriptions of the implementation process of general innovations and, in particular, managerial innovations

[1] https://www.merriam-webster.com/dictionary/implementation.

in the works of various authors, this process represents a separate element of the entire innovation process, and the decision to adopt an innovation does not guarantee the successful implementation of the innovation itself [27–30].

The authors [31] draw attention to the fact that the implementation process of innovations is cyclical since innovations are implemented after the ones implemented earlier. They proposed considering the implementation of a set of innovations within an organization rather than the implementation of a single innovation. In this regard, the implementation areas of managerial innovations in organizations become relevant.

There are some domestic and foreign studies of the implementation areas of managerial innovations. For example, a set of possible managerial innovations was presented in the report [32] by the Russian Managers Association, but without any clear logic for structuring these areas. Klevtsova [33] endeavored to structure the implementation areas, paying attention to the organizational structure, the introduction of new technologies, and the improvement of management techniques. Such an approach also seems controversial. Western researchers paid attention to studying specific types of managerial innovations such as structure and strategy, digital solutions, costing methods, agile method, etc. [34–38], considering various organization processes as the formation areas of managerial innovations.

Therefore, this area of research is of interest. A clear understanding of the priority implementation areas of managerial innovations is required depending on the type economic activity, the company's degree of internationalization, and its size for domestic and multinational companies operating in the Russian market.

2. METHODOLOGY AND EMPIRICAL BASE OF STUDY

For this study, *the main research question* (RQ) was formulated: In what key areas of management activities are managerial innovations implemented within domestic and multinational companies operating in the Russian market?

Two additional research questions were formulated to understand the specifics of the implementation areas of managerial innovations within domestic and multinational companies operating in the Russian market.

- RQ 1: What are the features of the implementation areas of managerial innovations depending on the types of economic activity, business size, and the company's degree of internationalization?
- RQ 2: What managerial innovations have been implemented in companies operating in the Russian market over the past three years?

The research questions were answered by analyzing the data obtained during the HSE research project "Study of managerial practices and innovations of Russian and global companies operating in Russia." Respondents in Moscow and Moscow region were polled using quantitative analysis methods in 2019–2020.

The respondents for the poll were selected using random non-repeated sampling with the following requirements:

– The sample should include respondents reflecting the age and gender characteristics of the employed population of Moscow.

– The sample should contain respondents with different work experiences.

– The respondents in the sample should differ by the level of their position in companies.

– The sample should contain respondents from companies with foreign capital and domestic exporting companies.

The diversification of companies in the sample is conditioned by their differences in implementing managerial innovations under the environment's peculiarities. Multinational companies have many distinctive features in implementing managerial innovations due to the existing international managerial practices outside the Russian market.

The sampling procedure with the above-mentioned requirements yielded a sufficiently representative base of respondents. Note that the empirical study involved 1025 employees from 791 companies. The age and gender structure of the sample of respondents is shown in Figure 1. This structure is close to that of Moscow's employed population,[2] indicating that the sample of respondents who took part in the empirical study is representative.

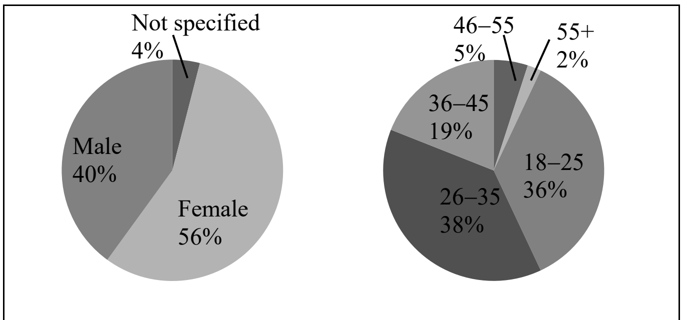

Fig. 1. Age and gender structure of the sample of respondents.

[2] Statistical report "Labor and employment of the population of Moscow in 2015". The Moscow Department of Labor and Social Protection, 2015.

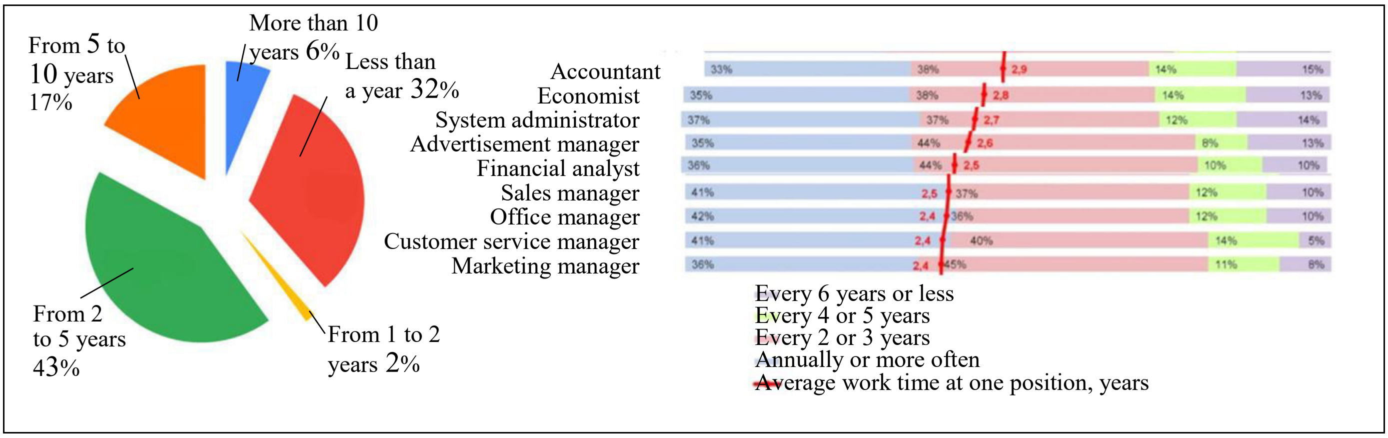

Fig. 2. Work experience structure of the sample of respondents.

The structure of various categories of employees with different work experiences in companies correlates with the data on the frequency of job changes among employees of domestic companies[3]; see Fig. 2.

The categories of employees differing by their gender, age, position, and work experience make the sample rather highly representative. Thus, we draw some conclusions about the characteristics of implementing managerial innovations at the company level, considering the specifics of individual groups of employees.

The position level structure of the sample of respondents is represented by all categories of employees (Fig. 3).

The structure of companies by their degree of internationalization is shown in Fig. 4. Since the sample included multinational companies and domestic exporting companies, we identified their inherent differences in the process of managerial innovations. These differences are due to another environment (global markets) in which the companies operate.

For answering RQ 1 on the implementation areas of managerial innovations, quantitative analysis methods were used. A closed list of processes (options) in an organization was formed, and the respondents were asked to select an appropriate option when answering each question of the poll. Based on practical experience, we identified the most innovative processes in the companies' activities, most often affected by managerial innovations. The list consisted of the following processes: internal communications, team building, negotiations, motivation, leadership, customer experience management, process management. Also, the respondents were offered the option "other" and the opportunity to specify an implementation area of managerial innovations, common in their company, but not included in the closed list.

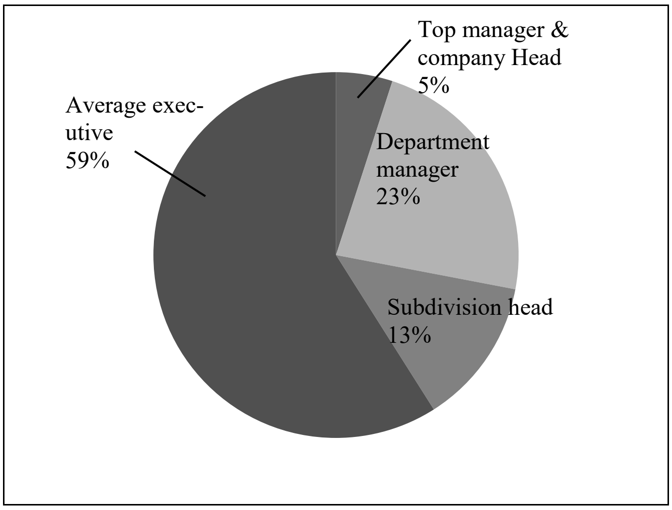

Fig. 3. Position level structure of the sample of respondents.

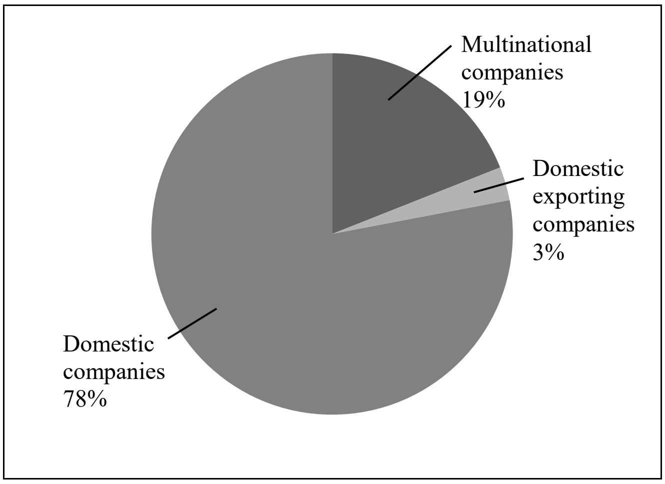

Fig. 4. The structure of companies by their degree of internationalization.

[3] https://www.superjob.ru/research/articles/111767/dolshe-vsego-na-odnom-meste-rabotayut-medsestry-i-uchitelya/

Process management as an implementation area of managerial innovations includes approaches and managerial practices for reducing the resources consumed by internal corporate processes. In other words, this type of managerial innovation is not aimed at increasing the productivity of processes (like the other implementation areas of managerial innovations mentioned in the study) but at reducing the cost of supporting and maintaining them (increasing the efficiency of processes). Within this study, the productivity of innovations of this type was not measured: this problem is very extensive and requires separate consideration.

To answer RQ 2, the respondents were asked an open-ended question on the most significant managerial innovations implemented in the company over the past three years. The response received within this poll block was grouped by the type of processes mentioned by the respondents and then analyzed for compliance with the implementation areas of managerial innovations within domestic and multinational companies over the past three years. As a result, we validated the closed list of the implementation areas of managerial innovations used in the poll.

3. KEY AREAS FOR IMPLEMENTING MANAGERIAL INNOVATIONS WITHIN DOMESTIC AND MULTINATIONAL COMPANIES

3.1. Identifying key areas of managerial innovations

According to the results of the empirical study, employee motivation is the most characteristic area for implementing managerial innovations within domestic and multinational companies operating in the Russian market (Figure 5): its share is 20% among all implementation areas mentioned by the respondents. Next, internal communications – 18% of all the respondents' answers – was ranked 2nd. Thus, every fifth innovation within the companies operating in the Russian market is implemented either to increase the efficiency of employee motivation, involving them in the process of achieving the company's goals, or to ensure internal communications. This conclusion is interesting and unexpected since, initially, customer experience management was considered a priority area for implementing managerial innovations within companies operating in the Russian market, closing the top 3 areas according to the empirical study results; see Fig. 5.

Initially, we considered process management within companies operating in the Russian market as a potential priority area for implementing managerial innovations. However, this process was ranked 4th in the list (14%), not entering the top 3 areas.

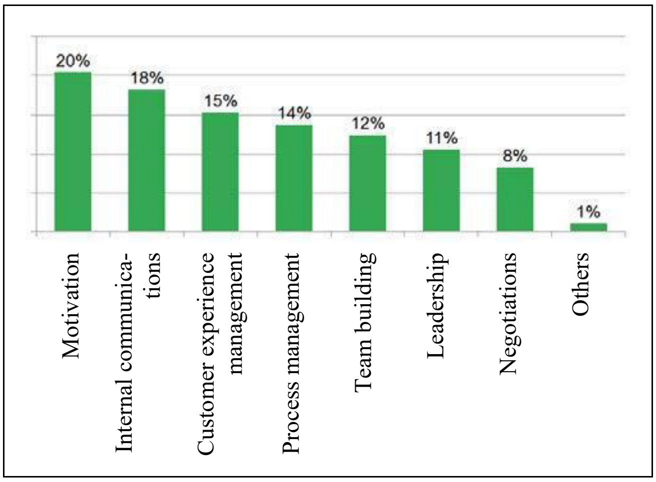

Fig. 5. Implementation areas of managerial innovations.

Team building (12%) and effective leadership (11%) have a slight difference by significance, but they are noticeably inferior to the leaders of the ranking list of the implementation areas of managerial innovations. Effective negotiation skills became an outsider in this study (8%).

For understanding the features and specifics of the implementation areas of managerial innovations for domestic and multinational companies operating in the Russian market, we analyzed these areas depending on various types of economic activity, business size, and the company's degree of internationalization. This analysis showed significant differences in the implementation areas of managerial innovations.

3.2. Features of the implementation areas of managerial innovations depending on the types of economic activity

The implementation areas of managerial innovations were examined by considering the following types of economic activity of the companies included in the empirical study:

– industrial production,
– transport, information and communication,
– wholesale and retail trade,
– hotel and restaurant business,
– publishing and printing activities,
– education and science,
– culture and sports,
– others.

According to the data analysis, the companies from the groups "industrial production," "education and science," and "culture and sports" differ by the significance of the implementation areas of managerial innovations from the general sample of the companies included in the empirical study; see Table 1 and Fig. 5.

Table 1

Implementation areas of managerial innovations by the types of economic activity

Type of economic activity	Implementation areas of MIs, %							
	Internal communications	Team building	Negotiations	Motivation	Leadership	Customer experience management	Process management	Others
Industrial production, including storage	26	11	8	16	8	13	15	2
Education and science	21	12	8	18	7	19	15	0
Culture and sports	14	16	11	20	13	14	11	0

For companies from the industrial production group, the most significant implementation area of MIs is internal communications (26%); motivation is noticeably less important, amounting to 16%. On the contrary, internal communications are most significant compared to the companies with other types of economic activity. In general, we can assert that for the companies from the industrial production group, there are some of the lowest values in all implementation areas of managerial innovations related to personnel: team building (11%), motivation (16%), and leadership (8%). We also emphasize a rather low degree significance for the areas related to customer interaction: negotiations (8%) and customer experience management (13%). Thus, the industrial production companies primarily focus on managerial innovations of internal processes, paying little attention to the processes related to personnel and customers, compared to the companies with other types of economic activity.

For companies from the education and science group, customer experience management is most significant (19%) among all companies considered. This process is of slightly greater importance than motivation (18%); see Table 1. Thus, among the companies operating in the Russian market, the companies related to education and science are most focused on customer experience management.

For companies from the culture and sports group, team building (16%) is more significant than process management (11%). Also, team building (16%) is more important than internal communications (14%) for the companies of this type; see Table 1. At the same time, note that process management for this group of companies is least significant compared to the companies with other types of economic activity. On the contrary, team building is most important compared to the companies with other types of economic activity. Simpler internal processes explain the obtained results for the companies of this group compared to other companies. Therefore, internal communications and process building are less important for them than team building and motivation of.

3.3. Features of the implementation areas of managerial innovations depending on business size

The structure of the respondents' answers depending on business size shows some differences in the importance of the implementation areas of managerial innovations compared to the general sample of companies; see Table 2 and Fig. 5.

Considerable differences were identified for negotiations. This area is most important for medium-sized companies (14%) and least for the large businesses (6%): the tougher competitive environment of medium-sized companies requires innovative approaches in negotiations with suppliers and buyers. In contrast, for the large businesses, the negotiations area is a more established business practice not requiring innovative approaches. Also, this difference in significance is explained by that many employees are not involved in the negotiation process in large companies, thereby not implementing managerial innovations in this area.

For medium-sized companies, approximately the same degree of significance is characteristic for all implementation areas of managerial innovations, except motivation (21%); see Table 2. According to the obtained results, the medium-sized businesses pay more attention to motivation and equally focus on all other implementation areas of managerial innovations.

Customer experience management as the implementation area of MIs is most important for small businesses (17%), while the least significance is noted for large businesses (13%); see Table 2. Such differences are explained by the levels of influence of their customers. For small businesses, the costs of consumers' switching are quite low, and customer loyalty is very important for this segment. This factor may be less significant for large companies due to the large volume of contracts, large discounts, and monopoly position.

3.4 Features of the implementation areas of managerial innovations depending on the company's degree of internationalization

The implementation areas of managerial innovations have some features depending on the company's degree of internationalization; see Table 3.

For the domestic exporting companies, internal communications (21%), customer experience management (17%), and negotiations (11%) are more significant than for other types of companies; see Table 3. These results are due to a tougher competitive environment for such companies and prompt decisions required to win and retain customers.

The multinational companies focus on customer experience management and process management noticeably less than the domestic ones (12% vs. 16% and 11% vs. 15%, respectively) when implementing managerial innovations. As we believe, the established practices of customer experience management and organization of internal processes reduce the need to implement managerial innovations in these areas for multinational companies. However, managerial innovations in leadership and team building are more typical for the multinational companies (Table 3). This situation is explained by a lower readiness for changing leadership styles among the domestic companies: their top management remains conservative when implementing new managerial practices in personnel management processes in general and leadership in particular.

Table 2

Implementation areas of managerial innovations by business size

Business size	Implementation areas of MIs, %							
	Internal communications	Team building	Negotiations	Motivation	Leadership	Customer experience management	Process management	Others
Large	19	13	6	22	12	13	14	1
Medium	14	13	14	21	11	14	13	1
Small	19	12	9	20	10	17	14	1

Table 3

Implementation areas of managerial innovations depending on the company's degree of internationalization

Company's degree of internationalization	Implementation areas of MIs, %							
	Internal communications	Team building	Negotiations	Motivation	Leadership	Customer experience management	Process management	Others
Multinational company	19	14	7	22	12	12	11	1
Domestic exporting company	21	13	11	21	7	17	10	0
Domestic company	18	12	8	20	10	16	15	1

Within the study, we analyzed the key features of the implementation areas of managerial innovations depending on the types of economic activity (Table 1), business size (Table 2), and the company's degree of internationalization (Table 3). According to the obtained results, the type of economic activity, business size, and the company's degree of internationalization affect and determine the specifics of the implementation areas of managerial innovations.

4. MANAGERIAL INNOVATIONS IMPLEMENTED WITHIN DOMESTIC AND MULTINATIONAL COMPANIES IN 2016–2019

In the study, the respondents were asked the following question: What are the most significant managerial innovations implemented within your company over the past three years? The results obtained (Fig. 6) reflect the general trend in the implementation areas of managerial innovations (Fig. 5). However, certain peculiarities do exist.

The respondents identified employee motivation as the most significant area of managerial innovations implemented within domestic and multinational companies operating in the Russian market. It amounted to 26% of the total number of the respondents' answers. Such results correlate with the poll data on the list of implementation areas of managerial innovations (Fig. 5), where the respondents rated the motivation process as most significant (20% of the total number of answers).

The advanced training of employees (17%) was ranked 2nd by significance among the managerial innovations implemented within the companies over the past three years. These results differ from the poll data on the implementation areas of managerial innovations (Fig. 5), in which the advanced training of employees was not highlighted. The reason is that we considered advanced training an integral part of innovations in other processes (motivation, internal communications, customer experience management, etc.). Nevertheless, the results obtained indicate the need to put the advanced training of employees in a separate block as one of the most significant areas for implementing managerial innovations.

Management system development (13%) was ranked 3rd by significance among the managerial innovations implemented over the past three years within domestic and multinational companies operating in the Russian market. We compared this process with the process management identified during the poll on the implementation areas of managerial innovations (Fig. 5). As mentioned above, process management as an implementation area of managerial innovations includes approaches and managerial practices for reducing costs (increasing the efficiency of processes). Thus, we conclude that process management is a fairly high priority but not the most significant implementation area of managerial innovations for companies operating in the Russian market.

Team building and communications (13%) were ranked 4th by significance among the managerial innovations implemented over the past three years. According to the respondents, internal communications, team building, and motivation are important among the managerial innovations implemented over the past three years.

The empirical study of the implemented managerial innovations confirms the earlier conclusion that the domestic and multinational companies operating in the Russian market focus primarily on internal processes of interaction and employee motivation.

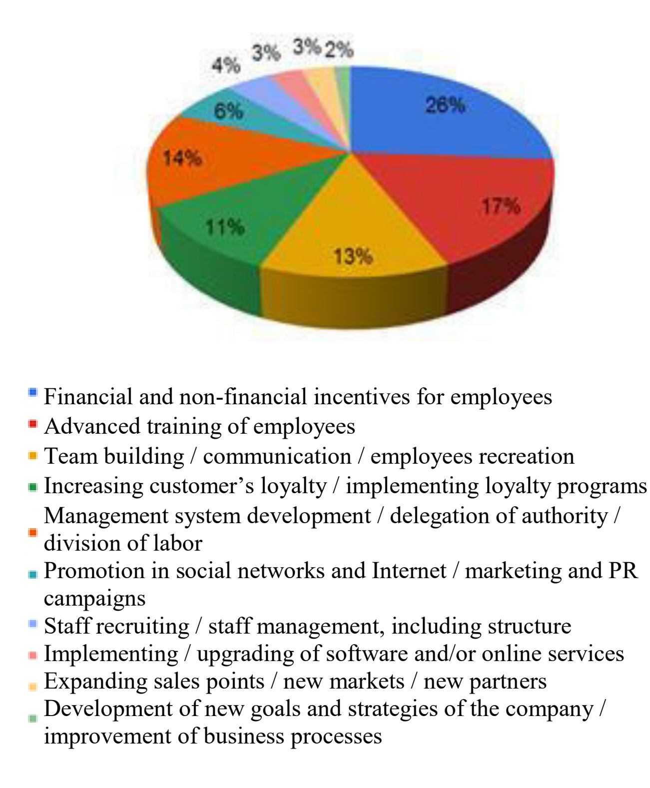

Fig. 6. Structure of most significant managerial innovations implemented within companies operating in Russian market in 2016–2019.

Implementing customer loyalty programs was ranked 5th by significance among the managerial innovations implemented within the domestic and multinational companies over the past three years. This process correlates with customer experience management, highlighted in the poll of respondents on the implementation areas of managerial innovations; see Fig. 5. The results obtained indicate that customer experience management and process management are important but not included in the top 3 implementation areas of managerial innovations within the companies operating in the Russian market.

In conclusion, note that the respondents did not indicate the negotiation process as one of the significant innovations implemented by the companies operating in the Russian market for 2016–2019. This process also received the least significance among the key implementation areas of managerial innovations within the domestic and multinational companies. Thus, the negotiation process, "soft skills," and competencies of employees in this area are not considered by the domestic and multinational companies as the most significant managerial innovations.

5. LIMITATIONS AND FURTHER RESEARCH

This study involved a representative base obtained by polling 791 domestic and multinational companies operating in the Russian market. Nevertheless, there are several limitations – as we believe – directions for further research:

- Expanding the geographical scope of the study by including other regions of the Russian Federation will reveal some regional differences in the implementation areas of managerial innovations.
- Supplementing the sample by companies from other groups (financial activity and real estate; transport, information, and communications) will reveal the specifics of managerial innovations for these types of economic activity.
- The case study of individual practical examples of implementing managerial innovations will reveal the distinctive characteristics of the managerial innovation process for particular types of managerial practices and companies.

CONCLUSIONS

This paper has studied, theoretically and empirically, the implementation areas of managerial innovations within domestic and multinational companies operating in the Russian market.

The theoretical part has analyzed the implementation process of managerial innovations. As proposed in the paper, the implementation of managerial innovations is treated as the process of deciding by an organization to start using new managerial practices, approaches, processes, and techniques and using them as well. The implementation process of managerial innovations has been assigned a new interpretation as two subprocesses: the decision to implement an MI and the direct implementation of an MI.

The empirical part has been conducted using a database obtained by polling 1025 employees from 791 domestic and multinational companies operating in the Russian market. The empirical study has yielded answers to the research questions on the implementation areas of managerial innovations; see Section 2.

The main research question has been to study the key implementation areas of managerial innovations within domestic and multinational companies operating in the Russian market. It has been decomposed into two second-level research questions (RQ 1 and RQ 2). According to the results obtained, domestic and multinational companies operating in the Russian market focus primarily on the internal processes of interaction and employee motivation.

RQ 1 has been to identify the features of the implementation areas of managerial innovations depending on the types of economic activity, business size, and the company's degree of internationalization. Among the features, let us highlight the following:

- The companies from the industrial production group primarily focus on managerial innovations in internal processes, paying insufficient attention to the processes related to personnel and customers compared to the companies with other types of economic activity.
- The medium-sized companies give higher priority to the process of motivation, being noticeably less (but equally) focused on all other implementation areas of managerial innovations.
- The well-established practices of customer experience management and internal processes reduce the need to implement managerial innovations in these areas for multinational companies compared to domestic companies.

RQ 2 has been answered by studying the managerial innovations implemented within the companies operating in the Russian market for 2016–2019. According to the results obtained, domestic and multinational companies focus on internal processes of interaction and motivation and separate the advanced training of employees. Moreover, two implementation features of managerial innovations characteristic of the companies operating in the Russian market have been revealed:

- Customer experience management and process management are considerable but not significant implementation areas of managerial innovations among the companies operating in the Russian market.
- There is no attention to developing managerial innovations in negotiations among the domestic and multinational companies operating in the Russian market.

The results obtained within this study are of practical importance. For example, industrial production companies should be more concerned with the processes related to personnel and customers. Internal processes and customer experience management require more effort from the domestic companies, as they lag behind the multinational companies in this area. Negotiations need special attention from the companies' management since soft skills are an important component for increasing the competitiveness of companies both in the Russian and global markets.

The outcomes of this study are useful for further research and the practice of innovative company management.

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BORDER ACTIVITIES AS A SYSTEM OF MEASURES AND ITS SCIENTIFIC SUPPORT

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Abstract. Border activities are aimed at ensuring national security in the border space. They can be treated as a system of preventive measures (border prevention and containment), security and control measures (border service and search), and protective and combat measures (special and combat actions, operational actions). Functions (stages of the activity cycle) are assigned to each type of boundary measure. Border measures can be implemented via operations carried out throughout the country, one or several federal subjects according to a single plan for achieving a specific goal. The problems of border activities are investigated in the science of border activities—the system of knowledge to ensure border security, build state border organizations, prepare and conduct border activities, and provide comprehensive support of border activities. The substantial aspects of the science of border activities are studied. The principles of border activities are systematized as follows: activity, the secrecy and surprise of actions, flexibility, complex use of forces and means, the continuity of actions in place and time, the concentration of main efforts on the key directions and tasks, interaction, international cooperation, the main link, the balance of security and freedom, deterrence, reliance on the local population, the primacy of preventive measures, a combination of traditional and new technologies, and a comprehensive assessment of border security.

Keywords: border activities, system of border measures, principles of border activities, preventive measures, security and control measures, protective and combat measures, basic border security models.

INTRODUCTION

Border activities are the activities (system of measures) carried out by state bodies, local government bodies, organizations, public associations, and citizens to ensure border security [1].

Border security (national security in the border space) is the process and result of the activities of state and social institutions to control and protect the interests of the state and society in the border space. The border space includes the state border and border territory, the underwater environment and airspace of the state, the exclusive economic zone, the continental shelf, and other maritime spaces within which the state has sovereign rights and exercises jurisdiction [1, 2].

Border security is an element of national security and defense potential. For example, border legions in the Roman Empire constituted 2/3 of the total number of the armed forces. Moreover, in the third century (years 212 and 284), the border security density (the number of border guards per kilometer of the border) ranged from 2 (Africa, Mauritania) to 15–20 (Syria, Germany, etc.). Built between 122 and 128 A.D., Hadrian's Wall included a whole system of military and border architecture elements. In the later period of the Roman Empire, the border security density was further increased. Border structures with walls or ramparts protected about one-tenth of the border, and a foothold system with fortresses, observation posts, and other security elements protected about two-tenths of the border. Those border equipment and protection measures allowed repelling many invasions of foreign tribes into the Roman (Byzantine) Empire [3]. For comparison, the border security density on the U.S.-Mexican border is 6.3 border guards per km. There are concrete fences, infrared cameras, and sensors along the border's perimeter. Unmanned aerial vehicles are constantly used, and over 20 thousand American border guards ensure the border's security [4].

Russia has the longest borders in the world. A wide range of threats and challenges in the border space requires applying a system of measures to neutralize them. This work is devoted to the scientific classification and characterization of the system of border measures.

1. BORDER ACTIVITIES AS A SYSTEM OF BORDER MEASURES

The "romantic" period [5] in the development of cybernetics and its first achievements motivated the leaders to organize research into using automation tools and mathematical models in border security management (the 1950s–1960s). The commanders and chiefs of staff were given courses on operations research and the implementation of operational-tactical calculations to improve the quality of state border security [6]. The mathematical models developed in the late 20th–early 21st century can be conditionally divided into three groups: game-theoretic models of using single border means, descriptive theoretical and probabilistic models of border security at the subdivision section, and aggregated models of border security at the regional section [6–8]. For estimating the model parameters, the border statistical data were used.

In the 1990s, the emphasis in the organization of border activities gradually shifted to using information technology and systemic studies of border security. In the 2000s, the concept of a border management system appeared [9, 10]. The book [9] specified the targets of border management: preventing and combating illegal migration, smuggling of weapons and drugs, the threat of terrorism, the spread of diseases and epidemics, and promoting the development of international trade, scientific and educational environment, and tourism. Border management is considered at the global, interstate, and national levels. Let us concretize the concept to the regional level.

The measures (actions or a set of actions and means for implementing or achieving something) to ensure border security are divided into legal, political, diplomatic, economic, defensive, border, customs, environmental, sanitary and epidemiological, ecological, and others [2].

Figure 1 shows the main elements of the system of border measures at the regional level. This diagram was developed by the authors, particularly based on the analysis of [1, 2, 11–15]. The system of border measures includes preventive (border prevention and containment), security and control (border service and search), and protective and combat measures (special and combat actions, operational and combat actions).

According to the border statistics analysis, the most important target of securing the USSR's state border—prohibiting any violations of the border and border regime—was achieved mainly using prevention and containment.

Border prevention is the direct and indirect impact on the border population in order to: involve citizens and organizations in defending and protecting the interests of the state in the border area; identify and eliminate the causes and conditions conducive to illegal activities in the border area; educate persons for preventing offenses in the border area.

Border containment is the impact of border forces and means on potential violators to make them refuse any illegal activities under the threat of being detained and punished.

Security and control measures and *protective and combat measures* aim to maintain legal regimes in the border space and neutralize the subjects of danger.

2. BORDER PREVENTIVE MEASURES

The subjects of preventive activities are as follows:

– the officials of border authorities;

– state bodies, federal executive bodies, local government bodies;

– vigilante and Cossack groups;

– media and art.

The objects of preventive activities are as follows:

– society as a whole, including public organizations and other associations of citizens;

– the local population of border areas;

– vigilante and Cossack groups;

– individuals and their family members;

– criminal communities.

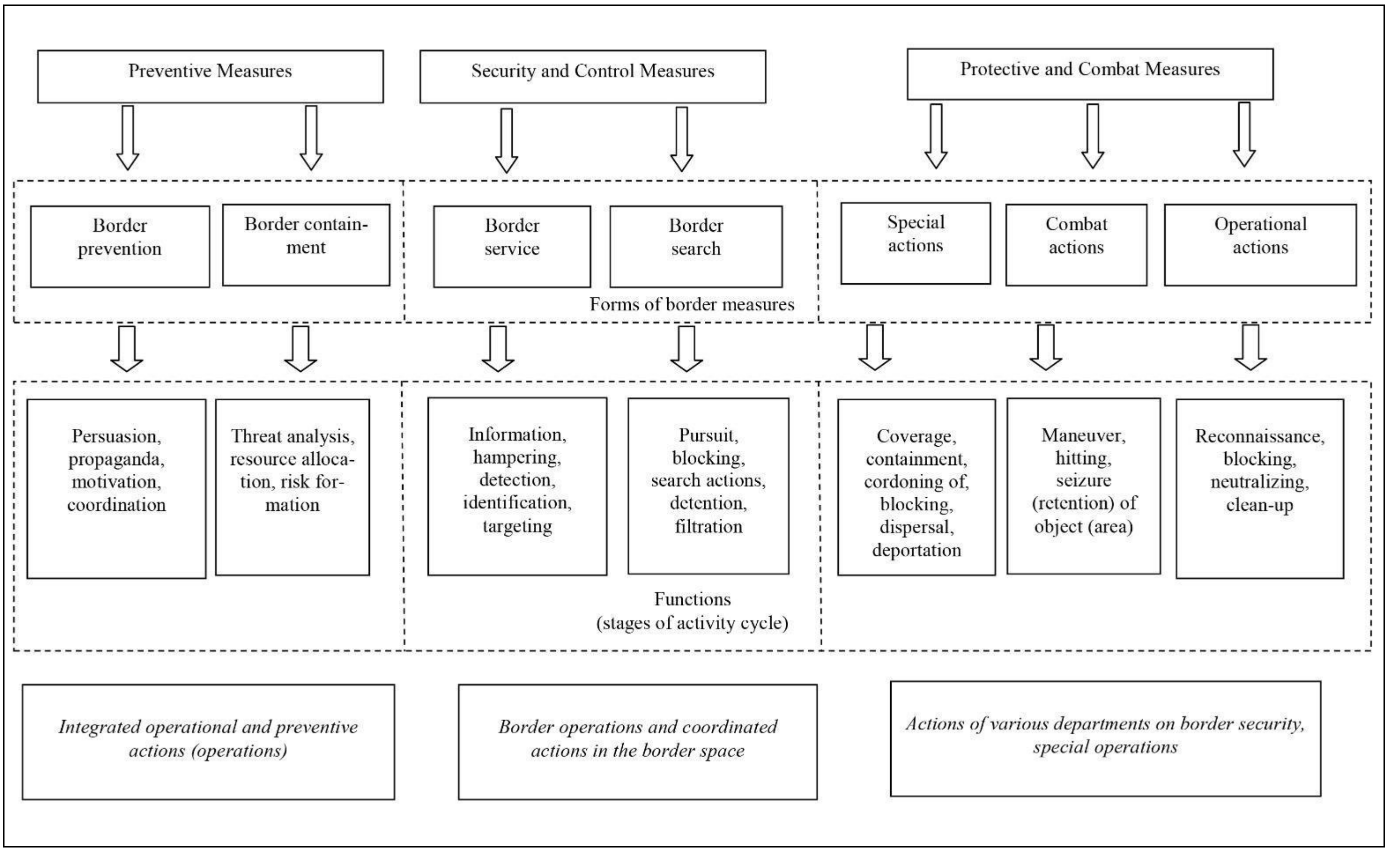

Fig. 1. Elements of the system of border measures.

2. BORDER PREVENTIVE MEASURES

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– criminal communities.

The tasks of preventive activities are as follows:

– forming a positive attitude towards the border authorities among the local population and increasing their loyalty;

– propagating the activities of border authorities and covering the issues related to border activities;

– involving the local population in the state border security and safety activities on a public basis, including motivation;

– increasing the efficiency of interaction with state and municipal authorities and executive authorities.

The forms of preventive activities include:

– information and propaganda activities;

– preventive work with the local population;

– coordination with state bodies, federal executive bodies, and local government bodies.

Depending on these forms, the following *methods of preventive activities* are distinguished*:*

• information and propaganda activities:

– covering the activities of border authorities through interaction with the media and art representatives to increase the loyalty of the local population;

– implementing counter-propaganda to neutralize negative informational impacts on society and social groups;

• preventive work with the local population:

– performing preventive and explanatory work to increase the level of literacy in the administrative and legal regime of the state border;

– performing preventive work to identify potential violators among the local population and prevent their illegal actions;

– motivating the local population to participate in the state border security and safety activities;

– performing educational work with youth and children in educational organizations.

• coordination with authorities and local government bodies:

– executing activities of coordinating bodies, for example, border commissions for the constituent entities of the Russian Federation;

– organizing the interaction between the subjects of preventive activities;

– organizing joint actions in the interests of border prevention.

The border prevention means[1] are as follows*:*

– publications in the media and Internet, including social networks and messengers;

– movies, literary works, etc.;

– preventive measures carried out by border officials among the local population;

– normative legal acts regulating the activities of vigilante and Cossack groups;

– administrative and legal regimes as a set of legal and organizational means.

For the first time, *border containment* was mentioned when fighting illicit trade (smuggling) in pre-revolutionary times. On the initiative of N.Kh. Bunge, the Minister of Finance of the Russian Empire, the Baltic customs cruiser flotilla was formed in 1873. The Minister of Finance described its activities in the following way: "The small number of arrests of smuggled goods by vessels is not a serious argument proving the flotilla's passivity. Concerning the practical results of the vessels' activities, they should be assessed not in quantitative terms when prosecuting illicit trade, i.e., not by the number of arrests of smuggled goods made by the vessels, but by its overall impact on illicit trade. The flotilla is a preventive guard. The effect of the cruiser flotilla is manifested mainly not in increasing the total amount of detained smuggled goods but in higher riskiness of illicit trade itself, heavier overhead costs, and extra time for choosing a convenient place to return to the shores, which finally reduces the profitability of illicit trade." [16, p. 363]. The main difference between containment and preventive activities lies in the narrower object of impact: containment measures are aimed at potential violators who plan illegal activities.

The border containment tasks are as follows*:*

– convincing potential violators in the inevitability and severity of punishment for illegal activities carried out across the state border;

– equipping the state border at a necessary level to prevent illegal activities;

– utilizing border forces and means effectively.

The containment measures include:

– informational measures, similar in forms and means to the preventive measures of information and propaganda, but differing in the goals and object of impact;

– measures for equipping the state border, including the use of visible informing, barrage, control, and other means;

– measures on using border forces and means (for example, the creation of sufficient security densities, in the entire section or single directions), lighting equipment, demonstrative actions, etc.

Particular indicators characterizing the efficiency of border containment are:

– the density of state border security (the number of guards per km);

– the density of installed (used) homogeneous technical means of border security (the number of units per km);

– the intensity of demonstrative actions near the state border;

– the maximum speed of violators and its reduction by barrage means;

– the maximum speed of border vehicles;

– the share of uncontrolled border sections;

– the density of informative signs and means, etc.

A promising scientific and practical task is to adapt the basic control mechanisms for managing preventive and containment measures and develop new ones.

3. SECURITY AND CONTROL MEASURES. PROTECTIVE AND COMBAT MEASURES

In the model law[2] "On Border Security" [1], *security and control measures* are understood as measures to maintain administrative and legal regimes in the border space (state border regime, border regime, etc.). *Protective and combat measures* (law-enforcement, military, reconnaissance and search, and other special measures) are intended to counter the existing threats and neutralize the subjects of danger.

The classification of such measures in other states may differ depending on the military-political, social, and economic situation in the border space and the national traditions of border security. Particularly in the United States, the complex of measures to ensure border security depends on the level of threats and is divided into [17]:

[1] A means is a technique or way of actions to achieve something.

[2] A model law is a legislative act of recommendatory nature that contains typical norms and provides normative guidance for legislation.

– Border Control (security from the illegal entry of people and goods through the border, perceived as a low-level threat);

– Border Safety (the measures implemented to protect against medium-level threats such as violence, criminals, smuggling, etc.);

– Border Security (the measures to counter-terrorism).

The border structure must be flexible to ensure border security. As noted by experts [17], law-enforcement functions at the border can potentially spill over into the functions of national defense. This fact will require the constant presence of the National Guard units at separate sections of the border and other measures.

When implementing security and control measures and protective and combat measures, *the subjects of control* are:

– federal government bodies;

– regional government bodies;

– officials of border authorities, etc.

When implementing security and control measures and protective and combat measures, *the objects of control* are:

– forces and means (subdivisions, organizations, etc.) allocated to maintain the administrative and legal regimes and neutralize the subjects of danger in the border space;

– regime legal means (regulations, acts of exercising the rights and obligations of subjects, law- enforcement acts, measures of encouragement and coercion, legal sanctions, methods, and techniques of administrative activities);

– territory where administrative and legal regimes are valid, etc.

Security and control measures and protective and combat measures *aim* to prevent any violations of the border regime and detain (neutralize) violators.

Security and control measures and protective and combat measures are managed using the principles of military strategy, border security and safety, and the functions of border forces and means [11].

4. SCIENCE OF BORDER ACTIVITIES

The science of border activities[3] is a system of knowledge to ensure border security, build border organizations, prepare and conduct border activities, and support them comprehensively (Fig. 2).

According to the analysis of Russian and foreign publications, the science of border activities includes such disciplines as border art (border policy, border strategy, and border tactics), border history, border statistics, the mathematical theory of border security control, legal foundations of border security and border activities, philosophy of border security, psychology and sociology of border activities, the theory of border training and education, the theory of comprehensive support of border activities, and the theory of development, application and operation of technical and special means of border activities. Note that this list is not exhaustive.

Border art is the theory and practice of preparing and conducting border activities. It includes state border policy, border strategy, and border tactics.

As a branch of social statistics, *border statistics* studies quantitative indicators related to the qualitative characteristics of such phenomena and processes as border security, border activities, and the results and consequences of operational and service activities and combat activities in the border space. It also explores the patterns of these phenomena in particular historical and regional conditions. The most important tasks of border statistics are the identification, acquisition, scientific processing, and analysis of statistical data, particular phenomena, and processes of border security and border activities.

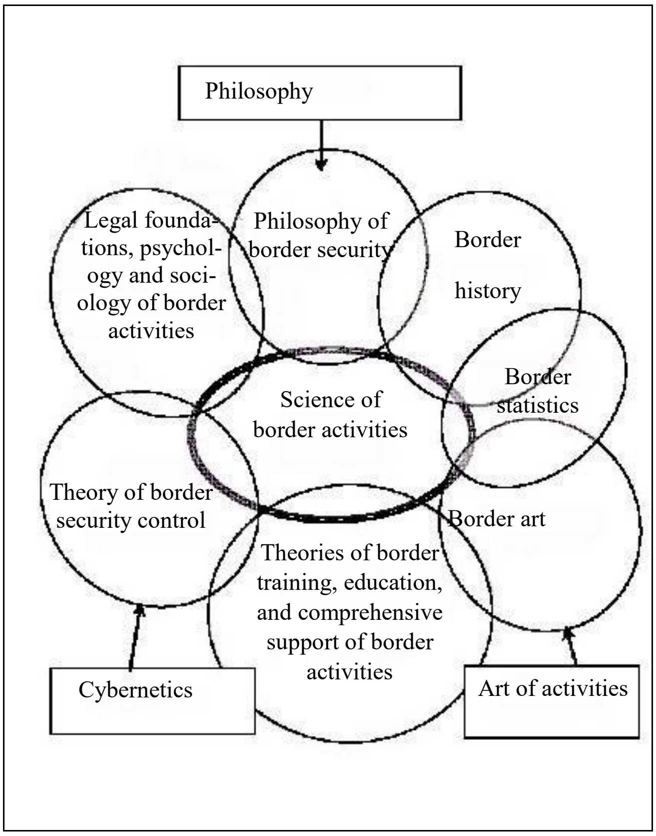

Fig. 2. Basic disciplines of the science of border activities.

[3] Some researchers also use the term "borderology."

The theory of border security control is reliable scientific knowledge about border security, border activities, and their control. It represents a system of interrelated statements and evidence, contains methods for explaining and predicting phenomena and processes in the subject area, and reduces the regularities revealed to a single unifying principle.

Figure 3 shows the subject area of the theory of border security control.

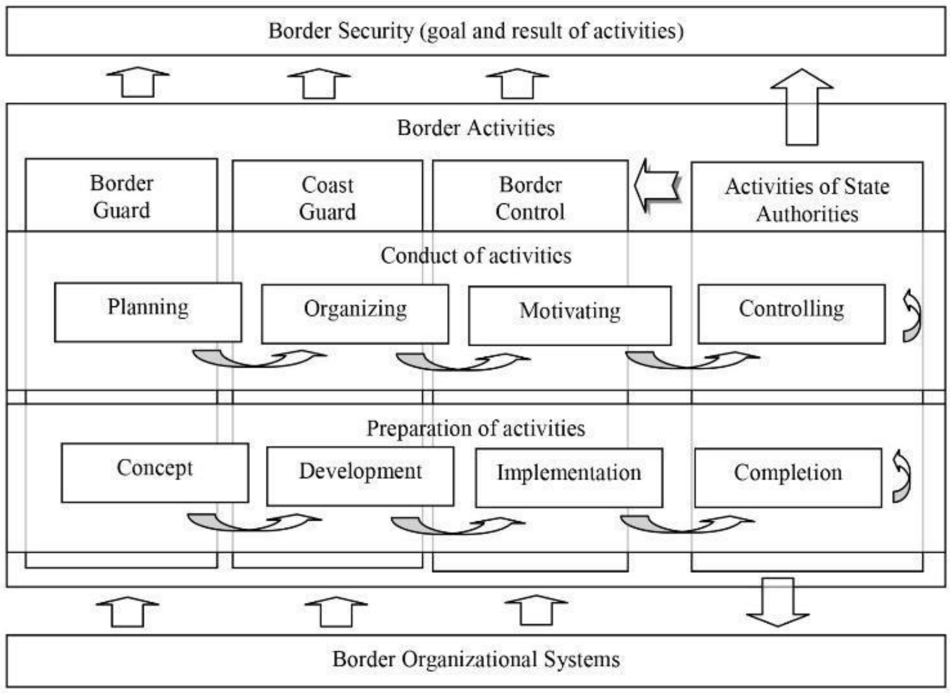

Fig. 3. The theory of border security control: subject area.

Border activities are aimed at securing and saving land and water (lakes and rivers) sections of the state border (border guard), protecting the economic and other interests of the state in the maritime border space (coast guard), passing persons, vehicles, and cargo through the state border (border control). Border activities include the coordinated (joint) actions of state authorities and control authorities.

There are two types of border activities: preparation of border activities (project activities) and conduct of border activities (process activities). They are organized as the sequential implementation of common stages (cycles).

Historically, the study of problems of activities and control was the prerogative of philosophy. Nowadays, philosophy is understood as reflexion over the foundations of all sciences: it does not consider particular problems of activities. The philosophy of border security can be defined as a science of the meaning of border activities and border security.

As is well known, the term "control science" is often unreasonably narrowed to the formal (mathematical) theory of control. However, control science includes a set of such theories [18]. The science of border activities also includes "strong" and "weak" disciplines (in the terminology of D.A. Novikov).

5. BASIC MODELS OF BORDER SECURIY

Let us present the basic models of border security and describe them in brief. (A basic model is an elementary model that can be extended by considering more factors and conditions.)

Model of regional security assessment. The security U_i of region i of a given state is assessed by the following formula [11]:

$$U_i = K_i \left(\frac{\zeta_i}{z_i} \right)^{\delta_i \mu_i},$$

where K_i specifies the level of social and economic development of region i; ζ_i and z_i are the sizes of regional (title) ethnos and regional population, respectively; $\delta_i > 0$ denotes the ethnic attraction parameter of the title ethnos; $\mu_i \geq 1$ is the interethnic heterogeneity parameter.

The model is used to assess security in regions with a mixed population and high interethnic differences (Transnistria, Abkhazia and South Ossetia, Nagorno-Karabakh, etc.), where conflicts on ethnic grounds periodically flare up. Such conflicts are very difficult to resolve.

Model of international migration. The migration flow M_{ij} from country i to country j takes into account migration laws and is estimated by the following formula [19]:

$$M_{ij} = k_{mi} \frac{(1 - R_{ij}) w_j D_j (V_j / V_i)}{(\mu_{ij})^2 \sqrt{r_{ij}}},$$

where k_{mi} denotes the migration parameter of country i (reflects social stability in a given state); w_j is the basic sovereignty of state j (the capacity of its migration market depending on the size of population and the area of country j); V_i and V_j are the per capita gross domestic products of countries i and j, respectively; D_j specifies the share of urban population in country j; r_{ij} gives the relative distance between countries i and j; μ_{ij} stands for the interethnic heterogeneity parameter of the countries; $0 < R_{ij} < 1$ is the degree of regime and legal restrictions on population migration from country i to country j.

Figure 4 shows the graph of long-term migration from Ukraine in 2011–2012.

Clearly, the largest migration flows are towards large countries with ethnically similar populations or countries with a high living standard and a significant capacity of the migration services market. The migration parameter k_{mi} characterizes the share of the country's population ready to leave (temporarily or permanently) for better living conditions. The value of this parameter increases during periods of the state's internal instability.

With the growth of migration flows, the flow of terrorists and smuggling grows, explaining the importance of migration forecasting for national and border security.

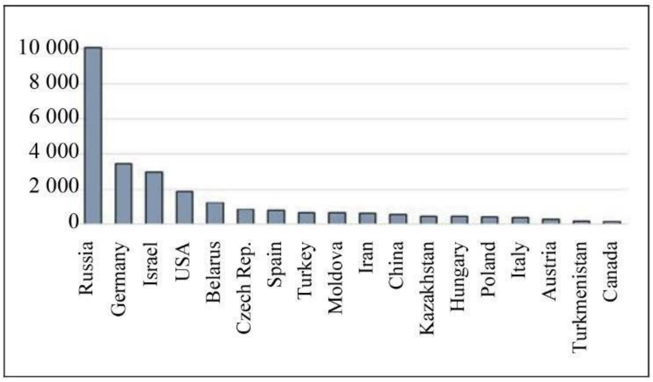

Fig. 4. Long-term migration from Ukraine in 2011–2012, people.

Model of social and informational impact. Border prevention largely depends on the actions of the border population and the awareness of citizens. Individuals form their picture of the world based on their perceptions (sensory-visual images produced by personal contact with reality, through the senses) and representations (sensory-visual images produced without the direct impact of the objects and phenomena of reality). For an indicator $\theta \in [\theta_0, \theta_1]$ (a probability $\theta \in [0, 1]$), a belief (perception) function $B(y, x, \theta)$ is defied under external impacts $y \geq 0$ ($x \geq 0$) directed to increase (decrease, respectively) the belief about this indicator. For probabilistic indicators (e.g., the probability of detention for violators, the degree of border cover, and others), the belief function [11] has the form

$$B(y, x, \theta) = \alpha B_+(y, \theta) + (1-\alpha) B_-(x, \theta),$$

$$B_+(y,\theta) = \frac{\theta \exp(z_y)}{1-\theta+\theta\exp(z_y)}, \quad z_y = \frac{k_y}{\nu+1} y^{\nu+1},$$

$$B_-(x,\theta) = \frac{\theta \exp(-z_x)}{1-\theta+\theta\exp(-z_x)}, \quad z_x = \frac{k_x}{\upsilon+1} x^{\upsilon+1},$$

where $0 < \alpha < 1$ is the optimism–pessimism parameter (the degree with which a particular individual will learn the impacts of definite direction); $\nu \geq 0$ and $\upsilon \geq 0$ are the modality parameter of the impacts; $k_x \geq 0$ and $k_y \geq 0$ denote the dimensionality coefficients of the impacts.

The model parameters were estimated and verified on social actions associated with the U.S. wars in Korea and Vietnam. According to military losses data and sociological surveys, the degree of modality for informational impacts lies within the range [0, 1], whereas the degree of modality for social impacts is greater than 1.

Note that for interval indicators (expected income, public welfare losses due to terrorism and drug trafficking, etc.), the belief function $B(\cdot)$ depends linearly on θ.

Model of illegal activities containment. The containment model is based on H. Becker's model (the utility of illegal activities) and the border production function. The goal of a border system is to maximize the efficiency of border security, i.e., the prevented losses $F(x, y)$ minus the expenses R on border security [10]:

$$F(x,y) = \sum_{i=1}^{k} u^i \left(y_0^i + \sum_{j=1}^{n} p_j(x_j) y_j^i \right) - R \to \max, \quad (1)$$

$$p_j(x_j) = 1 - \exp(-\lambda_j x_j),$$

where u^i denotes the expected losses incurred by the violator of group i; $p_j(\cdot)$ specifies the probability of violator's detention in section j; y^i_j is the number of violators i selecting section j (if $j = 0$, the violator refuses violation); x_j gives the resource allocated to section j; λ_j means the parameter of the border production function; k and n are the numbers of violators' groups and sections, respectively. (Losses are prevented if the violator refuses to intrude the border or if he is detained.)

Acting independently, the violators of group i maximize their utility:

$$f_i(x,y) = s^i y_0^i + \sum_{j=1}^{n} U_j^i y_j^i \to \max, \quad (2)$$

where s^i is the expected income from the illegal activities of violator i; U^i_j denotes the expected utility of violator i on section j, estimated within Becker's model.

For special cases, analytical and numerical solutions of the zero-sum two-player game with the payoff functions (1) and (2) were found.

Model of timely detection of border violation signs by patrol. Border security means are generally divided into posts (stationary control), sentries (control of a section of a limited length), and patrols (mobile control of a section). Assume that the patrol's task is to detect border violation signs before a lead time t_y. Then

the optimal mixed strategy of patrolling in the daytime (i = 1) or nighttime (i = 2) has the form

$$p_i = \frac{\rho_{-i} T_i}{\rho_1 T_2 + \rho_2 T_1},$$

where ρ_1 and ρ_2 are the probabilities of signs' detection in the daytime and nighttime, respectively; ρ_{-i} is the probability of signs' detection in the other time of the day; T_1 and T_2 specify the duration of daytime and nighttime, respectively. The game value—the probability of timely detection of border violation signs within 24 hours – is calculated as

$$\nu = \frac{n t_y \rho_1 \rho_2}{\rho_1 T_2 + \rho_2 T_1},$$

where n gives the number of patrols send to a section within 24 hours.

Model of victory in a battle. Border security measures (protective and combat measures) are aimed to suppress and resolve armed conflicts and military clashes. According to [20], side 1 wins in a battle with the estimated probability

$$p_x(x,\, y) = \frac{(\beta x)^m}{(\beta x)^m + y^m} = \frac{q^m}{q^m + 1},\ q = \frac{\beta x}{y}, \quad (3)$$

where m denotes the physical form parameter; x and y are the numbers of combat units of sides 1 and 2, respectively; $\beta > 0$ is the parameter of the combat (moral and technological) superiority of side 1 over side 2; q gives the ratio of their forces.

In particular, the model (3) can be used for solving the offense-defense game: find an optimal distribution of forces and means between tactical tasks and points of defense.

6. PRINCIPLES OF BORDER ACTIVITIES

Historically, border science emerged as an integral part of military science. Golovin [21] ascertained that the science of war will seek to discover laws, whereas the theory of military art generalizes war phenomena into principles. According to Golovin, principles are directly related to goal-setting and task-setting, representing the main idea and regulating creativity without imposing constraints on it.

Like the principles of any practical and managerial activities (e.g., the principle of ethics or legality), the principles of border activities were highlighted in [13, 16, 22–30]. Let us systematize and present them in the author's interpretation:

1. *The principle of activity.* The activity aims to create conditions preventing or limiting the actions of border violators and ensuring a high probability of their detection and impact on them. Activity is achieved by: continuously obtaining information about possible violations, techniques, and tricks used by violators; permanently searching for violation signs of the legal regimes in the border space; anticipating the actions of violators and imposing their will on them; showing courage and reasonable initiative in decision-making. Proactive actions will require comprehensive control and oversight. Creativity in security technologies and procedures is encouraged. New security systems must be quickly implemented for testing, and successful systems must be deployed across the entire border. The legal basis is reoriented from reactive approaches to preventive actions.

2. *The principle of secrecy and surprise of actions.* Secrecy and surprise allow achieving maximum results with the least consumption of forces, means, efforts, and time. Surprise is achieved by: keeping the plan of actions in secret; misleading the enemy about intentions; anticipating the enemy in actions; performing the assigned tasks promptly; using new means and methods of actions unknown to the enemy, particularly border cunning; camouflaging and countering enemy reconnaissance; satisfying the requirements of covert management and secrecy regime, etc.

3. *The principle of flexibility (mobility).* Flexibility is achieved by: excluding stereotypes in the forms and methods of actions; making the actions of forces and means mobile; performing timely maneuvers and quickly changing the forms of service actions to optimal ones.

4. *The principle of the integrated use of forces and means.* The integrated use of forces and means increases the efficiency of border operations by following a single concept and plan. Efforts should prevent, identify, preclude, and neutralize violators, reduce losses and destruction, guarantee fast response, and decrease alarms.

5. *The principle of focusing main efforts on key directions and tasks.* This principle expresses the ability to make decisions under a wide range of potential bor-

der security threats and prevent the existing vulnerabilities from passing into inevitable terrorist attacks and border violations. Violators can act anywhere and anytime. The round-the-clock continuous security of all border sections is impossible. The actions should not be reactive and excessive.

6. *The principle of continuous actions in place and time, in functions and tasks.* Continuity consists in the permanent implementation of actions of different forces and means coordinated in place and time. Under insufficient forces and means, continuity is achieved by using them in an order unpredictable for violators.

7. *The principle of the main link.* The main tasks of fighting violators are solved at the border guard, ship, crew, or employee level. Hence, it is required to increase intelligence capabilities at the local level, eliminate bureaucratic barriers to exchanging intelligence information, and develop and apply modern technologies.

8. *The principle of containment.* The prevention of border violations and criminal acts cannot be a criterion for the efficiency of border security measures due to the complexity of the assessment. The following goals should be quantified: the containment of violators (terrorists, smugglers, or illegal migrants), the detection of violators, and difficulties for criminal acts (reducing the potential benefit received by criminals).

9. *The principle of reliance on the local population* (through knowledge of traditions, culture, and language). This principle means educating the public, helping citizens assess the dangers and everyday risks, and creating a security culture. Local forces (vigilante and Cossack groups) should be prepared and used to ensure border security.

10. *The principle of coordination and interaction.* This principle includes the following: integrated efforts of departments, the public, and private individuals to fight cross-border threats at the President's level; an intelligent approach to ensuring border security using somewhat redundant security layers and proper consideration of interaction effects for implemented programs and measures.

11. *The principle of international cooperation.* International and bilateral cooperation of states fighting cross-border crime increases the efficiency of border activities.

12. *The principle of the primacy of preventive measures.* This principle expresses the primacy of preventive activities over law-enforcement ones; in preventive activities, the primacy of measures to provide social assistance to those in need over the legal restrictions and the priority of persuasion over coercion. Regional economic development plans should be aimed at preventing illegal economic activity. Efficient border security is achieved by early monitoring persons and goods from the point of departure and tracking the persons suspected of illegal cross-border activities.

13. *The principle of the security–freedom relation.* Security and freedom should not be converted into currency. Security measures should not restrict freedom. Freedom can be threatened by trying to eliminate all risks.

14. *The principle of combining traditional and new technologies, stationary and mobile border guard forces.* New technologies do not cancel the traditional ones (e.g., service dogs in border security) but supplement them. Reducing the costs of implementing contact and barrier functions is achieved through the efficient use of information and technology. The implementation of new technologies does not always cut the staff. New systems require experienced operators and quality control and test specialists, which is often neglected in budgeting.

15. *The principle of the integrated assessment of border security.* Resource allocation should be based on risk assessment. Rigorous analysis of security costs and security profits is difficult to implement because terrorist attacks and border violations with high social costs are rare, and the consequences of large-scale attacks are not easy to quantify. It is also difficult to estimate the consequences of drug trafficking and interethnic conflicts caused by uncontrolled migration in economic terms.

In the table below, these principles are associated with the types of border activities.

Table

Principles and types of border activities

Types of activities	Principles of border activities
Conduct of border activities	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 14, 15
Preparation of border activities	1, 7, 9, 10, 11, 12, 13, 14, 15

The principles of border activities, in combination with system analysis methods, allow solving many topical problems. Let us mention some of them: the integrated assessment and ranking of border forces and means according to their efficiency [31]; the formation of a promising image of border units; the equipment of the border.

At the tactical level, the following functions of border means are implemented in the border service: informing, tactical containment, barrage, detection, recognition, movement, targeting, detention, fixation of violation signs. Modeling involves criteria based on the principles of border activities. For the principle of the integrated application of forces and means, this criterion is the number of functions implemented by border means; for the principle of continuous actions in place and time, the continuity coefficient of using border means in different directions and time; for the principle of flexibility (mobility), the speed of motion and maneuvering of border means; for the principle of continuity in functions and tasks, the uniform distribution coefficient of border means by their functions.

Note that 20–50 years ago, the main tasks of border security were solved by a border post (subdivision). Modern means, technologies, and weapons create prerequisites for implementing the principle of the main link: a border patrol should now solve the tasks previously solved by a subdivision.

CONCLUSIONS

The concept of a border management system has been developed by studying border activities as a system of measures to ensure national security in the border space. The system of border measures at the regional level includes preventive measures (border prevention and containment), security and control measures (border service and search), and protective and combat measures (special and combat actions, operational actions). The criteria, tasks, forms, methods, and means of activities have been determined for border measures.

The science of border activities—a system of knowledge to ensure border security, build border organizations, prepare and conduct border activities, and support them comprehensively—has been structured. This science includes such disciplines as border art (border policy, border strategy, and border tactics), border history, border statistics, the mathematical theory of border security management, the legal foundations of border security and border activities, the philosophy of border security, psychology and sociology of border activities, the theory of border training and education, the theory of comprehensive support of border activities, and the theory of development, application, and operation of technical and special means of border activities.

The most important element in the science of border activities is the theory of border security control—reliable scientific knowledge about border security, border activities, and their control. It represents a system of interrelated statements and evidence, contains methods for explaining and predicting phenomena and processes in the subject area, and reduces the regularities revealed to a single unifying principle.

Extensions of the basic border security models create prerequisites for bettering decisions and increasing the efficiency of border activities.

While sciences strive to discover laws and patterns, border art generalizes phenomena, processes, and impacts to principles. The authors have systematized the principles of border activities.

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NEUROMODULATION AS A CONTROL TOOL FOR NEURONAL ENSEMBLES[1]

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Abstract. Control mechanisms for the rhythms of neuronal ensembles based on the neuromodulation effect are described and implemented. The biological mechanisms of neuromodulation are briefly outlined, and some aspects are highlighted to control the activity patterns of interconnected neurons forming ensembles. Within the suggested model, neuromodulation is a change in the neuron's properties responsible for its sensitivity to excitatory and inhibitory impacts (and, therefore, for its activity). This change is initiated by certain neurotransmitters (modulators), which indirectly influence the electrical activity of all neurons sensitive to them. The discrete asynchronous chemical interaction model of biological neurons in small neural networks is modified and extended to implement this control mechanism inherent in living organisms. The key effect of neuromodulation is the rapid functional reorganization of neural networks without changing their structural properties. Activity patterns are changed not via costly changes in the connections between neurons but by changing the chemical environment of the ensemble's neurons. The mechanism of neuromodulation is formalized. The new model is implemented in software, and several computational experiments are performed to change the gait of hexapods.

Keywords: neuron, neuromodulation, neurotransmitters, control, discrete modeling, generator of rhythmic activity.

INTRODUCTION

In neurobiology, there is the concept of a *central pattern generator* (CPG). It refers to a neuronal ensemble whose members jointly generate a certain motor program of the body. A motor program is understood as a time-ordered output activity transmitted to muscles, forcing them to contract and relax in a certain coordinated sequence that forms a motor pattern [1, 2]. Locomotor gaits are a good example of such patterns. For four legs, gallop, trot, amble, and step are often distinguished. The same neuronal ensemble is capable of generating different activity patterns. Some model examples in this paper will show how to switch between different patterns using the neuromodulation effect without restructuring the ensembles.

The neuromodulation effect is that neurotransmitters (chemical signaling molecules acting on neurons sensitive to them) can switch the network of interactions [3–6]. Anatomical connections between neurons indicate only the potential for their interactions. Real interactions are determined by molecules of neuromodulators, which change the composition and activity of neuronal ensembles [4]. In other words, anatomical connections are only a starting point for understanding the dynamics of ensembles [5]. An important role is also played by the fundamental diversity and heterogeneity of neurotransmitters and the types of neurons and their interactions [2, 7–9].

The overwhelming majority of biologically accurate mathematical models of neurons describe membrane potential dynamics [10–12]. The advantages of discrete models are interpretability and the reflection of neural interactions at a phenomenological level under a relatively low computational complexity. However, discrete models of biological neurons describing

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heterochemical interactions have not been developed so far.

The automata-based approach to modeling biological neurons was proposed in the monograph [13]. This paper presents an automaton model of a neuron that survives under conditions of limited nutrition. It is shown that when minimizing consumption, the system acquires memory and the mechanisms of behavior and feeling. The basic property of the neuron modeled below is its endogenous electrical activity: "A discharge in a neuron is needed by the neuron itself."

This paper modifies the discrete asynchronous chemical interaction model of neurons [14] to reproduce neuromodulation effects. In the previous version of the model, neurotransmitters only have an activating or inhibitory effect on neurons, i.e., increase or decrease the membrane potential. In the new version, two types of receptors are introduced for neurons as follows. The impact on the first-type receptors, as before, entails a change in the charge on the neuron membrane. The impact on the second-type receptors changes the sensitivity of the first receptors, thereby modulating the neuron's response to external impacts.

1. BIOLOGICAL MECHANISMS OF NEUROMODULATION

The main characteristic of neuron's activity is the electrical potential at its membrane. When the membrane potential exceeds some threshold, the neuron goes into an active, excited state. Excitation is transmitted from the neuron to other neurons and tissue cells via axons having terminals with synaptic endings. They contain neurotransmitter molecules whose function is to transmit signals between neurons chemically. When the excitement reaches the synaptic end, it undergoes rapid transformations, leading to the release of transmitters into the extracellular space. Near the synaptic end of the neuron that transmits the signal, there are dendrites or the body of the neuron that receives the signal. Their surface has receptors acting as signal receivers. The connection of a transmitter with a receptor sensitive to it causes a chemical reaction and intracellular transformations in the receiving neuron, which often change its membrane potential. Therefore, it is convenient to represent the nervous tissue at the cellular level as a network of electrically conductive elements. This approach to describing the nervous system is called electrophysiological. It resulted in many important discoveries and dominated neurosciences for the entire second half of the 20th century. The overwhelming majority of exact mathematical models of neurons are aimed precisely at describing membrane potential dynamics [10–12].

Nevertheless, chemical interactions between neurons can induce a wide range of intracellular effects not expressible by a direct change in the value of the membrane potential. Chemical interactions between neurons, not associated or indirectly associated with changes in the membrane potential, have a huge impact on the behavior of both individual neurons and their populations. The rich variety of such impacts is called "neuromodulation" [3–6]. Often, neuromodulation does not directly affect membrane potential dynamics but modifies endogenous and exogenous patterns of electrical activity. In this regard, it is especially interesting when studying the mechanisms of the emergence and maintenance of rhythmic activity in the nervous system. Neuromodulation can significantly vary rhythm parameters (e.g., the duration of the phases of activity and silence) and even transfer the neuron to the rhythmic mode from the non-rhythmic one [15].

The specifics of existing approaches to modeling natural neural systems restrict the possibilities of reflecting neuromodulatory impacts. Biophysically accurate modeling requires measuring the microconcentrations of various substances very finely in extremely small volumes of space and considering the geometric features of the extracellular space on the nanometer scale. Currently, models of this level of accuracy exist only for local areas of a neuron [16] and are impossible in practice, even for small groups of neurons.

In this paper, we develop the asynchronous model of multitransmitter interactions [14, 17]. The model has the following features: a discrete approach is used to model the behavior of neurons, and the neurons in the model exchange chemical signals extrasynaptically–through the common extracellular space (all signals are broadcast). However, any impact on a neuron is reflected in this model directly by a change in the membrane potential. Hence, the expressive power of the multitransmitter approach is considerably restricted. This paper introduces an additional modus of neural interactions into the asynchronous model by adding another type of receptors that would directly affect not the membrane potential but the weights of other receptors, thereby changing the neuron's sensitivity to certain input signals. As shown below, the introduced modifications allow implementing a fast and low-cost mechanism for controlling motor rhythms.

2. BASIC NOTIONS OF DISCRETE ASYNCHRONOUS MODEL

The basic model and the principles of its functioning were formally described in [17]. Some model examples of the rhythms generated by the nervous systems of various mollusks were presented in [14]. In this section, we briefly consider the model and the main definitions and notations used below.

A heterogeneous neural network is a system $\mathbf{S} = \langle\mathbf{N}, \mathbf{X}(t), \mathbf{C}, \mathbf{T}\rangle$ with the following notations: $\mathbf{N} = \{N_1, \ldots, N_n\}$ is the set of neurons; **X** is the extracellular space through which chemical neural interactions occur; $\mathbf{C} = \{c_1, \ldots, c_m\}$ is the set of transmitters; **T** denotes the continuous time in which the system is functioning.

The continuous time is divided into unequal intervals (time steps) by *events*. An event is a change in the state of at least one of the system's neurons (activation of a passive neuron or deactivation of an active neuron).

In each time step, neurons interact with the extracellular space **X**. Transmitters from the space **X** influence the behavior of neurons, which can be expressed in a change in their state of activity. In turn, a change in the neuron's state changes the transmitter composition of the space **X**. This approach allows describing both synaptic and nonsynaptic interactions [14].

2.1. Neuron Parameters

2.1.1. Receptors

Neuron N_i possesses many receptor slots, and each slot is characterized by sensitivity to some transmitter c_j and weight $w_{ij} \in \mathbf{R}$. A slot is a union of all receptors sensitive to transmitter c_j; its weight is the cumulative effect of these receptors. If the neuron is insensitive to transmitter c_j, it does not have a corresponding slot, and $w_{ij} = 0$. A weight $w_{ij} > 0$ ($w_{ij} < 0$) means that this transmitter has an excitatory impact (inhibitory impact, respectively) on the neuron. For all neurons, the receptor weights form the matrix $W = (w_{ij})_{n\times m}$.

2.1.2. Output activity of neurons

The activity of neuron N_i is given by a value $y_i(t) \in \{0, 1\}$: if $y_i(t) = 1$, the neuron is active in time step t; otherwise ($y_i(t) = 0$), the neuron is passive in time step t.

The neurons in the model are transmitter-specific: upon activation, each neuron releases the same transmitter c_j into the extracellular space. In the model without neuromodulation, the release is determined by a constant d_{ij}.

The output is represented by the matrix $D = (d_{ij})_{n\times m}$, in which $d_{ij} \geq 0$ is the rate of release of transmitter c_j by neuron N_i. Note that $d_{ij} = 0$ if neuron N_i does not release transmitter c_j. Due to the transmitter-specificity of neurons, each row of the matrix contains exactly one nonzero element. The value d_{ij} is assumed to be invariable during the release process.

2.1.3. Internal state of neurons

Neuron N_i has the membrane potential $U_i(t)$, which varies within a range $U_i^0 \leq U_i(t) \leq U_i^{\max}$. The neuron in the model is active if its membrane potential $U_i(t)$ exceeds a threshold P_i, often smaller than $U_i^{\max}$. The values U_i^0, $U_i^{\max}$, and P_i are specific for each neuron.

2.1.4. Types of neurons

The neurons in the model are heterogeneous. Each neuron is determined by the following characteristics:

– the transmitter it releases (see subsubsection 2.1.2);

– the set of receptors and their weights;

– the nature of endogenous activity, i.e., the ability for activation without external impacts.

The model implements three types of neurons with different types of activity (Fig. 1):

- *Tonic neuron* has permanent endogenous activity in the absence of inhibition. In the model, permanent activity is understood as the regular generation of spikes (nerve impulses) in equal time intervals.
- *Burst* (*oscillator*) neuron generates spike bursts in definite time intervals in the absence of inhibition. The frequency of spikes in bursts exceeds the frequency of spikes generated by the tonic neuron (Fig. 1a, b).
- *Reactive* (*passive*) neuron has no endogenous excitation. It is activated only under an excitation reaching the threshold.

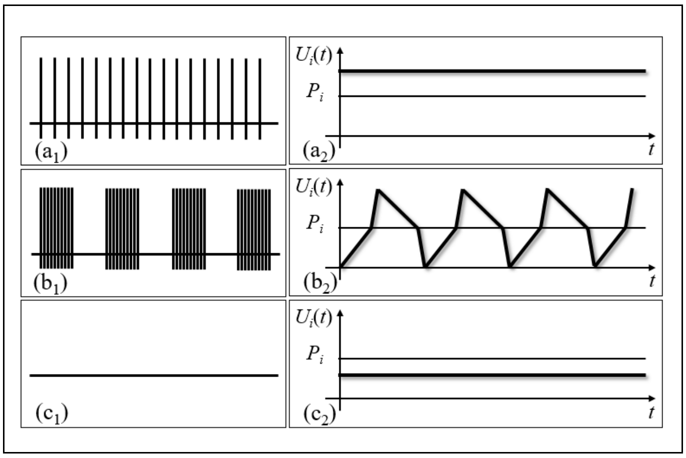

Fig. 1. Three types of endogenous activity. Diagrams with generated spikes (left column), and model approximations (right column): (a_1) tonic neuron with regular spikes; (a_2) constant membrane potential $U_i(t)$ exceeding threshold P_i; (b_1) oscillator's spike bursts; (b_2) piecewise linear approximation by four endogenous rates of change of membrane potential: two rates above threshold, and two rates below; (c_1), (c_2) reactive neuron with membrane potential below threshold.

A neuron is activated if its membrane potential has exceeded a threshold specific for each neuron. Activation occurs due to either endogenous activity or external impacts when the sum of the responses of the receptors (considering their weights) exceeds the threshold. In this case, the neuron releases a transmitter. Differences in the activation rates of tonic and burst neurons are implemented by setting specifying different values of the rates of release d_{ij}; see subsubsection 2.1.2.

For all three types of neurons, the endogenous dynamics of the membrane potential are given by linear functions. The left column in Fig. 1 schematically shows the membrane potential dynamics of the neurons generating impulses; the right column, its linear approximations used in the model.

2.1.5. Membrane potential dynamics

During each time step, the membrane potential of neuron N_i changes (increases or decreases) linearly, i.e., with a constant total rate:

$$U_i(t) = v_{ien}^{\alpha}(t) + s_i(t),$$

where $v_{ien}^{\alpha}(t)$ is the *endogenous rate* of change of the membrane potential given by a piecewise linear function; α is a parameter depending on the neuron's type of electrical activity (each type of neurons has a specific set of endogenous rates of change) and the current range of the membrane potential in this time step; $s_i(t)$ is the *exogenous rate* of change, equal to the power of the external impacts:

$$s_i(t) = \sum_{j=1}^{m} w_{ij} x_j(t), \tag{1}$$

where $x_j(t)$ is the concentration of the jth transmitter in the extracellular space (see subsection 2.2).

For different types of neurons, changes in the membrane potential were described in detail in [12].

2.2. Extracellular space

The state of the extracellular space in time step t is represented by a vector $X(t) = (x_1(t), \ldots, x_m(t))$, where $x_j(t) > 0$ is the total volume of transmitter c_j present during time step t; otherwise, $x_j(t) = 0$. The state of the extracellular space changes under each event: when a neuron is activated, the concentration of a neurotransmitter specific to it increases by d_{ij}; when deactivated, it decreases by the same value.

In what follows, we propose a modification of this model to reflect the neuromodulation effects.

3. FORMAL DESCRIPTION OF NEUROMODULATION

As shown in subsubsection 2.1.1, the receptor weights w_{ij} in the basic model are constant values. They contribute to the rate of change of the membrane potential according to formula (1). For reflecting the neuromodulation effect in the model, we introduce additional receptors responsible for neuromodulatory impacts. For neuron N_i, the weight of the modulatory receptor will be denoted by w_{ijk}^{β}, where the superscript β indicates the receptor's type. The weight w_{ijk}^{β} is the value by which the weight w_{ij} changes in the presence of transmitter c_k. We write the set of receptor weights of neuron N_i receptors responsible for neuromodulatory impacts as the matrix

$$\mathbf{W}_i^{\beta} = (w_{ijk}^{\beta})_{m\times m}.$$

Then the jth row of this matrix is a vector containing the weights of all receptors, sensitive to transmitter c_j, that change under the impact of transmitter c_k, $k = 1, \ldots, m$.

Formula (1) for calculating the external impact on a neuron (1) will be modified to

$$s_i(t) = \sum_{j=1}^{m} \left(w_{ij} + \sum_{k=1}^{m} w_{ijk}^{\beta} x_k(t) \right) x_j(t), \tag{2}$$

Using $\mathbf{W}_i = (w_{ij})_{1\times m}$ and $\mathbf{X}(t) = (x_j(t))_{1\times m}$, it can be written in the matrix form

$$s_i(t) = \mathbf{W}_i \mathbf{X}^{\mathrm{T}}(t) + \mathbf{X}(t)\mathbf{W}_i^{\beta}\mathbf{X}^{T}(t).$$

This means that before calculating the contribution of the jth transmitter to the external impact s on neuron N_i, the weight w_{ij} is summed up with the product of the jth row of the matrix $\mathbf{W}_i^{\beta}$ and the transmitter concentration vector $\mathbf{X}(t)$. All receptor types respond to the same set of transmitters. Such changes allow introducing interactions that modify the neuron's response to transmitters by specifying indirect impacts on the membrane potential.

All model parameters described in Sections 2 and 3 are illustrated Fig. 2a, b.

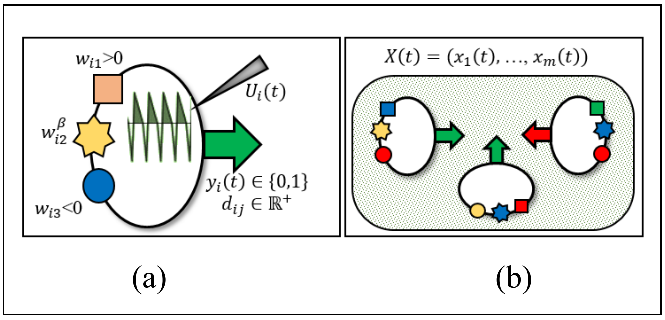

Fig. 2. (a) Neuron and its model parameters: excitatory, inhibitory, and modulatory receptors with weights $w_{i1} > 0$, $w_{i3} < 0$, and $\boldsymbol{w}_{i2}^{\beta}$, respectively; each receptor is sensitive to transmitter of one type; type of endogenous activity: dynamics of membrane potential type $U_i(t)$; neuron's release: type of neurotransmitter c_j and rate of release d_{ij}. (b) Interaction of neurons through common intercellular space characterized by neurotransmitter concentration vector.

Remark. Formula (2) has a quadratic term, which generally increases the number of parameters from $O(nm)$ to $O(nm^2)$. As a result, the problem of choosing appropriate parameters for implementing the system's desired behavior acquires a higher complexity. This paper will be restricted to studying a particular case in which modulatory impacts completely disable some receptors; see Section 4. One modulatory transmitter is added to zero the weights of given receptors. In this statement, the problem's dimension remains the same.

Fig. 3 illustrates the modulation of the impact of neuron N_2 on neuron N_1 by neuron N_3. Without the impact of neuron N_2, the membrane potential dynamics of neuron N_1 do not change.

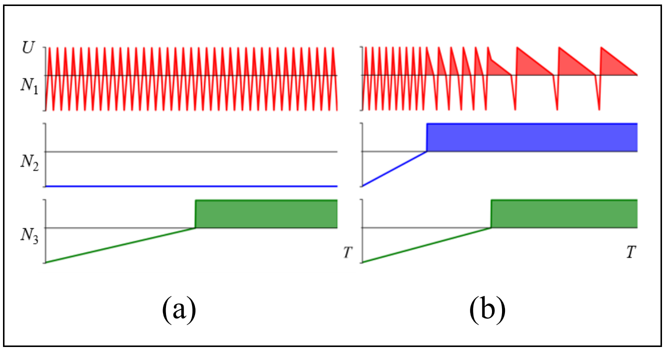

Fig. 3. Change in membrane potential under modulatory impact. Neuron N_3 modulates impact of neuron N_2 on neuron N_1.
(a) Modulatory impact changes the weight of receptor of neuron N_1 to the transmitter of neuron N_2; when neuron N_2 is silent, modulation does not affect the membrane potential of neuron N_1. (b) Neuron N_2 is activated earlier than neuron N_3 and slows down the oscillations of membrane potential of neuron N_1; when neuron N_3 is connected, oscillations slow down even more due to modulatory impact. Ordinate axis corresponds to membrane potential and abscissa axis to time.

4. OBJECT OF MODELING

Let us describe the gait switching mechanism of an abstract six-legged walking (hexapod) robot using neuromodulation. The motor programs that control walking differ in the number of legs on the ground at a given time. For example, four legs lean on the ground under a four-legged gait, and two are taking a step. As a rule, one, two, or even three legs take a step simultaneously. The more legs are involved, the higher the speed of movement will be. A three-legged (tripod) gait is considered optimal since the robot has three support points at each time, which provides stability.

The motor programs of hexapods are biologically inspired: insects are six-legged animals and have a fairly simple nervous system. Therefore, it is possible to study the mechanisms that control their walking [18, 19]. Figure 4 shows the three-legged gait of the fruit fly *Drosophila melanogaster*.

The tetrapod gait is similar: four legs are always on the ground, and two take a step. The idealized step alternation diagrams of these gaits are presented in Fig. 5.

5. GAIT SWITCHING USING NEUROMODULATION

Each leg of the animal performs two groups of mutually exclusive actions: moves the animal forward when it is on the ground, or steps forward. These groups consist of several simpler actions corresponding to flexion, extension, and movement of the limbs in different planes. In animals, the actions mentioned are implemented by contractions of various muscle groups; in robots, by switching on various servos.

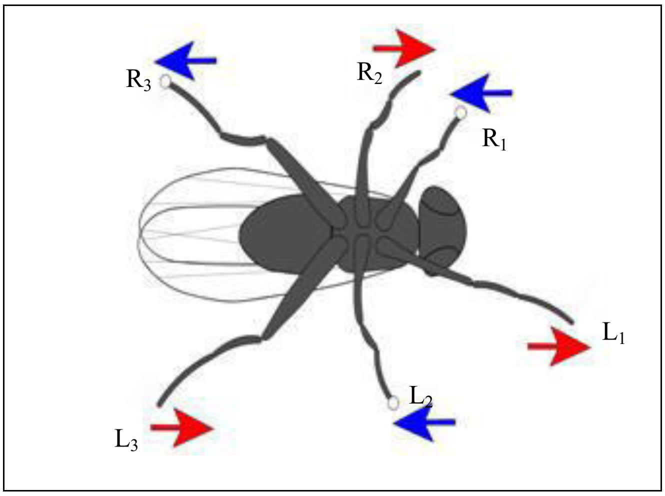

Fig. 4. Tripod gait of Drosophila, bottom view:
front and back right (R_1, R_3) and middle left (L_2) legs are on ground; front and back left (L_1, L_3), and middle right (R_2) feet take step.

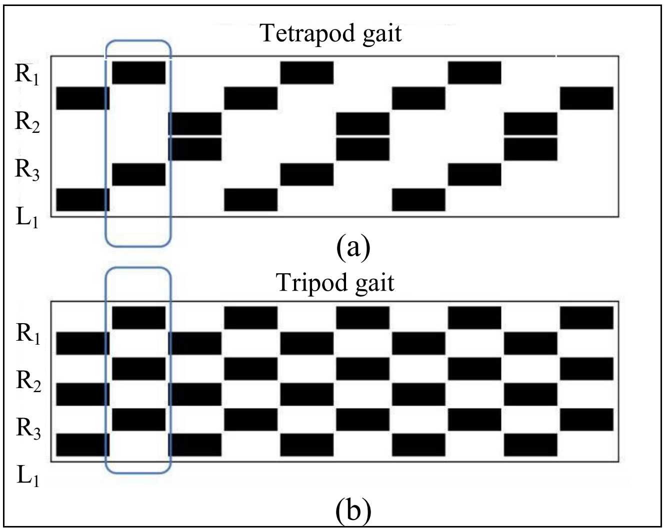

Fig. 5. Step alternation diagrams for Drosophila:
(a) tetrapod gait and (b) tripod gait.
(– ground, – step).

Since the action sequences for each leg are stereotyped, the asynchronous model will reproduce four- and three-legged gaits after assigning two neurons for each leg: one is active when the leg is on the ground, the other when the leg oversteps.

Let the neuron responsible for the movement of the supporting leg be *tonic* (active in the absence of external impacts), and the stepping one be *silent* (reactive). For their anti-phase activity, we create an excitatory connection from the support tonic neuron (*Supp*) to the step silent neuron (*Step*) and an inhibitory connection from the step neuron to the support one; see Fig. 6.

The membrane potential of such a pair of neurons will change as shown in Fig. 7. Tonic neuron *Supp* activates silent neuron *Step*. Neuron *Step* reaches the threshold, is activated, immediately inhibits neuron *Supp*, and remains active for some time, while its membrane potential decreases under the influence of endogenous forces, approaching the threshold from above. When neuron *Step* becomes silent, the inhibitory impact on neuron *Supp* is eliminated: it is activated, and the time step repeats.

Next, we build six pairs of such neurons with necessary connections so that the excitation pattern corresponds to the diagram of a four-legged gait (Fig. 5*a*).

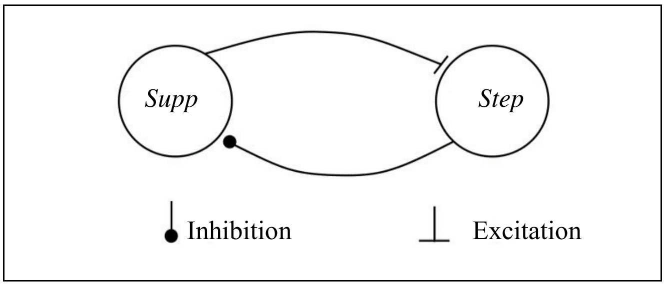

Fig. 6. Connection diagram of two antagonistic neurons controlling movements of one leg.

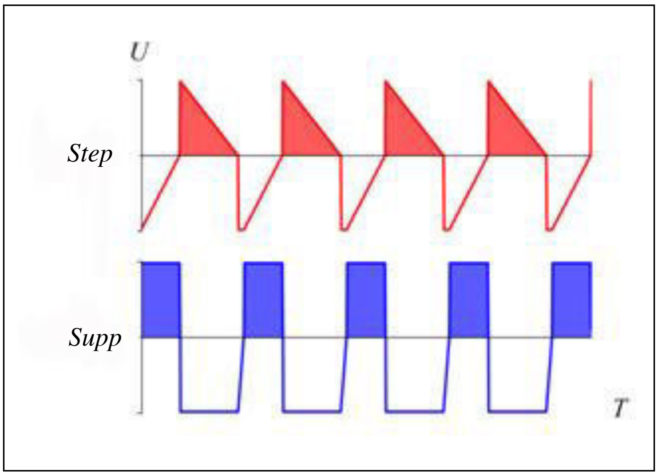

Fig. 7. Graphs of the membrane potential of model neurons with anti-phase excitation.
Ordinate axis corresponds to membrane potential and abscissa axis to time.

For this purpose, we introduce inhibitory connections from each step neuron on the right and left sides to two other step neurons on the same side. The corresponding diagram is shown in Fig. 8.

Here groups of neurons on the right and left sides are not connected with each other, and they are synchronized using the system parameters. They can be easily synchronized by making all support neurons silent and introducing one tonic neuron to excite them. However, this approach would considerably complicate the connection diagram, and Fig. 8 and 10 offer a simplified version.

Mutual inhibitory connections of the step neurons ensure that only one of them will be active on the left and right sides in each gait phase. The activation order is determined by the simulation parameters. The program-generated graphs of the membrane potentials of neurons are shown in Fig. 8. L_3*Step* is the first neuron activated on the left side. While active, it inhibits all other neurons *Step* on the left side. According to the simulation parameters, when the activity period of neuron L_3*Step* ends, L_2*Step* is activated first among the remaining step neurons. During the entire activity period, it inhibits the neighbors. Then neuron L_1*Step* switches on, and at the end of its activation period, the time step repeats. The order for the right side is symmetrical, with the only difference that neuron R_2*Step* is activated first.

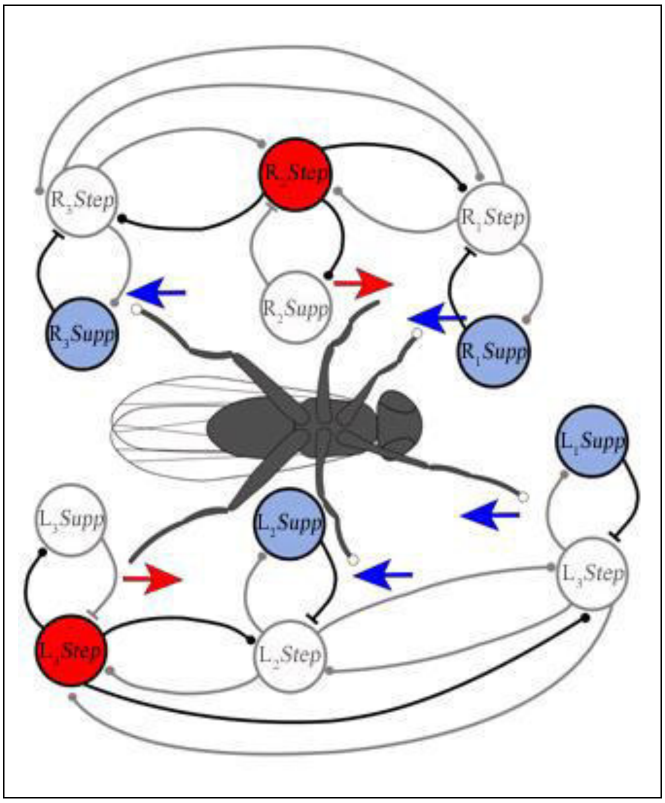

Fig. 8. Diagram of connections and activation of neurons in tetrapod gait phase.

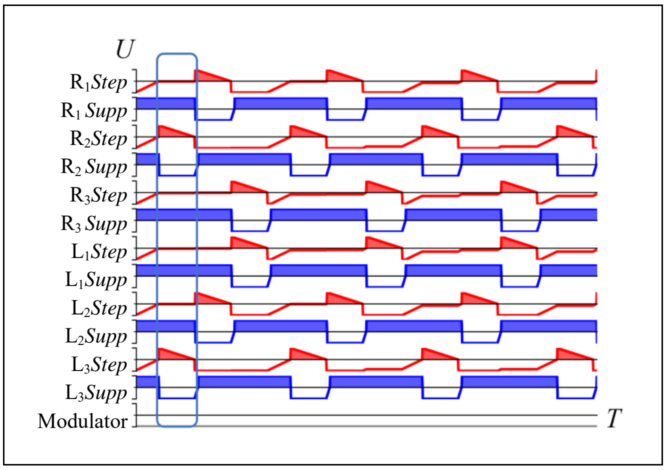

Fig. 9. Membrane potentials of neurons under tetrapod trait.
In each phase, only one step neuron is active on each side: two legs take step, and four are on ground. Modulatory neuron (silent) is required to adjust to tripod gait. Ordinate axis corresponds to membrane potential and abscissa axis to time.

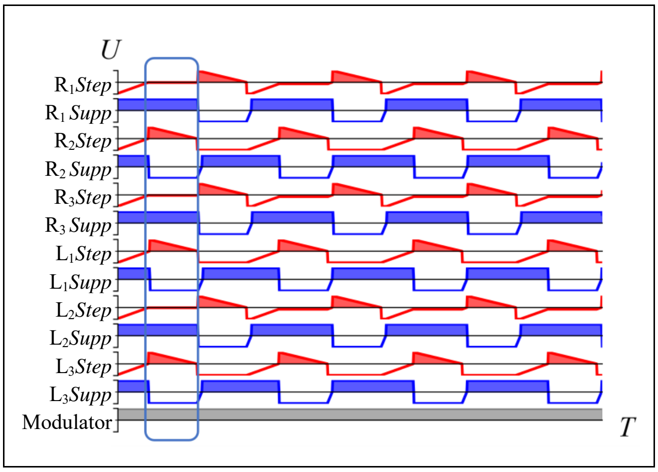

Fig. 11. Membrane potentials of neurons under tripod trait.
In each phase, three neurons are active on both sides. Modulatory neuron releases transmitter suppressing inhibitory connections between first and third neurons throughout operation. Ordinate axis corresponds to membrane potential and abscissa axis to time.

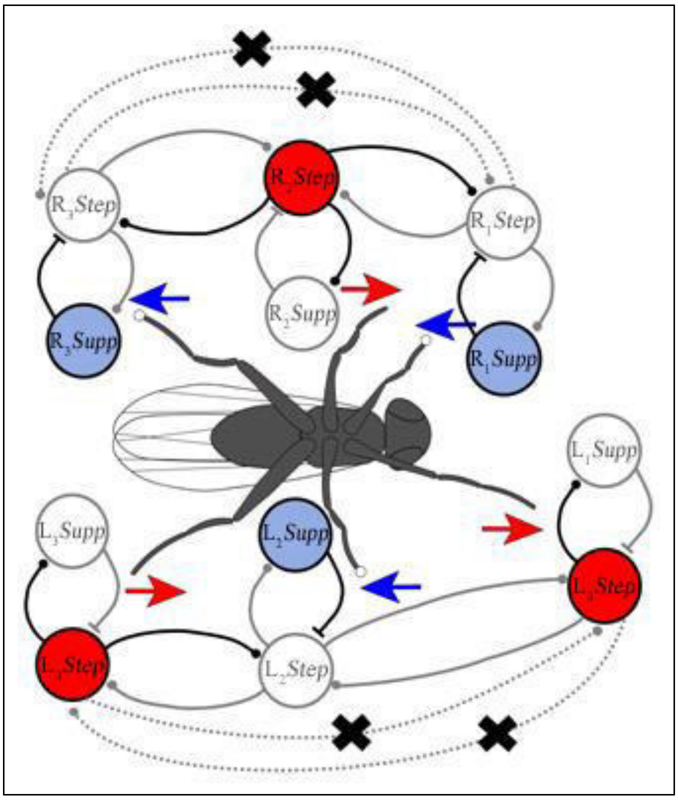

Fig. 10. Modulatory impact suppresses inhibitory connections between first and third neurons.

To harmonize this rhythm with the tripod gait diagram (Fig. 5*b*), it suffices to switch off the inhibitory connections between the first and third neurons on each side. This can be achieved by introducing a modulatory neuron, the transmitters of which switch off inhibitory receptors between the first and third neurons on each side (Fig. 10). As a result, the first and third neurons begin to activate synchronously, and the diagram corresponds to the tripod gait (Fig. 11).

CONCLUSIONS

This paper has proposed a formal description of the neuromodulation mechanism within the discrete asynchronous model of heterochemical neural interactions and demonstrated the results of switching the gait of hexapods.

The main and extremely significant neuromodulation effect is the rapid functional reconfiguration of neuronal circuits (both natural and artificial) without changing their structural properties. Thus, activity patterns can be changed not by long and costly changes in connections between neurons and not by switching between different neuronal circuits to perform different actions but by changing the chemical composition of the intercellular space within one neuronal ensemble. In the model, this is done by changing a single parameter. This mechanism greatly simplifies the control of gaits and other types of motor activity.

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28TH INTERNATIONAL CONFERENCE ON PROBLEMS OF COMPLEX SYSTEMS SECURITY CONTROL

In December 2020, the 28th International Conference on Problems of Complex Systems Security Control took place at Trapeznikov Institute of Control Sciences, Russian Academy of Sciences (RAS), Moscow. The conference organizers were the Ministry of Science and Higher Education of the Russian Federation, Trapeznikov Institute of Control Sciences RAS, Keldysh Institute of Applied Mathematics RAS, the RAS Scientific Council on the Theory of Controlled Processes and Automation, and the Ministry of the Russian Federation for Civil Defense, Emergencies and Elimination of Consequences of Natural Disasters.

The conference included the following sections:

- theoretical and methodological questions of security support;
- problems of economic and sociopolitical security support;
- problems of information security support;
- ecological and technogenic security;
- modeling and decision-making for complex systems security control;
- automatic systems and means of complex systems security support;
- legal aspects of complex systems security support.

At the conference, 130 authors from 42 organizations (Russia and some foreign countries) presented 91 papers.

According to the established tradition, the conference is held annually in the second half of December. The past 2020 will go down in history as the first year of the global fight against the COVID-19 coronavirus pandemic: its official date of appearance is November 17, 2019, when the first official diagnosis was made. The rapid spread of coronavirus around the world led to extremely serious consequences. (According to the World Health Organization data, by the end of 2020, the number of detected cases of this viral infection worldwide exceeded 80 million, and more than 1.6 million people infected with coronavirus died.) As a result, several negative processes and phenomena of the universal character were observed. The deepening of the global political and economic crisis caused by the pandemic, the destruction of international economic ties, the decline in production, and the growing social tension in many countries (particularly due to the imposed restrictive measures) significantly increased various kinds of risks and the emergence of new threats. At the same time, despite the changes in the world agenda associated with the coronavirus, acute problems and hotbeds of external and internal contradictions have not disappeared anywhere, like their reasons. Rather, on the contrary, the pandemic acted as a catalyst for the further exacerbation of the existing problems of socio-economic development both globally and nationally. All these factors considerably toughened the requirements to the quality and efficiency of security control in the broadest meaning, reflected in the topics and content of the conference papers.

The conference-opening paper "Imperatives of a new reality. The fate of capitalism. Risks of information and biological space" by *G.G. Malinetskii, V.V. Kul'ba, T.S. Akhromeeva, and S.A. Posashkova* was devoted to the analysis of modern threats to the development of human civilization, as well as measures to counter them. At present, the authors stated, the world is at the stage of transition from the industrial to post-industrial phase of development. This stage—in fact, a global bifurcation point—is associated with extremely serious threats, uncertainty, instability, and, at the same time, many unexpected opening opportunities.

The paper thoroughly analyzed the development prospects for a human civilization based on *Come On!: Capitalism, Short-termism, Population and the Destruction of the Planet*, a recent report to the Club of Rome, prepared for its 50th Anniversary in 2018. (This organization is well known for research into world dynamics and currently unites more than a hundred prominent representatives of world political, financial, cultural, and scientific elites.) The problems and prospects of the humanitarian and technological development of human civilization were considered in detail. The authors paid special attention to the extremely urgent problems of human society's fight against epidemics and pandemics.

The paper examined the reasons for the devastating response of most countries to the coronavirus pandemic, which led to a collapse of national economies, the destruction of a significant part of small and medium-sized businesses, an increase in unemployment, and social instability. Noting the tremendous successes of medical science in the last century, the authors emphasized the low level of readiness of most national health systems to deal with acute and large-scale epidemic problems. According to the authors, all these factors require recognizing healthcare as a priority sector of the economy and making carefully elaborated and radical changes in the management and financing of healthcare development, especially considering that by available forecasts, the coronavirus pandemic is far from the last on the foreseeable time horizon. (The consequences of the current pandemic have yet to be overcome.)

The paper "Pandemic, technologies, culture, and international stability" by *V.V. Tsyganov* considered a set of problems of civilizational development when reaching the global growth limits due to the bounded natural resources and the potential for their self-recovery, causing stagnation and social instability. Under these limits, the author emphasized, a significant part of the population of developed countries is in a state of depression. It is simultaneously strictly controlled by the global financial oligarchy, cultivating the consumer society values rigidly tied to its growth. The consequence is the strengthening of social instability in developed countries. Currently, a significant factor in reducing the level of consumption of the "golden billion" is the coronavirus pandemic and the associated restrictions on citizens' freedoms, a decrease in business activity, a drop in production, restrictions in the trade and service sector, etc. As noted in the paper, even the expected growth of the economy and, accordingly, consumption after the victory over the pandemic will be temporary: after a certain period, new global growth limits will be reached again and, accordingly, the period of the revival of the consumer society members will inevitably be replaced by the period of mass depression with all the ensuing consequences.

The author sees a fundamental way out of this situation in changing the value system of the "golden billion" (first of all, its middle class) from material to spiritual, which have no growth limits. However, note that this problem is unlikely to be resolved in the foreseeable future. In this situation, it is possible to somewhat reduce the severity of the emerging problems, according to the author, by creating an information technology for public security control under growth limits based on the behavioral models of society members. The paper emphasized that with the development of neurosciences, it is now possible to build models of humans considering their rationality, sensuality, and emotionality. These models are based on modern neurophysiological studies of the relationship between human behavior and hormonal characteristics. Currently, models of a far-sighted human, controlled by his desires, have been developed and used in sociological research, the development of public security systems, and high humanitarian technologies.

The paper "Models of SARS-CoV-2 virus spread and security control problems" by N.G. Kereselidze was of much interest. It presented the results of mathematical modeling for the spread of coronavirus based on the epidemic control protocol adopted by the healthcare system of the Republic of Georgia. As noted therein, the choice of strategy and tactics for fighting the pandemic is significantly influenced by several factors, the most important of which is the economy: a lockdown may become too expensive for the country's budget, which it may not be able to cope with. In this situation, the problem of assessing the need for quarantine measures and determining their starting time, volume and content, became urgent for protecting the population from infection, avoiding overloads of the healthcare system and, at the same time, preventing the crisis of the national economy, which would inevitably decrease the level of life of citizens.

According to the paper, this problem can be solved using the methodology of optimal control of dynamic systems. The model proposed by the author involves the apparatus of differential calculus to justify the need to introduce a lockdown (or the ability to refrain from strict quarantine measures) based on the number of detected coronavirus cases. Note that this is the first step towards solving an extremely difficult multidisciplinary task—assessing the need to introduce restrictive measures to fight the spread of coronavirus—which requires considering many epidemiological, medical, psychological, social, economic, and other factors.

Many conference papers, diverse by the subject, were devoted to a wide range of methodological and applied problems of managing the socio-economic development of Russia, its regions, and individual economic entities. Among them, we mention the following: “Experience and prospects of control of large-scale socio-economic projects development” by *N.I. Komkov*; “Fundamentals of a comprehensive assessment of risks of interstate integration entities” by *V.I. Medennikov*; “Tasks and specifics of organizing complex studies of the viability of rapidly changing Arctic systems” by *V.A. Putilov and A.V. Masloboev*; “Features of economic security control for the Arctic zone of the Russian Federation” by *R.V. Badylevich*; “Ensuring the reliability and security of information management systems for improving the survivability of the Russian energy system” by *E.P. Grabchak and E.L. Loginov*; “Crisis and security of socio-economic systems” by *A.Yu. Silant’eva, S.N. Grinyaeva, and I.V. Samarin*; “Systemic problems of public administration as a threat to national security” by *S.V. Zernov*; “Studying the security of control systems by analyzing their system parameters” by *D.A. Kononov*; “Decision support systems in the field of strategic planning and military security control. An approach to design and some features” by *Z.K. Avdeeva and S.V. Kovriga*; “Prospects for the development of domestic industrial companies” by *N.I. Komkov, A.A. Lazarev, and V.S. Romantsov*; “Multicriteria assessment of the economic security of an organization by criteria in quantitative and ordinal scales considering subjective probabilities” by *V.P. Korneenko*; “Applied issues of management in Russian integrated companies: risk factors and uncertainties” by *M.V. Krotova*; “Dividend policy of Russian companies as a factor in the development of the country’s economy” by *I.V. Cherenkov*; “Problems of calculating costs in the electric power industry in economic theory” by *V.I. Zhekov and N.V. Ivanov*; “A monitoring and security support system for natural resources” by *A.V. Golev*; “Optimal climatic models for ecological security support” by *V.G. Burlov, M.V. Mironova, A.I. Shershneva, and S.A. Shavurov*.

Traditionally, a large group of interesting papers was devoted to information security control and data protection from unauthorized access, relevant in the era of digitalization. Among them, we mention the following: “Assessment of information security under mixed uncertainty” by *G.S. Veresnikov and O.V. Ogorodnikov*; “Improving the efficiency of MANET mobile networks using data replication methods” by *S.K. Somov*; “A method for designing information security systems of complex objects” by *L.E. Mistrov*; “Analysis and assessment of information security risks of organizations” by *L.E. Sirotyuk*; “A risk assessment model for information security based on fuzzy logic” by *A.D. Kozlov and N.L. Noga*; “The emergence of dangerous situations when introducing digital twins at the objects of the fuel and energy complex and new development methods for special software to reduce these risks” by *A.V. Kryuchkov*; “On transferring information systems to domestic software” by *E.A. Kurako and V.L. Orlov*; “Communication problems in the digital information space” by *V.V. Muromtsev and A.V. Muromtseva*; “On control parameters in an information security monitoring model for complex systems” by *A.Yu. Maksimovskii*; “Modern issues of assessing data quality in the IT ecosystem” by *A.E. Mukhina*.

Following another tradition, the conference papers covered various topics on preventing and eliminating the consequences of man-made and natural emergencies and ensuring the secure (safe) and reliable operation of technological complexes and transport systems.

The paper “Methods for improving the safety of urban railways under centralized automatic control” by *L.A. Baranov and E.P. Balakina* was devoted to the problems of increasing the reliability and throughput of urban railway transport systems of a modern metropolis (subways and electric trains). The main attention was given to the development of schedule-interval algorithms for centralized control of urban train movements with a forecast of disturbances (e.g., exceeding the duration of station stops) considering the dependence of control constraints on the state of the transport system. Information about the forecasted disturbances (deviations of the durations of train stops from the planned ones) due to increased passenger traffic is generated at the output of an extrapolator developed by the authors based on Chebyshev polynomials. These algorithms minimize the number of restrictions imposed on the speed and duration of train stops.

Among the papers on the safe operation of transport systems, facilities, and their infrastructure, we mention the following: “Study of algorithms for optimizing the sequence and times of landing of aircrafts” by *E.L. Kulida and V.G. Lebedev*; “Advanced research and testing of swarms of air and ground vehicles for large-scale groups of joint autonomous systems in urban areas” by *M.V. Masyukov and S.A. Tyurin*; “Planning features for a safe transportation process on the Moscow Metro considering the operation of the Krasnaya Presnya depot of the Kol’tsevaya Line” by *A.I. Safronov*; “Modern challenges to the security of urban transport systems” by *V.G. Sidorenko*; “Choosing algorithms and parameters for an automatic speed control system of heavy long-haul trains based on the criterion of traffic safety" by *O.E. Pudovikov*.

Several interesting papers, diverse by the subject, considered various methodological and applied aspects of technogenic and industrial security and the reliable operation of technological complexes and systems. Among them, we mention the following: "Application features for situational-contextual visualization in monitoring and control systems" by *V.S. Nesterov and Yu.K. Bezgubova*; "Organizing compact visualization of information parameters in monitoring and control systems" by *A.M. Anokhin*; "Improving the technical safety of complex systems with a nuclear reactor" by *V.V. Leshchenko*; "IIoT vibration monitoring systems to support decisions on power equipment protection" by *O.B. Skvortsov*; "Modeling elastic stress waves in a ten-story building with half-plane base under non-stationary seismic action" by *V.K. Musaev*; "Analysis of the direct causes of accidents in a transport packaging container with radioactive materials when working with a crane" by *A.L. Yandreev*; "Optimizing the placement of detectors using the gradient method" by *A.A. Galyaev, A.S. Samokhin, and M.A. Samokhina*; "The law of synergy in the security support of high-risk facilities" by *T.A. Piskureva, L.A. Chernyakova, and A.N. Makhov*; "Decision support algorithms and models for determining an optimal number of employees in the territorial departments of the EMERCOM of Russia in the investigation of fire accidents" by *S.Yu. Karpov*; "Analysis and study of technological process hazards using the HAZOP method" by *A.Yu. Marusina, A.F. Akhmadieva, and M.A. Polyukhovich*; "An action algorithm for emergency response operations at PAO "Khimprom" by *M.V. Govor and A.Yu. Tumanov*; "Assessing the impact of human behavior characteristics on the time of evacuation using simulation" by *M.O. Avdeeva and K.A. Danilova*; "Methods for analyzing the safety of gas cylinder equipment at the stage of operation" by *A.A. Evstifeev*; "Analyzing the efficiency and safety of systems of individual heat supply stations" by *A.G. Bagoutdinova and V.L. Vorontsova*; "Increasing the reliability of a vital unit through the timely detection of a sudden failure of a product's structural element" by *S.A. Shilin*.

One of the distinguishing features of the conference was several interesting papers on the regulatory support of security control processes. Much attention of the participants was attracted to the paper "Web of Science and Scopus guarding the security of domestic science: the regulatory aspect" by S.A. Bochkarev. This paper presented the analysis of the requirements stipulated by Part 2 of Article 3 of the Federal Law no. 127-FZ "On Science and State Scientific and Technical Policy" dated August 23, 1996, obliging the state to ensure competition in the field of scientific activity and protect subjects of science from unfair competition. In the paper, from the standpoint of the federal legislation requirements, some bylaws of the Ministry of Education and Science were criticized for introducing the foreign concepts "Web of Science" and "Scopus" into the sphere of state jurisdiction. Moreover, as stated in the paper, their compliance with the concepts used in the mentioned Federal Law was not checked. These concepts denote the corresponding databases used as a source of information for calculating the place of the Russian Federation by specific weight in the total number of articles in the areas determined by the priorities of scientific and technological development. As emphasized by the author, Web of Science and Scopus are business projects managed by foreign commercial organizations (Clarivate Analytics, located in Pennsylvania, USA, and Elsevier, a European publishing house, respectively). They were not checked for participation in the illegal "sanctions policy" of Western countries against Russian society and the state.

In this regard, the paper raised pressing issues on the validity of decisions to assess the success of scientific organizations and employees by counting the number of publications in journals indexed by these databases as well as rules for forming a list of scientific periodicals where the results of research of scientific degree applicants must be published. As noted in the paper, these decisions impose an extensive list of requirements to Russian journals. However, there are no requirements for foreign periodicals included in the Web of Science and Scopus databases, which puts them in obviously unequal conditions. The absence of mutual obligations and legal guarantees of compliance with Russian legislation on the part of the foreign organizations mentioned, their branches and representative offices located on the territory of Russia, forces researchers and applicants to work with them outside the scope of Russian legislation, i.e., within foreign jurisdiction in the absence of a regulator in corporate relations.

The problems addressed in the paper are undoubtedly debatable. At the same time, an active discussion in the domestic scientific and expert communities will surely contribute to the competitive advantages of Russian science, increasing the efficiency of managing its development and eliminating negative factors and trends that reduce its independence and endanger its security.

Among other papers related to the legislature, we mention the following: “Management of the judicial system: retrospective and prospective aspects” by A.A. Timoshenko; “Technological gap in the field of new technologies and peculiarities of intellectual property protection – systems with reliable signs of artificial intelligence” by A.V. Rozhnov; “Legal regulation of the creation, maintenance, and operation of protective structures of civil defense” by E.K. Chalovskaya, I.O. Klochikhin, L.A. Belotserkovskaya; “Security of transport systems of the EAEU countries: regulatory and legal aspects of the new Silk Road” by Zh.I. Ismailov; “Risk management in a complex network based on arbitration award”; “Security support for complex systems within consistent regulatory acts” by T.Kh. Usmanova.

Unfortunately, it seems impossible to reveal (or even review) the content of all conference papers, interesting and diverse by the subject, due to objective limitations of this publication. The papers can be found in the conference proceedings[1] or on the official conference website: URL:https://iccss2020.ipu.ru/prcdngs.

In his closing remarks, the Conference Chair, Dr. Sci. (Eng.), Prof. V.V. Kul’ba announced plans to hold the 29th International Conference on Problems of Complex Systems Security Control, according to the established tradition, in December 2021 at Trapeznikov Institute of Control Sciences RAS. Please contact the Organizing Committee via phone + 7 495 198-17-20 (ext. 1407) or e-mail conf20@ipu.ru. The Technical Secretary of the conference is *Alla Farissovna Ibragimova*.

Academic Secretary of the Organizing Committee
A.B. Shelkov

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[1] Materialy 28-oi Mezhdunarodnoi konferentsii “Problemy upravleniya bezopasnost’yu slozhnykh sistem” (Proceedings of 28th International Conference on Complex Systems Security Control), December 16, 2020, Moscow, Kalashnikov, A.O. and Kul’ba, V.V., Eds., Moscow: Trapeznikov Institute of Control Sciences, 2020. (In Russian.)

ON THE 100TH ANNIVERSARY OF NAUM S. RAIBMAN'S BIRTH

February 4, 2021, marked the 100th anniversary of Naum Samoilovich Raibman's birth. An outstanding scientist, one of the "titans" of the golden age in the history of Trapeznikov Institute of Control Sciences, Russian Academy of Sciences (RAS).

Naum Samoilovich was born on February 4, 1921, in Medzhybizh, Khmelnytskaya oblast, Ukraine. His education at Moscow Machine Tool Institute (STANKIN) was interrupted by the Great Patriotic War. Together with fellow students, N.S. Raibman served in the Red Army. After the dismissal of students, he continued his studies in the institute. In 1943, after graduating from the institute, he was sent to Novosibirsk, where he worked as a technologist for several years, and then as a deputy head of a workshop at one of the defense plants.

In 1946–1950, N.S. Raibman was a postgraduate student at Moscow Aviation Technological Institute, where he continued to work after defending his candidate's dissertation. Then he delivered lectures at Ufa Aviation Institute and returned to Moscow in 1959. Since that time, he was the department head in one of the industry research institutes.

In 1959, Naum Samoilovich began working at the Institute of Automation and Remote Control, the USSR Academy of Sciences (now Trapeznikov Institute of Control Sciences RAS, hereinafter referred to as the Institute). At that time, the identification of control systems became the sphere of his scientific interests.

In 1965, N.S. Raibman defended his doctoral dissertation on identification and led a research group in the Laboratory of V.S. Pugachev. In April 1968, the group was reorganized into Laboratory No. 41 of the Institute.

Naum Samoilovich applied much effort to make the identification of control systems a separate line of research. Within this theory, under his leadership, new methods for identifying multidimensional, nonlinear, and time-varying objects, new methods for determining the structure of objects, and new methods for identifying distributed parameter objects were developed. The dispersion theory of statistically optimal systems was elaborated.

Combining the talent of a scientist and the technical erudition of an engineer, N.S. Raibman proposed the theory of adaptive control systems with an identifier (ASIs). This theory received a real embodiment: ASIs for controlling the accuracy of hot rolling of seamless pipes were adopted at many factories of the USSR.

In 1976, the research team of Laboratory No. 41 of the Institute, led by N.S. Raibman, was awarded the USSR State Prize for developing and successfully implementing a control system for a pipe rolling mill 160 at the Pervouralsk Novotrubny Plant (PNTZ) using a domestic computer UM1-NKh.

Naum Samoilovich devoted much time and effort to training young specialists for the USSR and Eastern European countries.

His scientific results were published in 7 books and 150 articles, which are of interest to researchers and students even today.

He was a member of the Scientific and Methodological Council of the Znanie Society in the RSFSR and an editor of Mir and Radio i Svyaz', well-known Soviet publishing houses.

Thanks to the active scientific and organizational activities of N.S. Raibman, many conferences on control theory – the local ones organized in the USSR and the global ones held by the International Federation of Automatic Control (IFAC) – began to include sections devoted to identification.

Naum Samoilovich and employees of his laboratory coordinated the branch of identification at leading international conferences and symposia held by IFAC, the Council for Mutual Economic Assistance (CMEA), and the European Economic Commission (EEC).

Employees of Laboratory No. 41 of the Institute, led by N. S. Raibman, organized and held the 4th IFAC Symposium on Identification and Estimation of System Parameters (Tbilisi, 1976). Also, the All-Union Symposia on Statistical Methods in Control (Moscow, Tashkent, Frunze, and Vilnius) and the All-Union Annual Seminars on Identification within the Cybernetics program were organized by them.

N.S. Raibman edited the translations of the best foreign books on identification published in the USSR.

Over the years, N.S. Raibman actively worked in the IFAC structures. During the last four years of his life, he was a member of the IFAC Advisory Committee.

Naum Samoilovich died suddenly on January 8, 1981, forty years ago. He left in the prime of his creative powers. Many researchers in different countries responded by scientific publications in his honor. The 6th IFAC Symposium on Identification and Estimation of System Parameters (Washington, 1982) was dedicated to his memory as well.

Today the life-work of N.S. Raibman – identification of control systems – is actively developed within traditional and new lines of research, particularly in Laboratory No. 41 of the Institute.

Naum Samoilovich was a man of extraordinary kindness, high intelligence, and bright giftedness. His entire adult life was devoted to developing domestic science, technology, and industry. He made an invaluable contribution to the theory of identification and control of complex systems.

Employees of Trapeznikov Institute
of Control Sciences RAS,
The Editorial Board and Editorial Office of the Journal