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CONTENTS

UNMANNED VEHICLES: A SURVEY OF MODERN SIMULATORS

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Abstract. This survey is devoted to popular simulators supporting rough terrain for unmanned vehicles, namely, Gazebo, CARLA, AirSim, NVIDIA Isaac Sim, and Webots. Their main capabilities related to terrain modeling, motion physics, and support for sensors and weather conditions are described. Particular attention is paid to the creation of realistic rough terrain scenes, the complexity of importing real maps, and interaction with other software platforms, such as Robot Operating System (ROS) and artificial intelligence (AI) systems. The main drawbacks of each simulator are analyzed: the labor intensity of creating detailed terrain and vehicle models, the high complexity of integrating real maps, and the dependence on powerful hardware. The survey also notes the complexity of interaction with various software solutions and the required knowledge of 3D modeling. Gazebo and Webots are remarkable for their good integration with ROS but require more effort to work with rough terrain. CARLA and AirSim provide high-quality visualization but have higher requirements for creating landscapes. NVIDIA Isaac Sim stands out for AI simulation support but is resource-intensive. The authors' experience in mapping vehicle trajectories and orienting in some simulators is presented.

Keywords: unmanned vehicles, simulators, rough terrain, tracked platforms.

INTRODUCTION

The development and testing of unmanned vehicles is one of the most difficult and urgent tasks of modern robotics and the automotive industry. To ensure the safety and reliability of autonomous systems, it is necessary to conduct large-scale tests in various conditions, including complex scenarios of interaction with the environment and other road users. However, conducting such tests in the real world leads to several challenges, including high costs, safety risks, and limited repeatability of experiments [1–3]. Simulators have long been an important tool in engineering and computer science. The first full-fledged vehicle simulators appeared as early as the 1990s and were intended to study particular aspects and characteristics of vehicle motion. With the development of computing power and modeling algorithms, simulators gradually became more sophisticated and realistic, allowing one to reproduce road conditions and interactions with other vehicles, pedestrians, and infrastructure.

In recent years, the development of *open-source* (OS) software communities has further contributed to the availability and flexibility of simulators. Platforms such as CARLA, Gazebo, and AirSim have become widespread due to their open-source nature and strong support from the development community. OS simulators allow researchers and engineers to customize and modify the simulation environment in accordance with the particular requirements of their projects, greatly accelerating the development and testing processes. Before the extensive evolution of OS simulators, some researchers used relevant computer games such as GTA [4].

This paper aims to review modern simulators used for the development and testing of unmanned vehicles, with a special focus on OS solutions supporting rough terrain simulation. We discuss different types of simulators and the fields of their application in the context of unmanned vehicles, particularly unique off-road transport platforms with electric drive of different layouts (wheel-tracked and ski-tracked, see Fig. 1). Significant datasets have already been collected to analyze the motion models of such mobile platforms within validating the elements of a conceptual distributed network of testing grounds for practicing the application scenarios of heterogeneous groups of electrically driven vehicles in difficult climatic and landscape conditions [5]. Based on these datasets, we plan to design autonomous motion control algorithms using

(a) (b)

Fig. 1. Tracked platforms: (a) wheel-tracked and (b) ski-tracked.

the 2D path planning methods proposed in [6], with a suitable adaptation to rough terrain.

1. THE MAIN TYPES OF SIMULATORS

Modern autopilots are extremely complex systems consisting of many interconnected modules, each performing definite functions. These systems include modules for localization, obstacle detection and tracking, traffic flow analysis, path planning, and path following. Each module requires a separate approach to development, testing, and optimization, making the use of simulators an integral part of the process of creating and improving autonomous vehicles. Different types of simulators are used to effectively model and test different operational aspects of autopilot systems, each focusing on particular tasks. Traffic flow simulators are intended to model and analyze vehicle interactions in urban and suburban environments; vehicle dynamics simulators focus on the physical behavior of a vehicle; sensor and perception simulators serve to model data from cameras, lidars, and other sensors. In addition, there are simulators for creating complex motion scenarios; they allow testing decision algorithms in various road situations.

In this section, we discuss the main types of simulators used to develop and test autopilot systems, their key features, and examples of the most popular solutions in each category.

1.1. Traffic Flow Simulators

Traffic flow simulators are intended to model the motion of multiple vehicles on roads, including their interactions, in order to analyze and optimize transportation systems. These simulators help to study the dynamics of traffic flows, the impact of different scenarios on congestion, the effectiveness of various road infrastructures, and the behavior of road users in different situations [7].

Key features:

• motion modeling for a large number of vehicles;

• support for different types of road networks and scenarios;

• the ability to integrate with transportation management systems (e.g., traffic lights);

• the analysis and visualization of traffic flows and congestion.

Examples of simulators:

• SUMO (Simulation of Urban Mobility) is one of the most widespread simulators for modeling traffic flows in urban environments. It supports the modeling of large urban networks and integration with other simulators [8].

• AIMSUN is a commercial simulator used for traffic flow analysis and management, with support for complex vehicle-to-vehicle interaction models [9].

1.2. Vehicle Dynamics Simulators

These simulators focus on modeling the dynamic performance characteristics of individual vehicles. They are used to analyze vehicle behavior in different conditions (acceleration, braking, steering on slippery surfaces, and interaction with road irregularities). Such simulators are important for the development and testing of control systems, particularly stabilization and autonomous driving systems [10].

Key features:

• the high-accuracy modeling of physical processes (chassis, suspension, engine, and brake system dynamics);

• modeling of vehicle-road surface interaction;

• support for different types of vehicles, including cars, trucks, and motorcycles;

• the ability to simulate extreme conditions (accidents).

Examples of simulators:

• CarMaker is an industry standard for vehicle dynamics modeling. It supports the testing of *advanced driver-assistance systems* (ADAS) [11].

• TruckSim is a special-purpose simulator for modeling the dynamics of heavy vehicles (trucks and buses) [12].

1.3. Sensor and Perception Simulators

These simulators focus on modeling the sensors used in autonomous vehicles (cameras, lidars, radars, and ultrasonic sensors). The main purpose of such simulators is to reproduce sensor data realistically, which can be used to develop and test perception and decision algorithms [13].

Key features:

• realistic modeling of sensor data in different environmental conditions;

• support for multiple sensor types and combinations;

• the ability to integrate with image and signal processing algorithms;

• testing and debugging perception systems in complex scenarios (poor weather conditions or poor lighting).

Examples of simulators [14]:

• CARLA provides a wide range of sensors and models for testing perception systems in urban environments.

• AirSim is a Microsoft simulator that supports the realistic modeling of sensor data and is used to develop autonomous drones and ground vehicles.

1.4. Complex Simulators

These simulators are intended to create and test complex traffic scenarios involving autonomous vehicles. They allow the modeling of various traffic situations and the interaction of an autonomous vehicle with other road users, which is especially important for developing decision systems. As a rule, they include a certain implementation of other types of simulators or have the possibility of integrating third-party solutions [15, 16].

Key features:

• support for creating complex scenarios with multiple vehicles and pedestrians;

• integration with decision and traffic control algorithms;

• the ability to simulate rare and extreme situations such as accidents or sudden obstacles;

• analysis and visualization of decisions made and their consequences.

Examples of simulators:

• CARLA is a flexible and extensible platform for training, testing, and validating autonomous driving systems; built on the Unreal Engine visualizer, CARLA offers highly accurate environments, realistic physics, and a full suite of sensors, including cameras, lidars, and radars [17].

• PreScan is used for the development and testing of ADAS systems and autonomous vehicle systems, including the modeling of complex scenarios and traffic situations.

• LGSVL Simulator supports the modeling of various scenarios and integration with autonomous control development platforms (Apollo and Autoware) [14].

• AutoDRIVE Simulator is a high-accuracy simulation platform developed using the Unity game engine; it includes a vehicle model equipped with a comprehensive set of sensors and actuators to facilitate research and training in autonomous vehicle technologies [18].

2. WORKING WITH ROUGH TERRAIN SCENES

The tasks set before modern unmanned vehicles require them to move on smooth, asphalted, and marked roads and, moreover, on rough terrain with different types of surfaces, ranging from dirt to snow [19]. To practice control, navigation, and localization algorithms, an appropriate simulator is required to implement motion over challenging terrain with a set of typical obstacles. Since the use of unmanned vehicles is not yet widespread, the number of appropriate simulators is significantly limited. Note the additional requirements for such simulators compared to those involving asphalted roads:

• The physical terrain model. It must support terrain with unevenness, different types of ground (gravel, sand, rocks), water, and other natural obstacles. This allows one to test the vehicle's response to slippery or loose surfaces as well as uphill and downhill slopes.

• Detailed vehicle models. It is important to consider the characteristics of different types of vehicles, whether they are autonomous cars, tracked vehicles, wheeled robots, or even drones capable of moving over difficult terrain. Simulators must accurately model suspension, grip, dynamometry, and other important aspects.

• Environment. Simulators must include various terrain types and simulate weather conditions (snow, fog, and wind) that can strongly affect vehicle handling. In the simulators discussed below, different weather conditions affect only the visual series acquired by sensors; the impact of these factors on grip, if possible, is explicitly shown in the simulator specifications.

The next subsections describe in detail several simulators that can be used to model motion over rough terrain.

2.1. Gazebo

This is one of the most popular simulators of robots and unmanned vehicles, supporting realistic physics and a high degree of customization. Due to its open-source code, the simulator is used in research and educational projects, as well as in commercial developments [20, 21]. It represents an open platform to integrate various physics engines (ODE, Bullet, Simbody, and others).

Advantages:

• Realistic physics. Gazebo supports challenging terrain with bumps, hills, rocks, and even water, allowing one to simulate scenarios of autonomous vehicle motion over rough terrain.

• Grip and suspension simulation. It is important for tracked or wheeled vehicles operating on rough terrain; one can accurately simulate how the vehicle will behave over challenging terrain.

• Support for different surface types. The simulator can simulate different types of ground, including slippery or loose surfaces (sand, gravel, and mud).

• Environment. One can add effects (rain, wind, and snow) that influence motion conditions over rough terrain.

• Integration with Robot Operating System (ROS). Gazebo is actively used in conjunction with ROS, which makes it convenient for developing and testing autonomous systems in real-world conditions.

Drawbacks:

• Scene creation. Although Gazebo provides tools for creating scenes, developing detailed rough landscapes can be time-consuming. Users often face the need to manually model challenging terrain elements.

• Map import. Maps can be automatically imported from real data (e.g., satellite data) via additional plugins, but this requires additional customization and does not necessarily provide sufficient detail for challenging terrain.

• The complexity of model creation. Creating detailed 3D vehicle models requires knowledge of 3D modeling and physical simulation. This process can be challenging for users unfamiliar with such tools.

• Integration with other systems. While Gazebo integrates well with ROS, integration with other frameworks may require additional effort. For example, the use of other physics engines or control interfaces may be limited without manual configuration.

Complexity: medium. Gazebo is oriented toward researchers, so a good understanding of ROS and simulation basics will be required to create and set up complex scenes.

2.2. CARLA

This is an open-source simulator developed for testing autonomous vehicles in urban environments [22]. However, it can be adapted to rough terrain, as it provides sufficient capabilities to create nonstandard landscapes [23].

Advantages:

• Environment customization. CARLA provides tools to create custom maps, allowing one to build challenging terrain with rough elements.

• Support for realistic motion physics. CARLA can simulate vehicle dynamics over rough terrain, including speed, grip, and stability control.

• The ability to work with different types of surfaces. Despite its urban orientation, CARLA can simulate grass, dirt, sand, and other types of surfaces.

• Weather conditions. Various weather conditions affecting grip and visibility can be simulated.

Drawbacks:

• Scene creation. CARLA is intended primarily for urban environments, and creating rough terrain scenes may require manual customization. The built-in maps do not include rough terrain, so it is necessary to import user maps and manually customize terrain.

• Map import. Importing real maps requires additional tools and modules, as well as skills in working with 3D graphics and geospatial data. Despite the declared support for rough terrain [24], no examples or publications on the application of this functionality have been found so far, making adaptation to rough terrain difficult.

• The complexity of model creation. CARLA contains ready-made vehicle models, but third-party tools such as Blender or Maya have to be used to create unique models. This can be challenging, especially if one needs to detail suspension and motion dynamics.

• Integration with other software solutions. This simulator integrates well with Python API for scripting, but interaction with other systems requires additional customization. There is no built-in ROS support, which can be disadvantageous for ROS projects.

Complexity: high for rough terrain. Considerable effort will be required to create detailed natural scenes and integrate real maps.

2.3. AirSim

This simulator was developed by Microsoft for drones and ground vehicles [25]. Its main advantage is integration with Unreal Engine, which allows one to create highly detailed 3D scenes, including rough terrain [26].

Advantages:

• Highly detailed terrain. Owing to Unreal Engine, AirSim can accurately simulate various challenging terrain types, from mountainous landscapes to dense forests.

• Support for realistic motion physics. Different types of vehicles can be simulated in AirSim, including wheeled and tracked platforms, allowing one to test motion over challenging terrain.

• Sensor customization. The simulator enables one to model various sensors (cameras, lidars, and GPS trackers), which is especially useful for testing operation in rough terrain conditions with low-level signals.

• Weather and lighting conditions. It is possible to simulate various weather conditions (rain, snow, and fog), which considerably complicate navigation over rough terrain.

Drawbacks:

• Scene creation. Since AirSim involves Unreal Engine, creating a scene requires working with the game engine's tools. Despite its visual power, Unreal Engine has a fairly high entry threshold for beginners. Creating complex landscapes and environments can be time-consuming and require serious 3D modeling skills.

• Map import. AirSim has no direct option for importing real geodata. Third-party tools are available, but their setup is complicated.

• The complexity of model creation. One has to use Unreal Engine or third-party 3D modeling programs to create detailed vehicle models. The process of integrating new models with physical simulation can be labor-intensive.

• Integration with other software solutions. AirSim provides API for working with Python and C++, but interaction with ROS or other systems will require additional effort. Integration with other frameworks is limited compared to Gazebo.

Complexity: high. Despite powerful visualization capabilities, the complexity of creating scenes and working with real maps makes AirSim more time-consuming for academic or commercial projects with rough terrain.

2.4. NVIDIA Isaac Sim

This robot simulation platform from NVIDIA is intended to support complex tasks of robotics and autonomous transportation. Based on the capabilities of the graphics processing unit (GPU), it allows one to simulate complex scenarios, including rough terrain. Also, custom extensions can be implemented with flexible functionality [27–29].

Advantages:

• Photorealistic environment. The PhysX engine and GPU acceleration allow accurately simulating vehicle motion physics over challenging terrain and various surfaces.

• Integration with real-world algorithms. Isaac Sim supports integration with deep learning and path planning algorithms to model and test complex scenarios over rough terrain.

• Support for different robot types. The simulator can model both wheeled vehicles and tracked robots or drones, thereby being versatile for autonomous motion tasks over rough terrain.

• Sensors. The simulation of complex sensor systems, including cameras, lidars, and GPS trackers, is supported to test robot performance in rough terrain conditions.

Drawbacks:

• Scene creation. Although Isaac Sim provides tools for creating complex scenes, modeling rough terrain will require considerable effort. The platform is oriented toward high-performance computing using GPUs, which can make development difficult for users without powerful hardware.

• Map import. It is possible to integrate real map models into Isaac Sim through 3D modeling, but the process involves third-party tools and can be quite complex, especially for the realistic simulation of terrain and natural conditions.

• The complexity of model creation. Creating new vehicle models and adapting them to motion physics requires extensive training and knowledge of 3D graphics. Incorporating suspension dynamics and complex mechanisms can be challenging for non-professionals.

• Integration with other software solutions. The platform integrates well with NVIDIA AI tools, but integration with ROS or other autonomous systems may require additional modules and customizations. This can be difficult for projects expecting quick and easy integration.

2.5. Webots

This free open-source robot simulator is used for educational and research purposes [30, 31]. It supports a wide range of robotic systems, including unmanned vehicles.

Advantages:

• Flexible terrain modeling. In Webots one can create custom models of challenging terrain, including mountains, hills, canyons, and other natural objects.

• Support for various vehicles. Webots can simulate the operation of both wheeled and tracked vehicles, allowing one to test motion algorithms over challenging terrain.

• Sensor modeling. Webots supports a wide range of sensors, therefore being suitable for testing autonomous systems over rough terrain using cameras, lidars, and GPS trackers.

• Weather conditions. One can simulate different weather conditions (rain, fog, etc.) that affect visibility and grip.

Drawbacks:

• Scene creation. Webots has a user-friendly interface for creating simple scenes; however, when it comes to complex rough terrain, one has to customize reliefs and surfaces manually. This can be a limitation compared to other (more advanced) simulators.

• Map import. Importing real maps is not directly supported: one has to use third-party tools to create complex reliefs and challenging terrain. Therefore, rough terrain modeling will be much more difficult compared to more powerful simulators.

• The complexity of model creation. Webots has a library of standard robots, but creating custom models requires using third-party 3D modeling tools. Built-in modeling tools are limited, which complicates work with unique vehicles.

• Integration with other software solutions. Webots supports integration with ROS and other popular frameworks. However, more complex tasks, such as deep integration with external AI systems, may require the development of additional modules.

2.6. Blender with OpenDroneMap photogrammetry

Blender is an editor for creating 3D computer graphics, including modeling, sculpting, animation, simulation, rendering, post-processing, and sound video editing. OpenDroneMap is a set of photogrammetry tools based on aerial images, which generates 3D maps of the captured terrain [32].

Advantages:

• Modeling. The platform provides almost unlimited capabilities for modeling and simulation of processes: there is a rich set of tools, from basic to highly specialized, to build a custom simulator [33].

• Map import. Plug-ins are available to import maps and project satellite images onto public elevation data.

• Animation. This platform has flexible options for creating animation and access to the Python API, where any properties can be changed.

• User-friendly interface. After creating the necessary auxiliary tools, the tasks of importing sensor records are reduced to one-button solutions with quick debugging of the results.

Drawbacks:

• Modularity. It is necessary to search (create, assemble) all parts of the simulation.

• GNSS support. There is no built-in binding of the 3D-editor base space to geographic coordinates: all work with real data requires the careful recalculation of coordinate frames and the preliminary preparation (processing) of records.

• Interaction physics. Note the complexity of simulating the physical interaction between vehicle and terrain. Blender is primarily intended for animation; its built-in physics emulation tools are often used to simplify the realization of artistic intent rather than to calculate physical loads.

Complexity: high. Despite powerful visualization capabilities, the primary complexity of creating scenes and working with real maps makes the simulator difficult to use in academic or commercial projects with rough terrain.

2.7. NVO73

This is a 3D simulator developed at the Trapeznikov Institute of Control Sciences, the Russian Academy of Sciences, based on Unreal Engine 5.2 and AirSim to simulate the joint operation of a group of unmanned ground, underwater, surface, and aerial vehicles [34]. Using AirSim as the core inherits all its advantages and drawbacks. There is an adapted and simplified assembly for viewing the records of electric vehicle paths, along with the operational characteristics of the power unit.

Advantages:

• Easy viewing control. There are a convenient slider with a timeline and start/stop buttons.

• Support for heterogeneous vehicles. The platform supports four types of unmanned vehicles: ground, aerial, surface, and underwater.

• The variety of animations. One can visualize engine revolutions through wheel rotation animation as well as change lighting and weather conditions.

• Simplified run and import of records. The entire environment runs with a single executable file and reads the prepared record file from a predefined folder.

• A built-in algorithm corrects the path record by elevation, linking the motion to the map surface.

Drawbacks:

• No ability to add terrain models. New maps can be added only with developer involvement.

• No display of the entire path. For each time instant, only the current position of the vehicle is shown, making it difficult to assess the whole record.

• Surface referencing ignores recorded elevation data, making the simulation highly dependent on the quality of the terrain model.

• No documentation. All information about the simulator can be obtained only from several papers or directly from its developers, which causes difficulties during the installation procedure and further use.

2.8. Comparison of the Simulators Considered

Each of the above simulators provides some rough terrain modeling capabilities. Which simulator should be chosen? The answer depends on the specifics of the simulation project. Gazebo and AirSim are suitable for the highly detailed simulation of physical conditions; CARLA can be adapted for nonstandard tasks; NVIDIA Isaac Sim offers powerful GPU simulation capabilities; Webots is a simple and flexible tool for educational purposes. Gazebo and Webots are easier to integrate with ROS but may require more manual work to create complex natural landscapes. In turn, CARLA and AirSim are good for high-quality visualization but require significant effort to create rough terrain scenes and import real data. NVIDIA Isaac Sim offers strong support for AI simulations but requires high-performance resources and complex customization to create a scene. Building a terrain model with OpenDroneMap and simulating motion records in Blender are suitable for visualizing terrain and paths but provide no ready-made tools for simulating physical interactions. Most of these simulators support interaction with external tools, making it possible to reproduce the behavior of a real object in a simulation environment.

To select the simulator, we define the main parameters under comparison.

• RTF (*Real-Time Factor*), an index to evaluate the speed of simulation relative to real time. It is widely used in robotics and autonomous transportation systems to analyze performance and computational efficiency.

• Sensor refresh rate, representing the frequency of updating the readings of sensors (lidars, cameras, and GPS trackers). The higher this rate is, the more accurate the simulation process will be (of course, at the cost of increasing the load on the processor and graphics units).

• Graphics engine. It is responsible for rendering images. Unreal Engine and Omniverse RTX provide high-quality graphics but require powerful hardware. OGRE and OpenGL are simpler and more lightweight.

• Physics engine. It is responsible for simulating motion dynamics, collisions, friction force, suspension reactions, and other physical effects. The more powerful the engine is, the more realistically the objects will behave.

• The maximum number of vehicles (robots). It shows how many objects can be modeled simultaneously. This number is limited by the computing power of the processor and video adapter.

The simulators considered in the survey are compared by these parameters in Table 1.

The final expert assessment of the capabilities and technical specifications of the simulators is given in Table 2; scores 1, 2, and 3 correspond to the least complicated, medium, and most complicated ones. According to Table 2, it seems impossible to identify a simulator excelling than the others by all parameters.

3. PRACTICAL APPLICATION OF SIMULATORS

We have to simulate the motion of wheel-tracked and ski-tracked platforms (Fig. 1), primarily to analyze their application scenarios in the sections of a distributed network of testing grounds located in the mountainous areas of the Murmansk region and the Caucasus [5]. To solve scientific and applied tasks, the simulator must satisfy the following requirements:

• support for rough terrain motion, including various types of surfaces (dirt, snow, and sand);

• the availability of ready-made vehicle models, close by characteristics to those in Fig. 1, or the possibility of adding a custom model;

• Python integration;

• shareware (no need to purchase a license);

• the ability to display motion based on real vehicle data.

Table 1

Simulators for unmanned vehicles: a comparison of characteristics

Characteristics	Gazebo	CARLA	AirSim/NVO73	NVIDIA Isaac Sim	Webots
Physics engine	ODE, Bullet, Simbody, DART	Unreal Engine PhysX	Unreal Engine PhysX	NVIDIA PhysX	ODE
RTF	1.0–2.0	~1.0 (depends on the video adapter)	~1.5 (depends on settings)	1.0 (high GPU demands)	1.0–3.0 (lightweight)
Sensor refresh rate	100–1000 Hz	~100 Hz (cameras, lidars)	120 Hz (lidars), 30–60 Hz (cameras)	240 Hz (lidars), 60 Hz (cameras)	~100 Hz
Graphics engine	OGRE (basic)	Unreal Engine 4	Unreal Engine 4 (5.2 in the case of NVO73)	Omniverse RTX (high detail)	Built-in OpenGL-based
The maximum number of vehicles	50+ (optimized)	~20–50 (depends on settings)	10–30 (depends on GPU)	100+ (with RTX acceleration)	10–50 (optimized for mobile robots)

Table 2

Simulators for unmanned vehicles with rough terrain support: expert assessments (a lower score corresponds to a higher priority)

Criterion	Gazebo	CARLA	AirSim/NVO73	NVIDIA Isaac Sim	Webots	ODM+Blender
Scene creation	2	3	3	2	1	2
Map import	2	3	3	3	3	2
The complexity of model creation	2	3	3	3	1	2
Integration with other software solutions	1	1	1	2	2	1
Rough terrain support	1	3	1	1	2	1
Display of real object's logs	1	1	3	2	1	2
Equipment requirements	1	2	3	3	1	2

According to the survey results and Table 2, it is impossible to identify one simulator excelling the others in all parameters. The most suitable simulators are Webots, Isaac Sim, and Gazebo. However, Isaac Sim requires high-performance video adapters; Gazebo leads by all parameters, except for the complexity of creating scenes (being inferior to Webots). We do not care so much about the ability to create custom scenes, so further experiments and testing will be done using Gazebo. This simulator offers tools for modeling the physical behavior of a vehicle, working with sensors, and visualizing the collected data. The tracked platforms under consideration have complex dynamics, especially when moving on rough terrain. Therefore, Gazebo with its realistic physics engine allows testing important aspects such as track grip, suspension behavior, and the effects of different types of ground (sand, dirt, and rocks) [35].

We tried two different tools to display real data recorded on a motorcycle or tracked platform:

– the NVO 73 simulator based on Unreal Engine [34, 36];

– Blender with OpenDroneMap photogrammetry.

The functionality of NVO 73 is still very limited, which prevents it from satisfying all the requirements for visualization tools. However, it is possible to reproduce motion from a pre-recorded log file. The interface of this simulator is shown in Fig. 2.

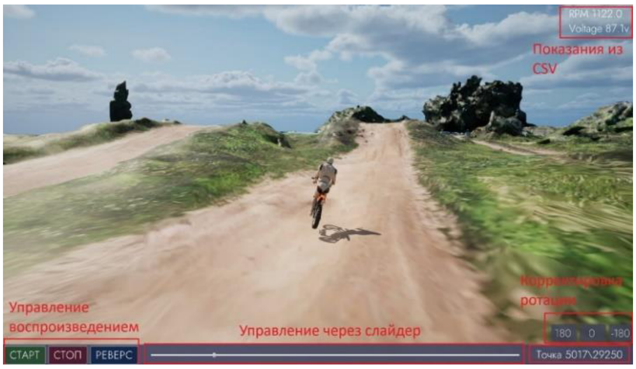

Fig. 2. The interface of the NVO 73 simulator.

Rendering motion paths on terrain models from OpenDroneMap in Blender turns out to be quite visual (Fig. 3), also demonstrating the record of body movements. However, the synchronization of records is very labor-intensive, as it requires manual corrections for all initial positions (the terrain model and satellite navigation coordinates, and the mutual location of the vehicle and body) [36]. Without developing a physical motion model and binding the vehicle to the ground surface and the body to the vehicle, the reconstruction process yields periodic significant discrepancies with the control video record: the vehicle flies into the sky or falls under the ground, whereas the body turns and shifts in different directions.

Fig. 3. Rendering motion in Blender with ODM.

CONCLUSIONS

This survey has been devoted to popular simulators for modeling various aspects of autonomous vehicles. Special attention has been paid to simulators supporting rough terrain: Gazebo, CARLA, AirSim, NVIDIA Isaac Sim, and Webots. Each of these simulators has unique capabilities, suitable for different applications, and some limitations as well.

For example, Gazebo and Webots stand out for easy integration with ROS and low hardware requirements but, at the same time, need significant effort to create complex rough terrain scenes. CARLA and AirSim offer high-quality visualization and flexibility for custom scenarios; however, the complexity of setting up rough terrain and high hardware load can be limiting factors for these platforms. In turn, NVIDIA Isaac Sim demonstrates outstanding AI and GPU-accelerated simulation capabilities but has high computational demands and complexity of customization. The final choice of an appropriate simulator depends on project goals, available hardware, and the level of simulation detail required. Regardless of the choice, the use of simulators significantly accelerates the development and testing of autonomous systems, minimizing the risks and costs of real-world testing. For tasks involving rough terrain, one should consider the simulator's capabilities and, moreover, the complexity of integrating real data (relief and motion records), which is especially relevant for research and optimization of autonomous control algorithms.

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REFERENCES

1. Karunakaran, D., Berrio, J.S., and Worrall, S., Challenges of Testing Highly Automated Vehicles: a Literature Review, *Proceedings of 2022 IEEE International Conference on Recent Advances in Systems Science and Engineering (RASSE)*, Tainan, 2022, pp. 1–8. DOI: 10.1109/RASSE54974.2022.9989562
2. Beringhoff, F., Greenyer, J., and Roesener, C., Thirty-One Challenges in Testing Automated Vehicles: Interviews with Experts from Industry and Research, *Proceedings of 2022 IEEE Intelligent Vehicles Symposium (IV)*, Aachen, 2022, pp. 360–366. DOI: 10.1109/IV51971.2022.9827097
3. Lou, G., Deng, Y., Zheng, X., et al., Testing of Autonomous Driving Systems: Where Are We and Where Should We Go?, *Proceedings of the 30th ACM Joint European Software Engineering Conference and Symposium on the Foundations of Software Engineering*, Singapore, 2022, pp. 31–43.
4. Martinez, M. and Sitawarin, C., Beyond Grand Theft Auto V for Training, Testing and Enhancing Deep Learning in Self-Driving Cars, *arXiv:1712.01397*, 2017.
5. Korgin, N.A. and Meshcheryakov, R.V., The Conceptual Project on Creating a Distributed Network of Testing Grounds for Practicing the Application Scenarios of Heterogeneous Groups of Electrically Driven Vehicles in Difficult Climatic and Landscape Conditions, *Trudy 11-oi Vserossiiskoi nauchnoi konferentsii "Sistemnyi sintez i prikladnaya sinergetika"* (Proceedings of the 11th All-Russian Scientific Conference "System Synthesis and Applied Synergetics"), Nizhnii Arkhyz, Rostov-on-Don: Southern Federal University, 2022, pp. 197–202. (In Russian.)
6. Makarov, M.I., A Local Path Planning Algorithm for Avoiding Obstacles in the Frenet Frame, *Control Sciences*, 2024, no. 3, pp. 56–61.
7. Mitrohin, M.A., Alyaev, A.O., Lobanov, R.I., and Semenkin, M.V., Investigation of the Influence of Lighting Objects

Control Algorithms on the Characteristics of Road Traffic at Intersections, *Transport Automation Research*, 2024, no. 3, pp. 282–295.
8. Lim, K.G., Lee, C.H., Chin, R.K., et al., SUMO Enhancement for Vehicular Ad Hoc Network (VANET) Simulation, *Proceedings of 2017 IEEE 2nd International Conference on Automatic Control and Intelligent Systems (I2CACIS)*, Kota Kinabalu, 2017, pp. 86–91.
9. Barceló, J., Barceló, P., Casas, J., and Ferrer, J.L., AIMSUN: New ITS Capabilities, *Proc. Eur. ITS Conf.*, Bilbao, Spain, 2001, pp. 1–10.
10. Ghafarian, M., Watson, N., Mohajer, N., et al., A Review of Dynamic Vehicular Motion Simulators: Systems and Algorithms, *IEEE Access*, 2023, vol. 11, pp. 36331–36348.
11. Ziegler, S. and Höpler, R., Extending the IPG CarMaker by FMI Compliant Units, *Proceedings of 8th International Modelica Conference*, Dresden, 2011, pp. 779–784.
12. *TruckSim Overview*. URL: https://www.carsim.com/products/trucksim/index.php. (Accessed September 30, 2024.)
13. Silva, I., Silva, H., Botelho, F., and Pendao, C., Realistic 3D Simulators for Automotive: a Review of Main Applications and Features, *Sensors*, 2024, vol. 24, no. 18, art. no. 5880.
14. Li, Y., Yuan, W., Zhang, S., et al., Choose Your Simulator Wisely: a Review on Open-Source Simulators for Autonomous Driving, *IEEE Transactions on Intelligent Vehicles*, 2024, vol. 9, no. 5, pp. 4861–4876.
15. Cantas, M.R. and Guvenc, L., Customized Co-simulation Environment for Autonomous Driving Algorithm Development and Evaluation, *arXiv:2306.00223*, 2023.
16. Holen, M., Knausgard, K., and Goodwin, M., An Evaluation of Autonomous Car Simulators and Their Applicability for Supervised and Reinforcement Learning, *Proceedings of International Conference on Intelligent Technologies and Applications*, Grimstad, 2021, pp. 367–379.
17. May, J., Poudel, S., Amdan, S., et al., Using the CARLA Simulator to Train a Deep Q Self-Driving Car to Control a Real-World Counterpart on a College Campus, *Proceedings of 2023 IEEE International Conference on Big Data*, Sorrento, 2023, pp. 2206–2210.
18. Tanmay, V.S., Chinmay, V.S., and Ming, X., AutoDRIVE Simulator: a Simulator for Scaled Autonomous Vehicle Research and Education, *Proceedings of the 2021 2nd International Conference on Control, Robotics and Intelligent System (CCRIS '21)*, Qingdao, 2021, pp. 1–5.
19. Hanevold, M., Path Following Model Predictive Control of a Differential Drive UGV in Off-Road Terrain, *Master of Informatics Thesis*, Oslo: University of Oslo, 2022.
20. Zheng, H., Smereka, J.M., Mikulski, D., et al., Bayesian Optimization Based Trust Model for Human Multi-Robot Collaborative Motion Tasks in Offroad Environments, *International Journal of Social Robotics*, 2023, vol. 15, no. 7, pp. 1181–1201.
21. Koenig, N. and Howard, A., Design and Use Paradigms for Gazebo, an Open-Source Multi-Robot Simulator, *Proceedings of 2004 IEEE/RSJ International Conference on Intelligent Robots and Systems (IROS)*, Sendai, 2004, vol. 3, pp. 2149–2154.
22. Dosovitskiy, A., Ros, G., Codevilla, F., et al.. CARLA: An Open Urban Driving Simulator, *Proceedings of Conference on Robot Learning*, Mountain View, 2017, pp. 1–16.
23. Han, I., Park, D.H., and Kim, K.J., A New Open-Source Off-Road Environment for Benchmark Generalization of Autonomous Driving, *IEEE Access*, 2021, vol. 9, pp. 136 071–136 082.
24. *Let's go off-road!* URL: https://carla.org/2023/04/21/avl-off-road-simulation. (Accessed September 30, 2024.)
25. Shah, S., Dey, D., Lovett, C., and Kapoor, A., Airsim: High-Fidelity Visual and Physical Simulation for Autonomous Vehicles, *Proceedings of the 11th International Conference on Field and Service Robotics*, Cham: Springer, 2018, pp. 621–635.
26. Jansen, W., Verreycken, E., Schenck, A., et al., COSYS-AIRSIM: a Real-Time Simulation Framework Expanded for Complex Industrial Applications, *Proceedings of 2023 Annual Modeling and Simulation Conference (ANNSIM)*, Hamilton, 2023, pp. 37–48.
27. Richard, A., Kamohara, J., Uno, K., et al., Omnilrs: A Photorealistic Simulator for Lunar Robotics, *Proceedings of 2024 IEEE International Conference on Robotics and Automation (ICRA)*, Yokohama, 2024, pp. 16901–16907.
28. Jacinto, M., Pinto, J., Patrikar, J., et al., Pegasus Simulator: an Isaac Sim Framework for Multiple Aerial Vehicles Simulation, *Proceedings of 2024 International Conference on Unmanned Aircraft Systems (ICUAS)*, Chania, 2024, pp. 917–922.
29. Ellis, K., Zhang, H., Stoyanov, D., and Kanoulas, D., Navigation among Movable Obstacles with Object Localization Using Photorealistic Simulation, *Proceedings of 2022 IEEE/RSJ International Conference on Intelligent Robots and Systems (IROS)*, Huntington Place, 2022, pp. 1711–1716.
30. Zea, D., Toapanta, A., and Perez, V.H., Intelligent and Autonomous Guidance through a Geometric Model for Conventional Vehicles, in *Innovation and Research:* A *Driving Force for Socio-Econo-Technological Development*, Cham: Springer, 2020, pp. 78–93.
31. Couceiro, M.S., Vargas, P.A., and Rocha, R.P., Bridging the Reality Gap between the Webots Simulator and E-puck Robots, *Robotics and Autonomous Systems*, 2014, vol. 62, no. 10, pp. 1549–1567.
32. Zhang, C. and Maga, A.M., An Open-Source Photogrammetry Workflow for Reconstructing 3D Models, *Integrative Organismal Biology*, 2023, vol. 5, no. 1, art no. obad024.
33. *OpenDroneMap + blender = Fun*. URL: https://smathermather.com/2019/12/30/opendronemap-blender-fun-part-2/. (Accessed September 30, 2024.)
34. Amosov, O.S., Amosova, S.G., and Kulagin, K.A., Modeling of a Virtual Testing Ground for Practicing Joint Navigation of a Group of Heterogeneous Unmanned Vehicles, *Trudy 34-oi konferentsii pamyati vydayushchegosya konstruktora giroskopicheskikh priborov N.N. Ostryakova* (Proceedings of the 34th Conference in Memory of Outstanding Designer of Gyroscopic Devices N.N. Ostryakov), St. Petersburg, 2024, pp. 117–120. (In Russian.)
35. Dobrokvashina, A., Lavrenov, R., Bai, Y., et al., Servosila Engineer Crawler Robot Modelling in Webots Simulator, *International Journal of Mechanical Engineering and Robotics Research*, 2022, vol. 11, no. 6, pp. 417–421.
36. Trefilov, P., Kulagin, K., and Mamchenko, M., Developing a Flight Mission Simulator in the Context of UAVs Group Control, *Proceedings of 2020 13th International Conference "Management of Large-Scale System Development" (MLSD)*, Moscow, 2020, pp. 1–4. DOI:10.1109/MLSD49919.2020.9247692
37. Bazenkov N.I. and Pyzh'yanov, A.A., A Data Acquisition System for Reconstructing Motorcyclist Movements, *Trudy 14-go Vserossiiskogo soveshchaniya po problemam upravleniya* (Proceedings of the 14th All-Russian Meeting on Control Problems), Moscow, 2024, pp. 1776–1780. (In Russian.)

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POLYNOMIAL REGRESSION OF EXPERT ESTIMATES OF COMPLEX QUALITY

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Abstract. The multicriteria ranking problem of objects with several useful qualities is considered. Relating to the field of multicriteria optimization, this problem also arises when management decisions are chosen among several alternatives. The goal of this study is to develop a solution method based on calculating complex (generalized mean) quality indicators that represent polynomials from the class of normalized mean functions. The latter belong to strictly monotonic, shift-invariant aggregation operators. Such polynomials are called SPs for short. For example, the weighted arithmetic mean indicators of complex quality are SPs of degree 1. Apparently, SPs have all the properties of such linear functions that are essential for multicriteria ranking. Within the method presented, called the interactive approximation of expert estimates, we SPs of arbitrary degree for calculating complex quality indicators. This approach is similar to the expert-statistical method for determining weights. It provides the best root-mean-square approximation of any number of expert estimates, reducing their uncertainty and increasing their mutual consistency during the expertise procedure. The SPs of degrees 1, 2, and 3 are described below. The interactive approximation method of expert estimates is tested for SPs of degree 2 in the problem of calculating a complex quality indicator for smartphones ranked by seven partial qualities.

Keywords: multicriteria optimization, decision-making, normalized mean function, shift-invariant polynomial, aggregation operator, weight coefficient, complex indicator, expert estimate.

INTRODUCTION

The problem of choosing the most preferable object considering its various useful properties (*qualities*) belongs to multicriteria optimization. By assumption, the qualities of an object are measured by some indicators $q_i \in [0, 1]$, $i = 1, \ldots, n$. A higher value of an indicator q_i corresponds to a more preferable object by quality i [1]. This object can be a way of action in a decision situation, and then the role of its partial qualities is played by the estimates of utility from different viewpoints [2]. The problem causes no fundamental difficulties if all partial qualities can be ordered by importance so that the more important quality has an overwhelming priority over the less important one. In this case, multicriteria optimization is reduced to sequential optimization by a single criterion. However, if all qualities are comparable in importance, no completely objective method exists for solving the choice problem. One can state that the most preferable object must be Pareto optimal [3]. Although this statement narrows the search domain, it leaves room for uncertainty. To eliminate the uncertainty, it is necessary to involve various kinds of subjective (fuzzy) estimates, which can come from experts or the decision-maker (DM).

The most commonly used method is the weighted averaging of all indicators q_i, yielding the so-called complex quality indicator $q \in [0, 1]$:

$$q = \sum_{i=1}^{n} w_i q_i \tag{1}$$

(also known as the aggregation, integral, etc. indicator, essentially representing a generalized (mean) characteristic). Here, the numbers $w_i > 0$ are called weight coefficients (or simply weights) and satisfy the condition $\sum_{i=1}^{n} w_i = 1$. The function (1) is often called the utility function or the value function. Functions of the form (1) are used owing to their simplicity and following tradition. However, there is no convincing scientific evidence that such functions are preferable to others. Formula (1) is indispensable only in some special multicriteria optimization problems. For example, the utility function should have the form (1) if the partial and complex preferences underlying the corresponding rankings satisfy the von Neumann–Morgenstern axioms; see Theorem 1 in [4, *subsection 4.2.2*]. According to the axioms, the rankings involve uncertain objects for which only the distributions of their matching probabilities with specific objects are known (the so-called mixtures or lotteries). The von Neumann–Morgenstern theory is applicable to decision problems with outcome uncertainty, but it becomes useless in the certainty case. Within another approach [5, *subsection 3.6.3*], the indicator q can have formula (1) if for each pair of criteria measured by indicators q_i and q_j, the complex rank by q_i and q_j does not depend on the positions of these objects in the rankings by all other criteria. Obviously, this condition fails in the general case. Thus, it is not obligatory to calculate the complex quality indicator $q(q_1,..., q_n)$ by formula (1).

The concept of weights w_i is considered intuitively clear and expresses the relative importance (significance) of partial qualities measured by indicators q_i [1]. However, there exists no conventional definition of weights with a unique rule for their calculation. Over one hundred methods yielding different values of weights were described in the review [6]. In this regard, some experts consider the convolution of partial indicators to be an insufficiently objective method. In a polemically sharp form, this opinion was expressed by A.I. Orlov's phrase: games on developing a generalized quality indicator are not objective [7]. Within such a paradigm, all values of partial indicators (criteria) q_i in the aggregate should be used in optimization. V.D. Noghin developed a method for correcting indicators q_i based on comparisons of their relative importance [3]; this approach actually reduces the Pareto set. V.V. Podinovskii's method of N-models [8] is to compare partial qualities (or utility criteria) quantitatively by their importance, but without using formula (1). In fact, the method of N-models deals with ranking by partial qualities (criteria). In this paper, we claim that the problem of constructing an average estimate is the objective basis for calculating the complex indicator $q = f(q_1,..., q_n)$.

According to formula (1), each weight w_i expresses the relative increment of the value of the complex indicator q that corresponds to the increment of the value of q_i under the fixed values of all other partial indicators. Generalizing this definition, we arrive at the concept of a *functional weight* ρ_i, which was introduced in the monograph [9]:

$$\rho_i = \frac{\partial q}{\partial q_i}. \tag{2}$$

The transparent idea of adopting the derivatives (2) as weight coefficients has a long history. Back in 1907, remarkable shipbuilder and scientist A.N. Krylov investigated the issue of estimating the complex quality of a warship considering four partial qualities: armor protection, firepower, speed, and range [1, *p. 46*]. He expressed the ratio of the weights of these qualities by the following proportion (on the left of the approximate equality sign):

$$\frac{w_i}{w_k} = \frac{\Delta_i q / \delta}{\Delta_k q / \delta} \approx \frac{\partial q / \partial q_i}{\partial q / \partial q_k} = \frac{\rho_i}{\rho_k}, \tag{3}$$

where $\delta = \Delta q_i = \Delta q_k$ are equal small increments of two partial indicators under fixed values of the other indicators. Here, the complex indicator q characterizes tonnage, and its increments $\Delta_i q$ and $\Delta_k q$ are calculated from the "average warship." With the expression (3), Krylov was close to the concept of the functional weights (2).

The derivatives (2) are used in multicriteria optimization algorithms [10], but they are not identified with weights. A preference function $P(F_1,..., F_n)$ dependent on criteria F_i, (unnormalized quality indicators) was considered in the paper [11]. The derivatives $\partial P / \partial F_i$ were treated as measures of the importance of the criteria F_i. This idea is close to the concept of a functional weight, although the authors [11] did not reject the weights w_i, discriminating between the preference function and the utility function of the form (1) as its linear approximation. In this case, the condition $\sum_{i=1}^{n} w_i = 1$ is not imposed on the weights, and their relations are determined from those of the derivatives $\partial P / \partial F_i$.

This paper considers complex quality indicators $q(q_1,..., q_n)$ described by shift-invariant monotonic polynomials (SPs), with application to multicriteria ranking. The SPs of degrees 1, 2, and 3 are discussed in detail. We propose an algorithm for constructing SPs of degree 2 and a related interactive approximation method for the expert estimates of complex quality, which expresses the polynomial coefficients through the estimates. The algorithm increases their mutual consistency and reduces the uncertainty. The latter appreciably simplifies the task of experts, thereby improving the objectivity degree of their data.

1. STRICTLY MONOTONIC SHIFT-INVARIANT MEAN POLYNOMIALS

Restricting the range of indicators q and q_i to the closed interval $[0, 1]$ is due to their application in the ranking problem with the *partial* ranks of objects by each quality i. Let the ranks be ascending, i.e., the best object has rank N and the worst one rank 1. The position r_i^j of object j in ranking i is the integer part of the number $q_i^j(N-1)+1$, where $q_i = q_i^j$ is the corresponding quality indicator. Similarly, the position r^j of object j in the overall (complex) ranking is the integer part of the number $q^j(N-1)+1$, where $q = q^j$.

We consider the number of positions N to be the same for all rankings and large enough to rank all objects comparable to each other by each quality $i = 1,..., n$. Hypothetical objects that can occupy any positions in the corresponding rankings are also considered. *Finer rankings* (with intermediate positions in the original rankings) are required for the correct interpretation of quality indicators. The number of such positions between r_i and $r_i + 1$, as well as between r and $r + 1$, is $10^m - 1$ if the products $q_i(N - 1)$ and $q(N - 1)$ are computed to m decimals. Adding to the real objects all *conceivable* objects of the same kind corresponding to any sets of positions in fine rankings by qualities i, we obtain a *hypercubic population*. The term "hypercubic" indicates that the value sets of indicators q_i corresponding to the objects of this population are uniformly distributed in an n-dimensional hypercube. We will rank these objects in the (fine) overall ranking under a given number $m \ge 1$.

The following fundamental proposition is accepted without proof as a natural property of any overall ranking with a single scale for the corresponding partial rankings. A priori it cannot be excluded that this property will fail for some overall rankings. Such rankings are not studied below. Also, different scales can be used to rank objects by partial qualities.

Proposition (the equal shift principle). *For any object from the hypercubic population occupying positions* $r_1,..., r_n$ *and r in partial and overall (thin) rankings, an object with positions* $r+10^{-m},..., r_n+10^{-m}$ *in partial rankings occupies position* $r+10^{-m}$ *in the overall ranking.*

Based on this principle, any equal shifts of object's positions in partial rankings are associated with exactly the same shift of its position in the overall ranking. What properties should a complex indicator $q = f(q_1,..., q_n)$ have in order to be used for multicriteria ranking?

The object with the worst qualities $q_1 = ... = q_n = 0$ should occupy the lowest position in the overall ranking and the one with the best qualities $q_1 = ... = q_n = 1$ the highest position. Therefore, it is required that $f(0, ..., 0)$ and $f(1, ..., 1) = 1$. Hence, the function $f(\boldsymbol{q})$, $\boldsymbol{q} = (q_1,..., q_n)$, maps the hypercube Q^n into the closed interval [0, 1], where

$$Q^n = \left\{\boldsymbol{q} \in \mathbb{R}^n : 0 \le q_i \le 1\ \forall i\right\}. \tag{4}$$

Another natural property is called monotonicity:

$$\forall \boldsymbol{q}, \boldsymbol{q}' \in Q^n\ (\forall i \in \{1,..., n\}\ \ q_i' \ge q_i) \Rightarrow f(q_1',..., q_n') \ge f(q_1,..., q_n). \tag{5}$$

In other words, an object not worse by all partial qualities will be not worse on average.

Any function $q = f(\boldsymbol{q})$ with the above properties is called an aggregation operator [12]. If, in addition to property (5), we have

$$\big((\forall i \in \{1,..., n\}\ \ q_i' \ge q_i)\ \&\ (\exists i \in \{1,..., n\}\ \ q_i' > q_i)\big) \Rightarrow f(q_1',..., q_n') > f(q_1,..., q_n), \tag{6}$$

then the function f is said to be strictly monotonic. The practical meaning of condition (6) seems obvious.

The property $f(a, a,..., a) = a\ \ \forall a \in [0, 1]$ is called idempotency [12]: if an object occupies the same position $r = r_i = a(N - 1) + 1$ in each partial (fine) ranking, it will occupy position r in the overall ranking. For the linear function (1), idempotency is equivalent to $\sum_{i=1}^{n} w_i = 1$ and strict monotonicity implies $w_i > 0$ for all i.

For monotonic aggregation operators, idempotency is equivalent to the property

$$\forall \boldsymbol{q} \in Q^n\ \ \min_{i=1,...,n} q_i \le f(\boldsymbol{q}) \le \max_{i=1,...,n} q_i,\quad \boldsymbol{q} = (q_1,..., q_n). \tag{7}$$

Property (7) is known as the compensatory property [12]. Such $f(\boldsymbol{q})$ will be called *mean functions*. An important property of the aggregation operators $q = f(\boldsymbol{q})$ (1) is referred to as shift invariance [12]:

$$\forall \boldsymbol{q} \in Q^n\ \ \forall \Delta\boldsymbol{q} \in R^n\ \ (\boldsymbol{q} + \Delta\boldsymbol{q} \in Q^n\ \&\ \Delta q_1 = ... = \Delta q_n) \Rightarrow \Delta q = \Delta f = \Delta q_i. \tag{8}$$

Condition (8) corresponds to the equal shift principle and looks very natural, although not all mean functions are shift-invariant. For example, the following functions are not: the weighted geometric mean $f(\boldsymbol{q})=\prod_{i=1}^{n} q_i^{w_i}$, the root mean square (RMS, quadratic mean) $f(\boldsymbol{q})\sqrt{\sum_{i=1}^{n}(q_i)^2/n}$, and the harmonic mean $f(\boldsymbol{q})=n\left(\sum_{i=1}^{n} 1/q_i\right)^{-1}$. Note that the condition $f(\mathbf{0})=0$ (by definition, any aggregation operator) and shift invariance imply idempotency.

So, according to the equal shift principle, strictly monotonic shift-invariant aggregation operators should be used to construct multicriteria rankings. This class of operators includes *normalized mean functions* (NMFs), introduced in [13] under the impression of [9]. (The basic ideas were also presented in [14].) In [15], the definition of NMFs was somewhat strengthened to strict monotonicity on all faces of the hypercube Q^n (4) of dimensions up to $n-1$.

Definition 1. *A normalized mean function (NMF) is any continuously differentiable function $f(\boldsymbol{q})$ defined on the hypercube Q^n such that $f(\mathbf{0})=0$ and*

$$\forall \boldsymbol{q}\in Q^n\left(\sum_{i=1}^{n}\frac{\partial f}{\partial q_i}(\boldsymbol{q})=1 \ \&\ \forall i\in\{1,\dots,n\}\right.$$
$$\left.\left[\frac{\partial f}{\partial q_i}(\boldsymbol{q})\ge 0 \ \&\left(\frac{\partial f}{\partial q_i}(\boldsymbol{q})=0 \Rightarrow q_i\in\{0,1\}\right)\right]\right). \tag{9}$$

The differential equation in (9) is a generalization of the condition $\sum_{i=1}^{n} w_i=1$ to the functional weights (2). Functions of the form (1) belong to the class of NMFs. Now we check that each continuously differentiable function $f(\boldsymbol{q})$ on the hypercube Q^n representing a strictly monotonic shift-invariant aggregation operator belongs to the class of NMFs.

Let us take an arbitrary set of closed intervals $[q_i, q_i+\Delta q]\subseteq[0,1]$, $i=1,\dots,n.$ By the Lagrange formula of finite increments, for some $\theta\in(0,1)$ we obtain

$$\Delta q = f(q_1+\Delta q,\dots, q_n+\Delta q)-f(q_1,\dots,q_n)$$
$$=\sum_{i=1}^{n}\frac{\partial f}{\partial q_i}(q_1+\theta\Delta q,\dots, q_n+\theta\Delta q)\Delta q, \tag{10}$$
$$\sum_{i=1}^{n}\frac{\partial f}{\partial q_i}(q_1+\theta\Delta q,\dots, q_n+\theta\Delta q)=1.$$

Note that the number Δq can be as small as desired. Therefore, the set of points $\boldsymbol{q}+\theta\Delta\boldsymbol{q}$ for condition (10) is everywhere dense in the hypercube Q^n. Hence, at each point Q^n the sum of the derivatives $\partial f/\partial q_i$ equals 1.

Thus, the class of NMF functions is not essentially new, but the very natural equation (9) had not been considered in the context of generalized mean functions until [14]. This equation gives a new description for shift-invariant aggregation operators and, moreover, an efficient method for calculating polynomials that are NMFs. They were termed monotonic shift-invariant mean polynomials (SPs) in [15]. Any continuous function on the hypercube Q^n can be approximated by a polynomial with any accuracy. Therefore, from the practical point of view, complex quality indicators in the form of polynomials are sufficient to calculate multicriteria average rankings. From this point onwards, we consider complex indicators $q=f(\boldsymbol{q})$ representing SPs.

The expert-statistical method stands out [16] among many methods for calculating weights w_i. The core of this method consists in the following. Consider a given sample of K real objects with partial indicators $q_i=q_i^j$, $i=1,\dots,n,$ $j=1,\dots,K.$ This sample will be called an *empirical population*. In the presence of an expert estimate $q=q_0^j$ of the complex indicator for each object j, the coefficients w_i are selected so as to approximate the estimates q_0^j using formula (1). For this purpose, it is necessary to solve the optimization problem

$$\sum_{j=1}^{K}\left(q_0^j-\sum_{i=1}^{n} w_i q_i^j\right)^2 \to \min,\ \sum_{i=1}^{n} w_i=1,\ w_i\ge 0\ \ \forall i.$$

The weights w_i can be treated as the coefficients of a polynomial $f(q_1,\dots,q_n)$ of degree 1 that depend on the estimates q_0^j. Raising the degree of the polynomial improves the accuracy of hitting the expert estimates and allows increasing their number. If the polynomial belongs to the class of SPs, it will have *all* the properties of the functions (1), which ensure multicriteria ranking based on the equal shift principle.

Thus, the one-time expertise procedure of objects from the empirical population gives a complex quality indicator formula that can be applied to all objects from the hypercubic population: there is no need to use organizationally complex and costly expertises each time.

Note that the complex quality indicator q determines the ranking of objects of the virtual hypercubic population. Objects from the empirical population are endowed only with those quality indicators, both partial and complex, that they have in the hypercubic population. Except for one example in Section 4, this paper does not consider quality indicators for objects from the empirical population that are defined independently of the hypercubic one. For example, the indicator q_i for the best object, by quality *i*, from the empirical population does not necessarily equal 1: this object may be not the best in the hypercubic population.

A formal objection can be raised in favor of the SPs of a degree above 1: such a function $q = f(q_1,\ldots,q_n)$ transforms the indicators $q_1,\ldots, q_n$ measured by the scale of intervals or ratios into an indicator q not measured by any of these scales. However, restricting the quality scale values to the interval $[0, 1]$ precludes its linear transformations.

In multicriteria ranking problems, quality indicators (value, utility, or preference functions) are placed on an ordinal scale. *All possible* values of the indicators q and q_i on the interval $[0, 1]$ are allowed, and the admissible transformations of their scales are limited to strictly increasing continuously differentiable functions $q = \psi(p)$ and $q_i = \varphi_i(p_i)$ that map the interval $[0, 1]$ into itself. If $q = f(\boldsymbol{q})$ is an NMF and this transformation of the variables q and q_i into the variables p and p_i satisfies the relation

$$\sum_{i=1}^{n} \frac{\partial f}{\partial q_i}(\varphi_1(p_1),\ldots, \varphi_n(p_n)) \frac{\partial \varphi_i}{\partial p_i}(p_i) = \frac{\partial \psi}{\partial p}(p),$$

where $p = \psi^{-1}\left(f(\varphi_1(p_1),\ldots, \varphi_n(p_n))\right)$, then the function $p = \psi^{-1}\left(f\left(\varphi_1(p_1),\ldots,\varphi_n(p_n)\right)\right)$ is an NMF. Such *admissible* transformations of the indicators q and q_i into the indicators p and p_i, $i = 1,\ldots, n$, preserve the rankings (ranks) consistent with them.

Consider the NMF (1) as an example of an admissible transformation of scales $[0, 1]$. Note that this function is an SP of degree 1. As is easily verified, for any strictly increasing continuously differentiable function $\varphi(p)$, the transformations $q = \varphi(p)$ and $q_i = \varphi(p_i)$ $\forall i \in \{1,\ldots,n\}$ are admissible in the above sense. The corresponding complex indicator $p = \varphi^{-1}\left(\sum_{i=1}^{n} w_i \varphi(p_i)\right)$ belongs to the class of NMFs but is not an SP. It is nothing else than the Kolmogorov–Nagumo weighted mean.

2. THE SPS OF DEGREES 2 AND 3

An arbitrary polynomial of degree 2 can be written as

$$f(q_1,\ldots, q_n) = \sum_{i=1}^{n} a_i q_i^2 + \sum_{1 \le i < k \le n} b_{ik} q_i q_k + \sum_{i=1}^{n} c_i q_i. \quad (11)$$

For each $i \in \{1,\ldots, n\}$, the functional weight (2) is represented as

$$\rho_i = \frac{\partial f}{\partial q_i} = 2a_i q_i + \sum_{1 \le k < i} b_{ki} q_k + \sum_{i < k \le n} b_{ik} q_k + c_i. \quad (12)$$

The vertices of the hypercube Q^n (4) have the form $(B_1,\ldots,B_n)$, $B_i \in \{0, 1\}$. (There are 2^n vertices in total.) The functional weights ρ_i are linear functions of the qualities $q_1,\ldots, q_n$, so the function (11) possesses strict monotonicity iff, for any vertex $\boldsymbol{B} = (B_1,\ldots,B_n)$ of the hypercube Q^n,

$$\begin{gathered} \forall i \in \{1,\ldots, n\} \\ \rho_i(\boldsymbol{B}) \ge 0 \ \ \& \ \ \rho_i(B_1,\ldots, B_{i-1}, 0, B_{i+1},\ldots, B_n) \\ + \rho_i(B_1,\ldots, B_{i-1}, 1, B_{i+1},\ldots, B_n) > 0. \end{gathered} \quad (13)$$

Condition (13) is equivalent to the following relation holding for any point $\boldsymbol{q} = (q_1,\ldots, q_n) \in Q^n$:

$$\begin{gathered} \forall i \in \{1, \ldots, n\} \\ \left[\rho_i(\boldsymbol{q}) \ge 0 \ \ \&\left(\rho_i(\boldsymbol{q}) = 0 \Rightarrow q_i \in \{0, 1\}\right)\right]. \end{gathered}$$

Condition (13) is valid if $\rho_i(\boldsymbol{B}) \ge \varepsilon$ for some $\varepsilon > 0$ and any vertex $\boldsymbol{B}$.

Under condition (13), from the vertex $(0,\ldots,0)$ we obtain that all $c_i \ge 0$. The differential equation in formula (9) is equivalent to the system of equations

$$\begin{gathered} \forall i \in \{1,\ldots, n\} \\ 2a_i + \sum_{1 \le k < i} b_{ki} + \sum_{i < k \le n} b_{ik} = 0, \ \sum_{k=1}^{n} c_k = 1. \end{gathered} \quad (14)$$

Thus, the coefficients of the SP (11) can be determined from equations (14) under conditions (13). It is necessary that all coefficients b_{ik} and b_{ki} do not vanish simultaneously. (Otherwise, the degree of the polynomial decreases.)

By varying the free parameters b_{ik}, $i<k$, and c_l, $l<n$, $(n(n+1)/2-1$ parameters in total), we can ap-

proximate the expert estimates $q = q_0^j$ by solving the optimization problem (15) for the coefficients of the polynomial (11) under conditions (13) and (14):

$$\sum_{j=1}^{E}\left(\sum_{i=1}^{n} a_i \left(q_i^j\right)^2 + \sum_{1\le i<k\le n} b_{ik} q_i^j q_k^j + \sum_{i=1}^{n} c_i q_i^j - q_0^j\right)^2 \to \min, \tag{15}$$

where E is the number of objects under expertise (i.e., the size of the empirical population). The absolute error of this approximation is estimated by

$$\Delta = \max_{j=1,\dots,E}\left|\sum_{i=1}^{n} a_i \left(q_i^j\right)^2 + \sum_{1\le i<k\le n} b_{ik} q_i^j q_k^j + \sum_{i=1}^{n} c_i q_i^j - q_0^j\right|. \tag{16}$$

If the condition $\Delta << 1$ fails, the function (11) may be useless in practice.

The best approximation of the estimates q_0^j using the SPs of degree 2 can be found as follows. Expressing $a_1, a_2, \dots, a_n$ and c_n through the other parameters in formula (14) and substituting them into the objective function of problem (15), we obtain the quadratic function $S\left(b_{12}, b_{13}, \dots, b_{1n}, b_{23}, \dots, b_{2n}, \dots, b_{n-1,n}, c_1, c_2, \dots, c_{n-1}\right)$. The positive definite function S achieves its smallest value at a single point

$$\tilde{X} = \left(\tilde{b}_{12}, \tilde{b}_{13}, \dots, \tilde{b}_{1n}, \tilde{b}_{23}, \dots, \tilde{b}_{2n}, \dots, \tilde{b}_{n-1,n}, \tilde{c}_1, \tilde{c}_2, \dots, \tilde{c}_{n-1}\right) \in \mathbb{R}^{n(n+1)/2-1},$$

which is determined by the system of equations

$$\frac{\partial S}{\partial b_{ik}} = 0,\ \frac{\partial S}{\partial c_l} = 0,\ 1 \le i < k \le n,\ 1 \le l \le n-1. \tag{17}$$

It follows from formula (14) that $\partial a_i / \partial b_{ik} = \partial a_k / \partial b_{ik} = -1/2$ and $\partial c_n / \partial c_l = -1$, all other partial derivatives being zero. Hence, with $q^j = q_0^j$, system (17) is equivalent to

$$\begin{cases} \sum_{j=1}^{E}\left(\sum_{i=1}^{n} a_i \left(q_i^j\right)^2 + \sum_{1\le i<k\le n} b_{ik} q_i^j q_k^j + \sum_{i=1}^{n} c_i q_i^j - q^j\right)\cdot\left(q_s^j - q_t^j\right)^2 = 0,\ 1 \le s < t \le n \\ \sum_{j=1}^{E}\left(\sum_{i=1}^{n} a_i \left(q_i^j\right)^2 + \sum_{1\le i<k\le n} b_{ik} q_i^l q_k^j + \sum_{i=1}^{n} c_i q_i^j - q^j\right)\cdot\left(q_l^j - q_n^j\right) = 0,\ 1 \le l \le n-1. \end{cases} \tag{18}$$

This system, containing $n(n+1)/2-1$ equations, yields the unique point $\tilde{X}$. Due to conditions (14), the point determines the coefficients of the polynomial (11) describing the regression of the expert estimates q_0^j with the smallest approximation error $\sigma = \sqrt{S\left(\tilde{X}\right)/E}$.

Also, condition (13) should be verified. If it holds, the resulting SP $f(\boldsymbol{q})$ of degree 2 is optimal by the RMS error (deviation) σ. However, it is necessary to estimate the error (16) to judge the practical applicability of the complex indicator $q = f(\boldsymbol{q})$.

If condition (13) fails, the function (11) is not an SP due to violation of strict monotonicity. In this case, we can numerically solve problem (15) under conditions (13) and (14).

An arbitrary polynomial of degree 3 can be written as

$$f(q_1, \dots, q_n) = \sum_{i,j,k=1}^{n} \alpha_{ijk} q_i q_j q_k + \sum_{i,j=1}^{n} \beta_{ij} q_i q_j + \sum_{i=1}^{n} \gamma_i q_i, \tag{19}$$

where the tensors α_{ijk} and β_{ij} are symmetric. For each $k \in \{1, \dots, n\}$, we find the functional weight

$$\rho_k = 3\sum_{i,j=1}^{n} \alpha_{ijk} q_i q_j + 2\sum_{i=1}^{n} \beta_{ik} q_i + \gamma_k.$$

The differential equation (9) for the function (19) is equivalent to the system of equations

$$\sum_{k=1}^{n} \alpha_{ijk} = 0,\ \sum_{k=1}^{n} \beta_{ik} = 0,\ \sum_{k=1}^{n} \gamma_k = 1,\ 1 \le i, j \le n. \tag{20}$$

The polynomial (19) satisfying equations (20) will belong to the class of SPs iff, for any point $\boldsymbol{q} = (q_1, \dots, q_n) \in Q^n$,

$$\forall k \in \{1, \dots, n\}$$
$$\left[\rho_k(\boldsymbol{q}) \ge 0\ \&\ \left(\rho_k(\boldsymbol{q}) = 0 \Rightarrow q_k \in \{0, 1\}\right)\right].$$

Note that α_{ijk} are required not to vanish all simultaneously.

3. THE INTERACTIVE APPROXIMATION OF EXPERT ESTIMATES

Polynomials from the class of SPs can be used to approximate best the expert estimates q_0^j of complex quality (construct the polynomial regression), reduce their uncertainty, and increase their mutual consistency. The corresponding algorithm with the SPs of degree 1 was proposed in [17] and described in [15].

Increasing the mutual consistency of expert estimates means improving their approximability by some NMF (see Definition 1). According to the Weierstrass–Stone theorem, any NMF can be approximated, as accurately as desired, by a polynomial of a sufficiently large degree. Obviously, with an arbitrarily small error, this polynomial belongs to the class of SPs. Thus, a polynomial from the class of SPs expresses the average indicator $q = q(q_1,\ldots, q_n)$ most adequately to expert estimates.

Consider the case of SPs of degree 2. Let the estimates q_0^j be assigned numbers $j=1,\ldots,E$ in the order of the sequential expertise of objects from the empirical population. The range of all possible values of the complex quality indicator for the next object depends on the previous estimates. This range is given by

$$q_{\min}^j < q_0^j < q_{\max}^j, \tag{21}$$

where $q_{\min}^j$ and $q_{\max}^j$ are the minimum and maximum values of the function

$$g_j\left(q^1,\ldots, q^E\right) = q^j,\ \left(q^1,\ldots, q^E\right) \in \bar{\Omega}_j,$$

on the closure $\bar{\Omega}_j$ of the set Ω_j of all points $\left(q^1,\ldots, q^E\right) \in \mathbb{R}^E$ such that

$$q^l = q_0^l\ \forall l \in \{1,\ldots, j-1\}\ \&\ 0 \le q^k \le 1\ \forall k \in \{j,\ldots, E\},$$

and all inequalities (13) hold for any set of binary numbers $B_i \in \{0; 1\}$ and any numbers a_i, b_{kl}, and c_i satisfying systems (14) and (18). The signs of the strict inequality in (21) are due to that the values $q_{\min}^j$ and $q_{\max}^j$ are calculated on the closure $\bar{\Omega}_j$ of the set Ω_j and the expert estimates q_0^j must correspond to points from the set Ω_j.

Obviously, equations (14) and inequalities (13) define a convex polyhedron $\wp$ in the space $\mathbb{R}^{n(n+1)/2-1}$ of the parameters $b_{ik}, c_1,\ldots,c_{n-1}$, $1 \le i < k \le n$. The set Ω_j is the intersection of the $(E-j+1)$-dimensional plane

$$\left\{\left(q^1,\ldots, q^E\right) \in \mathbb{R}^E : q^l = q_0^l\ ,\ l = 1,\ldots, j-1\right\}$$

with the polyhedron Ω_1, which is the preimage of the polyhedron $\wp$ under a linear mapping ω: $Q^E \to \mathbb{R}^{n(n+1)/2-1}$, where Q^E is the hypercube in the space $\mathbb{R}^E$: $0 \le q^j \le 1$. The mapping ω is defined by the system (18) of equations linear in q^j, where the coefficients a_i and c_n are expressed through $b_{ik}, c_1,\ldots, c_{n-1}$ according to formula (14). Hence, $\bar{\Omega}_j$ is a convex compact polyhedron in $\mathbb{R}^E$. It is necessary to check that $\bar{\Omega}_j \ne \varnothing$ for each $j = 1,\ldots, E.$

For zero values of all b_{ik} and any positive values of c_i, conditions (13) are true due to system (14), so the corresponding point $\tilde{X} \in \wp$. It is easy to observe that $\tilde{X}$ is an inner point of the polyhedron $\wp$. Hence, $\dim \wp = n(n+1)/2-1$. For any $\tilde{X} \in \wp$ under conditions (14), system (18) has the solution

$$q^j = \sum_{i=1}^n a_i \left(q_i^j\right)^2 + \sum_{1\le i<k\le n} b_{ik} q_i^j q_k^j + \sum_{i=1}^n c_i q_i^j,\ \ j = 1,\ldots, E.$$

Therefore, $\Omega_1 = \omega^{-1}(\wp) \ne \varnothing$. The above formula, together with conditions (14), defines a linear mapping $\mathbb{R}^{n(n+1)/2-1} \to \mathbb{R}^E$. If its matrix $\mathfrak{I}$ has rank E, then $\dim\Omega_1 = E.$ In this case, the convex polyhedron $\bar{\Omega}_1$ has a non-empty interior. By the construction procedure of the sets $\bar{\Omega}_j$, each of them also has a non-empty interior. In addition, the linear function $g_j\left(q^1,\ldots, q^E\right)$ achieves non-coincident maximum and minimum on $\bar{\Omega}_j$. If the rank condition fails for the matrix $\mathfrak{I}$ (which is unlikely under $E \le n(n+1)/2-1$), the algorithm will not surely produce the desired result. However, even in the case $E > n(n+1)/2-1,$ the algorithm can be effective; see Example 1 in Section 4.

Each estimate q_0^j is chosen by an expert (or a group of experts) subjectively but from the objectively given numerical interval (21). The following condition must be satisfied as well:

$$\min\left\{q_1^j,\ldots, q_n^j\right\} \le q_0^j \le \max\left\{q_1^j,\ldots, q_n^j\right\}. \tag{22}$$

Now we discuss the geometric meaning of limiting the values of the indicator q_0^j to the interval between $q_{\min}^j$ and $q_{\max}^j$.

The convex non-empty polyhedron $\Omega_1 \subset \mathbb{R}^E$ consists of such points $\left(q^1,\ldots, q^E\right)$ that the corresponding solution (a_i, b_{kl}, c_i) of system (18) under conditions (14) determines the polynomial (11) belonging to the class of SPs. This SP minimizes the objective function in (15), thereby providing the best RMS approximation of the expert estimates $q_0^j = q^j$ for $j=1, \ldots, E$. If $q^1 \in (q_{\min}^1, q_{\max}^1)$, then there exists a point $\left(q^1, q^2, \ldots, q^E\right) \in \Omega_1$. Let $q_0^1 \in (q_{\min}^1, q_{\max}^1)$. Then the convex non-empty polyhedron $\Omega_2 \subset \mathbb{R}^E$ is the intersection of the polyhedron Ω_1 with the $(E-1)$-dimensional plane $q^1 = q_0^1$. This polyhedron consists of all points $\left(q_0^1, q^2, \ldots, q^E\right)$ that can be used, due to the expression (18), to construct the SP providing the best approximation of the estimates $q_0^1, q_0^j = q^j$ for $j=2,\ldots,E$. If $q^2 \in (q_{\min}^2, q_{\max}^2)$, then there exists a point $\left(q_0^1, q^2, \ldots, q^E\right) \in \Omega_2$. Let $q_0^2 \in (q_{\min}^2, q_{\max}^2)$. Then there exists a point $\left(q_0^1, q_0^2, q^3, \ldots, q^E\right) \in \Omega_1$, and the convex non-empty polyhedron $\Omega_3 \subset \mathbb{R}^E$ is the intersection of the polyhedron Ω_2 with the $(E-2)$-dimensional plane $q^1 = q_0^1$, $q^2 = q_0^2$. This polyhedron consists of all points $\left(q_0^1, q_0^2, q^3, \ldots, q^E\right)$ that can be used to construct the SP providing the best approximation of the estimates $q_0^1, q_0^2, q_0^j = q^j$ for $j=3, \ldots, E$. Continuing by analogy, we finally obtain a set of expert estimates $q^j = q_0^j$ for $j=1,\ldots,E$ such that the corresponding solution (a_i, b_{kl}, c_i) of system (18) under conditions (14) will determine the SP (11) providing the best RMS approximation of the estimates q_0^j.

Remark. Condition (22) holds automatically if the number of free coefficients of the SP satisfying system (18) is not smaller than the number of expert estimates E. These estimates can be then interpolated instead of being regressed, so the minimum value of the objective function in problem (15) will be 0. In other words, there exists an SP (11) such that $\forall j \in \{1,\ldots, E\}$ $q_0^j = f\left(q_1^j, \ldots, q_n^j\right)$. This immediately implies condition (22). In the case of the SPs of degrees 2 and 3, the number of free coefficients is $n(n+1)/2-1$ and $n(n+1)(n+2)/6-1$, respectively. ♦

Let us summarize the above considerations.

The algorithm for calculating the coefficients of the SP of degree 2 is the following sequence of actions. (This algorithm, called *interactive approximation*, also gives a procedure for obtaining the necessary expert estimates.)

Step 1: Randomly number, from 1 to E, the objects of the empirical population. Carry out the expertise procedure sequentially, in ascending order of the numbers. If, in a certain pair of objects, one object is worse subjected to expertise than the other, let its number be higher. However, this condition is not essential. Assign the value 1 to j.

Step 2: Calculate the values of $q_{\min}^j$ and $q_{\max}^j$ using the above method. Ask an expert (or experts) to estimate the value of the complex quality indicator q_0^j for object j by choosing an appropriate number from the interval (21) under condition (22). In this case, the experts should estimate the average quality of the object with respect to all objects from the hypercubic population instead of those from the empirical population. (In particular, this means that the best and worst objects of the empirical population do not necessarily have the quality indicators $q=1$ and $q=0$, respectively.)

Step 3. If $j<E$, then increase the number j by one and revert to Step 2; otherwise proceed to Step 4.

Step 4: From the systems of equations (18) and (14), determine the values of the coefficients a_i, b_{kl}, c_i of the desired SP (11). Note that conditions (13) are automatically valid for it.

The SP produced by this algorithm globally minimizes the RMS deviation σ of the expert estimates q_0^j from the calculated values of the indicators $q=q^j$ for all objects from the empirical population.

This raises the following question: how objective can be expert estimates of the complex quality of objects that make up the empirical population? The ranking of these objects is not enough, as it is necessary to estimate them with respect to the *entire* hypercubic population. The latter is mostly hypothetical, although it contains all real objects. The best object B from this population has the indicators $q_1 = \ldots = q_n = q = 1$ whereas the worst one W the indicators $q_1 = \ldots = q_n = q = 0$. Thus, the task of experts is to estimate the complex quality q by comparing an object with B and W as well as with the real objects already estimated. It can also be compared with hypothetical objects that have the indicators $q_1 = \ldots = q_n = q = 0.5$, etc. Thus, the expertise of the indicator q does not require keeping in mind the entire hypercubic population (which can be immense).

The ability of experts to assign quality estimates objectively is by no means obvious, but it seems quite an acceptable assumption. An interesting method for estimating quality indicators based on the pairwise comparisons of alternatives was proposed by Thurstone [18]. This method proceeds from an implicit assumption that experts can give objective estimates of indicators with the Gaussian distribution. In another study [3, *p. 54*], experienced employees of a large hospital were asked to subjectively estimate each of 12 *hypothetical* hospital wards on a scale from 0 to 100 points. According to the results, professionals (experts) can develop highly reliable, subjective estimation models [3].

Within the interactive approximation method, each expert estimate $q = q_0^j$ is selected from some interval $[a, b] \subset [0, 1]$. The latter may be small, which will significantly simplify the expertise procedure.

4. THE INTERACTIVE APPROXIMATION OF EXPERT ESTIMATES: NUMERICAL EXAMPLES

Consider some examples of the interactive approximation of expert estimates. Being abstract, the first serves as a simple illustration of the method. The second example has a real interpretation.

Example 1. Let the empirical population consist of three objects numbered by $j = 1, 2, 3$ and characterized by partial quality indicators $\left(q_1^1; q_1^2; q_1^3\right) = (0.578;\ 0.361;\ 0.437)$ and $\left(q_2^1; q_2^2; q_2^3\right) = (0.701;\ 0.762;\ 0.675)$. It is required to obtain complex quality estimates q_0^j (via an expertise procedure) and find an SP $f(q_1, q_2)$ of degree 2 so that the RMS deviation σ_0 of the estimates q_0^j from the complex quality values $q^j = f\left(q_1^j, q_2^j\right)$ will be minimum among all such deviations σ calculated for all possible SPs of degree 2. Note that such a polynomial will be uniquely defined.

An arbitrary SP of degree 2 with two variables has the form

$$f(q_1, q_2) = \alpha (q_1 - q_2)^2 + (1-\beta) q_1 + \beta q_2,$$

where the coefficients α and β satisfy the system of inequalities

$$\alpha \neq 0,\ 2|\alpha| \leq \beta \leq 1 - 2|\alpha|,\ 0 < \beta < 1. \tag{23}$$

The polyhedron $\wp$ in the space $\mathbb{R}^2$ of the parameters α, β is given by system (23). In this case, system (18) becomes

$$\begin{cases} \sum\limits_{j=1}^{3} \left(\alpha \left(q_1^j - q_2^j \right)^2 + (1-\beta) q_1^j + \beta q_2^j - q^j \right) \cdot \left(q_1^j - q_2^j \right)^2 = 0 \\ \sum\limits_{j=1}^{3} \left(\alpha \left(q_1^j - q_2^j \right)^2 + (1-\beta) q_1^j + \beta q_2^j - q^j \right) \cdot \left(q_1^j - q_2^j \right) = 0. \end{cases}$$

Substituting the numerical values of q_1^j and q_2^j yields the system of equations

$$\begin{cases} 0.029\alpha + 0.08\beta - 0.015 q^1 - 0.161 q^2 \\ \quad - 0.057 q^3 + 0.092 = 0 \\ -0.08\alpha - 0.232\beta + 0.123 q^1 + 0.401 q^2 \\ \quad + 0.238 q^3 - 0.32 = 0. \end{cases}$$

Therefore, the coefficients of the desired polynomial are

$$\begin{cases} \alpha = 9.617 - 14.303 q^1 - 12.234 q^2 - 13.199 q^3 \\ \beta = -4.675 + 5.435 q^1 - 2.471 q^2 + 5.551 q^3. \end{cases} \tag{24}$$

The three-dimensional polyhedron Ω_1 in the space $\mathbb{R}^E = \mathbb{R}^3\left(q^1, q^2, q^3\right)$ is given by the system of inequalities (23), where the variables α and β are expressed through q^1, q^2, q^3 by (24). In addition, $0 \leq q^j \leq 1$ for each $j = 1, 2, 3$.

Calculating the maximum $q_{\max}^1$ and minimum $q_{\min}^1$ values of the function $g_1\left(q^1, q^2, q^3\right) = q^1$ on the (compact and convex) polyhedron $\bar{\Omega}_1$, we obtain the trivial values $q_{\max}^1 = 1$ and $q_{\min}^1 = 0$. Under condition (22), the range for expert estimates is $0.578 < q_0^1 < 0.701$. Suppose that the experts have chosen the estimate $q_0^1 = 0.652$. (In reality, this estimate would probably be $q_0^1 = 0.65$.)

The two-dimensional polyhedron Ω_2 in the space $\mathbb{R}^E = \mathbb{R}^3\left(q^1, q^2, q^3\right)$ is given by system (23) and the equality $q^1 = q_0^1$, where the variables α and β are expressed through q^1, q^2, and q^3 (24). Thus, $0 \leq q^j \leq 1$ for each $j = 2, 3$. Calculating the maximum $q_{\max}^2$ and minimum $q_{\min}^2$ values of the function $g_2\left(q^1, q^2, q^3\right) = q^2$ on the (compact and convex) polygon $\bar{\Omega}_2$, we obtain $q_{\max}^2 = 0.751$ and $q_{\min}^2 = 0.377$. Condition (22) produces a slightly wider interval, $(0.361, 0.762)$; so the final range is $0.377 < q_0^2 < 0.751$. Suppose that the experts have chosen the estimate $q_0^2 = 0.6$.

The one-dimensional polyhedron Ω_3 in the space $\mathbb{R}^E = \mathbb{R}^3\left(q^1, q^2, q^3\right)$ is given by system (23) and the equalities $q^1 = q_0^1$ and $q^2 = q_0^2$, where the variables α and β are expressed through q^1, q^2, and q^3 (24). Thus, $0 \leq q^3 \leq 1$. Calculating the maximum $q_{\max}^3$ and minimum $q_{\min}^3$ values of the function $g_3\left(q^1, q^2, q^3\right) = q^3$ on the interval $\bar{\Omega}_3$, we obtain $q_{\max}^3 = 0.599$ and $q_{\min}^3 = 0.561$. Condition (22) produces a wider interval, $(0.437, 0.675)$;

therefore, the range is $0.561 < q_0^3 < 0.599$. Suppose that the experts have chosen the estimate $q_0^3 = 0.584$.

Substituting the estimates q_0^j into system (24) yields the following coefficients of the desired SP: $\alpha = -0.051$ and $\beta = 0.622$. Thus, the complex quality index q providing the best RMS approximation of these expert estimates is given by

$$q = f(q_1, q_2) = -0.051(q_1 - q_2)^2 + 0.378 q_1 - 0.622 q_2 .$$

Note that this polynomial of degree 2 is unique, so the indicator q is well-defined. The polynomial $f(q_1, q_2)$ approximates the expert estimates q_0^j with the error $\Delta = 0.002$ (16).

Example 2. For ten smartphones (Table 1), a complex ranking by seven consumer qualities was constructed in [19].

The parameters in Table 1 are numbered into partial quality indicators $q_i = q_i^j$, characterizing price, capacity, RAM, battery, display diagonal, front camera, and CPU frequency, respectively, for smartphone j, $i=1,\ldots,7$ and $j=1,\ldots,10$. These indicators are presented in Table 2.

Let us calculate an SP $f(\boldsymbol{q})$ of degree 2 approximating the expert estimates q_0^j of the complex quality indicators $q = q^j = f(\boldsymbol{q}^j)$, $j = 1,\ldots,10$, and $\boldsymbol{q} = (q_1, q_2, q_3, q_4, q_5, q_6, q_7)$. The set of points $\boldsymbol{q}$ with all $q_i \in [0, 1]$ is the hypercube Q^7. Its vertices are points $(B_1, B_2, B_3, B_4, B_5, B_6, B_7)$ where $B_i \in \{0; 1\}$ (128 vertices in total). The desired polynomial (11) for $n = 7$ is given by the system of equations (14). Its strict monotonicity (5), (6) is valid iff all conditions (13) hold at each vertex of the hypercube Q^7.

Table 1

The parameters of smartphones [19]

No.	Model	Consumer qualities						
		Price, USD	Capacity, GB	RAM, GB	Battery, mAh	Display diagonal, cm	Front camera, Mp	CPU frequency, GHz
1	Redmi 7a	85	16	2	4000	13.84	12	2
2	Samsung Galaxy A10	105	32	2	3400	15.75	13	1.6
3	Samsung J6 Plus	171	64	4	3300	15.24	13	1.4
4	Oppo k1	197	64	4	3600	16.28	16	1.95
5	Realme 3	124	64	3	4230	15.8	13	2.1
6	Redmi Note 7S	131	32	3	4000	16	48	2.2
7	Honor 10 Lite	160	64	6	3400	15.77	13	2.2
8	Realme 5i	144	128	4	5000	16.56	12	2.2
9	Redmi 8a	98	32	3	5000	15.8	12	1.95
10	Redmi k20 pro	355	128	6	4000	16.23	48	2.84

Table 2

The partial quality indicators of smartphones

No.	Model	q_1	q_2	q_3	q_4	q_5	q_6	q_7
1	Redmi 7a	0.000	0.000	0.000	0.206	0.000	0.000	0.208
2	Samsung Galaxy A10	0.037	0.071	0.000	0.029	0.351	0.014	0.069
3	Samsung J6 Plus	0.159	0.214	0.250	0.000	0.257	0.014	0.000
4	Oppo k1	0.207	0.214	0.250	0.088	0.449	0.056	0.191
5	Realme 3	0.072	0.214	0.125	0.274	0.360	0.014	0.243
6	Redmi Note 7S	0.085	0.071	0.125	0.206	0.397	0.500	0.278
7	Honor 10 Lite	0.139	0.214	0.500	0.029	0.355	0.014	0.278
8	Realme 5i	0.109	0.500	0.250	0.500	0.500	0.000	0.278
9	Redmi 8a	0.024	0.071	0.125	0.500	0.360	0.000	0.191
10	Redmi k20 pro	0.500	0.500	0.500	0.206	0.439	0.500	0.500

Using the Solution Search add-in in Excel, we implemented the above algorithm (see Section 3) to find the ranges (21) for q_0^j. According to the Remark of Section 3, condition (22) automatically holds here: for $n=7$, the SP (11) has 27 free parameters, exceeding the number of expert estimates $E=10$. The centers of the intervals (21) were chosen as the estimates q_0^j. In reality, they would be assigned by experts within the intervals (21).

Based on the estimates q_0^j (Table 3) and the system of equations (18), the coefficients a_i, b_{ik}, and c_l of the polynomial (11) were found under conditions (14). As a result, the following SP was obtained:

$$\begin{aligned} f(\boldsymbol{q}) = & -0.017q_1^2 - 0.032q_2^2 - 0.032q_3^2 - 0.13q_4^2 \\ & -0.049q_5^2 - 0.033q_6^2 - 0.093q_7^2 + 0.034q_1q_4 \\ & +0.063q_2q_4 + 0.023q_3q_4 + 0.013q_3q_5 \\ & +0.028q_3q_7 + 0.067q_4q_6 + 0.073q_4q_7 \\ & +0.086q_5q_7 + 0.034q_1 + 0.063q_2 + 0.064q_3 \\ & +0.259q_4 + 0.238q_5 + 0.067q_6 + 0.275q_7. \end{aligned} \tag{25}$$

Table 3

The ranges of the expert estimates of complex quality

No.	Model	$q_{\min}^j$	$q_{\max}^j$	q_0^j
1	Redmi 7a	0.000	0.208	0.104
2	Samsung Galaxy A10	0.015	0.210	0.113
3	Samsung J6 Plus	0.055	0.117	0.091
4	Oppo k1	0.186	0.249	0.218
5	Realme 3	0.223	0.264	0.244
6	Redmi Note 7S	0.236	0.299	0.268
7	Honor 10 Lite	0.198	0.227	0.212
8	Realme 5i	0.350	0.371	0.360
9	Redmi 8a	0.227	0.286	0.257
10	Redmi k20 pro	0.398	0.424	0.411

The polynomial (25) approximates the expert estimates q_0^j with the error $\Delta=0.013$ (16). The overall ranking of the smartphones constructed using the quality indicator $q=f(\boldsymbol{q})$ is shown in Table 4.

Table 4

The overall ranking of the smartphones based on the SP (24)

Rank	Model	q^j, calculated estimate
1	Samsung J6 Plus	0.091
2	Redmi 7a	0.104
5	Samsung Galaxy A10	0.113
6	Honor 10 Lite	0.212
3	Oppo k1	0.218
8	Realme 3	0.244
7	Redmi 8a	0.257
9	Redmi Note 7S	0.268
4	Realme 5i	0.360
10	Redmi k20 pro	0.398

CONCLUSIONS

This paper has presented and theoretically justified the idea of calculating complex quality indicators (multidimensional functions of value, preference, or utility) that are expressed through partial indicators by shift-invariant monotonic polynomials (SPs). Methods have been described to obtain SPs of degrees 2 and 3.

This study has been focused on developing an algorithm for calculating an SP of degree 2, called the interactive approximation of expert estimates. This polynomial minimizes the RMS deviation between expert estimates and the calculated values of complex quality indicators for objects from a sample (empirical population). The latter should be representative of the hypercubic population or some part of it. The concept of a hypercubic population has been introduced to give a quite definite meaning to the concept of a complex quality indicator, linking it to multicriteria ranking.

The method under consideration narrows the range of possible values for the estimates, which are otherwise assigned at the discretion of experts. This reduces uncertainty, simplifying the work of experts and increasing the objectivity of their judgments. Also, the mutual consistency of expert estimates is improved. The coefficients of the desired SP are expressed through the resulting expert estimates in a single procedure, called interactive approximation and carried out by the algorithm described in the final part of Section 3.

The fundamental feasibility of the interactive approximation method of expert estimates has been confirmed by numerical examples in Section 4. This method needs to be generalized to the case of SPs of degree 3 and higher. Also, its practical usefulness should be verified, including real expertises.

REFERENCES

1. Azgal'dov, G.G. and Raikhman, E.P., *O kvalimetrii* (On Qualimetry), Glichev, A.V., Ed., Moscow: Izdatel'stvo standartov, 1973. (In Russian.)
2. Petrovskii, A.B., *Teoriya prinyatiya reshenii* (Decision Theory), Moscow: Akademiya, 2009. (In Russian.)

3. Noghin, V.D., *Reduction of the Pareto Set: an Axiomatic Approach*, Cham: Springer, 2018.
4. Mirkin, B.G., *Problema gruppovogo vybora* (The Problem of Group Choice), Moscow: Nauka, 1974. (In Russian.)
5. Keeney, R.L. and Raiffa, H., *Decisions with Multiple Objectives: Preferences and Value Tradeoffs*, New York: Wiley, 1976.
6. Anokhin, A.M., Glotov, V.A., Pavel'ev, V.V., and Cherkashin, A.M., Methods for Determination of Criteria Importance Coefficients, *Avtom. Telemekh.*, 1997, no. **8**, pp. 3–35. (In Russian.)
7. Orlov, A.I., Expert Estimates, *Industrial Laboratory. Diagnostics of Materials*, 1996, vol. 62, no. 1, pp. 54–60. (In Russian.)
8. Podinovskii, V.V., *Idei i metody teorii vazhnosti kriteriev v mnogokriterial'nykh zadachakh prinyatiya reshenii* (Ideas and Methods of the Theory of Criteria Importance in Multicriteria Decision Problems), Moscow: Nauka, 2019. (In Russian.)
9. Bryzgalin, G.I., *Vvedenie v teoriyu kachestva* (Introduction to Quality Theory), Volgograd: Volgograd Polytechnic Institute, 1988. (In Russian.)
10. Geoffrion, A.M., Dyer, J. S., and Feinberg, A., Interactive Approach for Multicriterion Optimization, with an Application to the Operation of an Academic Department, *Manage. Sci., Appl.*, 1972, vol. 19, pp. 357–368.
11. Marler, R. and Arora, J., The Weighted Sum Method for Multi-Objective Optimization: New Insights, *Structural and Multidisciplinary Optimization*, 2010, no. 41, pp. 853–862.
12. Calvo, T., Kolesárová, A., Komorníková, M., and Mesiar, R., Aggregation Operators: Properties, Classes and Construction Methods, in *Aggregation Operators. New Trends and Applications*, Heidelberg: Physica-Verlag, 2002, pp. 3–106.
13. Zot'ev, D.B., Normalized Averages Functions and the Aggregation Problem of Quality Indicators, *Handbook. An Engineering Journal*, 2009, no. 5 (146), pp. 43–48. (In Russian.)
14. Bryzgalin, G.I., Theory of Qualities and System Applications, *Handbook. An Engineering Journal*, 2009, no. 5 (146), pp. 57–63. (In Russian.)
15. Zot'ev, D.B., On Normalized Average Criteria Interpolating Expert Estimates, *Handbook. An Engineering Journal*, 2012, no. 7 (184), pp. 50–56. (In Russian.)
16. Aivazyan, S.A., Bukhshtaber, V.M., Enyukov, I.S., and Meshalkin, L.D., *Prikladnaya statistika. Klassifikatsiya i snizhenie razmernosti* (Applied Statistics. Classification and Dimension Reduction), Moscow: Finansy i Statistika, 1989. (In Russian.)
17. Zot'ev, D.B., On Problem of Weight Quotients Determination on Base of Expert Estimations, *Industrial Laboratory. Diagnostics of Materials*, 2011, no. 1 (77), pp. 75–78. (In Russian.)
18. Litvak, B.G., *Ekspertnaya informatsiya: metody polucheniya i analiza* (Expert Information: Acquisition and Analysis Methods), Moscow: Radio i Svyaz', 1982. (In Russian.)
19. Goswami, S.S. and Behera, D.K., Evaluation of the Best Smartphone Model in the Market by Integrating Fuzzy-AHP and PROMETHEE Decision-Making Approach, *Decision*, 2021, no. 1(48), pp. 71–96.

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DYNAMIC ANISOTROPY-BASED CONTROLLER DESIGN FOR TIME-INVARIANT SYSTEMS WITH MULTIPLICATIVE NOISE

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Abstract. This paper considers a linear discrete time-invariant system with multiplicative noise and a control input under an external disturbance from a special class. The plant's dynamics are described in the state space. The class of external disturbances contains a set of stationary Gaussian sequences with a bounded mean anisotropy. The anisotropic norm of the closed-loop control system is chosen as the performance criterion. It is required to design a dynamic link-based control scheme under which the anisotropic norm of the closed-loop control system will be bounded by the minimum possible threshold. At the first stage of solving this problem, the controller's dynamics are written out and the plant under consideration is augmented. The boundedness criterion of the anisotropic norm in terms of matrix inequalities is used to derive sufficient conditions for the existence of a solution of a convex optimization problem to minimize the upper bound of the anisotropic norm. A special change of variables is performed in the resulting inequalities to eliminate the nonlinear dependence on the unknown controller matrices. After a linearizing invertible change of variables, the optimization problem is solved numerically using standard methods. At the last stage, the desired controller matrices are calculated in the state space to ensure the bounded anisotropic norm of the closed-loop control system.

Keywords: linear discrete time-invariant systems, anisotropy-based theory, dynamic control, LMI, convex optimization.

INTRODUCTION

The active development of automatic control theory in the 20th century caused the creation of tools for attenuating external disturbances, which has become one of the most important problems in this theory. Since the approach to attenuating Gaussian disturbances with the linear quadratic performance criterion had been pioneered [1], many methods for dealing with external disturbances were proposed. Some of them are tuned to the case of known stochastic characteristics of input signals. On the other hand, the $\mathcal{H}_\infty$-optimization approach [2] offers a way to parry the "worst-case" disturbance. However, $\mathcal{H}_\infty$ control has excessive conservatism, and optimal $\mathcal{H}_2$ controllers are sensitive to small parameter variations. Therefore, they turn out non-robust, and control optimality is violated accordingly. Despite the fundamental differences between $\mathcal{H}_2$- and $\mathcal{H}_\infty$-theories, some studies combining the two methods were published [3–6].

One branch of control theory, investigating ways to attenuate external disturbances, was developed by I.G. Vladimirov about thirty years ago [7, 8]; it covers both the $\mathcal{H}_2$- and $\mathcal{H}_\infty$-optimal control theories as limiting cases. This theory, called the anisotropy-based (control) theory by the author, offers a stochastic approach to $\mathcal{H}_\infty$ control and, at the same time, has a close terminological connection to information theory. The central concept of the anisotropy-based theory is the anisotropy of a random vector, which originally corresponded to the relative entropy of the normalized distribution function of a random vector on the unit sphere with respect to the uniform distribution. Thus, for the uniform distribution, the anisotropy value is zero, and the denser the distribution becomes along certain axes, the higher the anisotropy value will be, up to infinity. This concept was later modified [9]. Recently, the anisotropy of a random vector is understood as the Kullback–Leibler divergence between two probability density functions (PDFs), one belonging to a fixed random vector and the other to a Gaussian family of random vectors with zero mean and a scalar covariance matrix. With this definition, one can give a simple geometric interpretation of anisotropy: it measures the difference (distance) between a random vector and a set of centered Gaussian vectors with a scalar covariance matrix. The performance criterion in the anisotropy-based theory is related to the anisotropic norm, a stochastic analog of the $\mathcal{H}_\infty$ norm of a dynamic system.

Within the theoretical framework based on the anisotropy of random vectors, many analysis and design problems were solved for both time-varying [9] and time-invariant systems [10]. However, until recently, only linear and deterministic plants were considered. The first attempt to study stochastic plants from the anisotropic point of view was undertaken in [11]. The analysis therein replaced the approach proposed in [12, 13], which involved the majorants of norms for systems with multiplicative noise.

Systems with multiplicative noise are an important example of stochastic systems. They describe mechanical, hybrid, and biological systems, financial models, and many other objects and processes [14, 15]. An anisotropy-based robust performance analysis of time-varying systems was presented in [16], and time-invariant systems were considered in [17]. The problem of constructing an output estimator for a time-varying system was successfully solved in [18], and the adjacency matrix of a sensor network with dropouts was tuned in [19]. In view of the results obtained within the framework of anisotropic analysis for time-invariant systems [17], the control design problem can also be posed and solved. Below, we consider a dynamic controller and formulate a convex optimization problem to calculate its gain matrices in the state space. The matrix inequalities are linearized using the procedure described in [20]. The developed controller can be applied to the automatic control of any moving objects. Section 1 provides a summary of the anisotropy-based theory. In Section 2, we describe the system and problem statement; in Section 3, the solution of this problem. The results of numerical simulation are demonstrated in Section 4.

1. THEORETICAL BACKGROUND

This section recalls the basic concepts of the anisotropy-based theory for time-invariant systems. More detailed information can be found in [10, 21–23].

The anisotropy of a random vector W from the space $\mathbb{R}^m$ with a PDF f is given by

$$\mathbf{A}(W) = \min_{\lambda>0} \mathbf{D}(f \,||\, p_\lambda),$$

where

$$\mathbf{D}(f \,||\, p_\lambda) = \mathbf{E}\left[\ln \frac{f}{p_\lambda}\right]$$

represents the relative entropy (or the Kullback–Leibler information divergence) with respect to a reference PDF of the form

$$p_\lambda(x) = (2\pi\lambda)^{-m/2} \exp\left(-\frac{||x||^2}{2\lambda}\right),$$

which is chosen Gaussian with zero mean and a scalar covariance matrix λI_m, where I_m denotes an identity matrix of order m. From this point onwards, the notation $\mathbf{E}[\cdot]$ corresponds to the expectation operator and $\|\cdot\|$ is the Euclidean vector norm. The anisotropy of a random vector is not a norm due to violating the axioms of norms. At the same time, anisotropy is a measure of the closeness of a random vector to vectors obeying the standard Gaussian distribution.

Consider the extended vector composed of elements of a random vector sequence $\{w_k\}$:

$$W_{s:t} = (w_s^{\mathrm{T}}, w_{s+1}^{\mathrm{T}}, \ldots, w_t^{\mathrm{T}})^{\mathrm{T}}, \; s \le t.$$

For the extended vector $W_{s:t}$, the limit

$$\overline{\mathbf{A}}(W) = \lim_{N\to\infty} \frac{\mathbf{A}(W_{0:N-1})}{N}$$

is called the mean anisotropy of the sequence $\{w_k\}$ [10]. The anisotropy-based theory introduces a particular performance criterion, known as the anisotropic norm. First, we consider the mean-square gain

$$Q(Z,W)=\sqrt{\frac{\mathbf{E}(|Z|^2)}{\mathbf{E}(|W|^2)}},$$

where Z and W are the output and input of a linear system with a transfer matrix $F\in\mathbb{C}^{p\times m}$, respectively. The expression

$$\sup_{W\in\mathcal{L}_2^m} Q(Z,W)=\sqrt{\max_{1\le k\le m}\lambda_k(F^{\mathrm{T}}F)}=\|F\|_\infty$$

is the definition of the $\mathcal{H}_\infty$ norm, where $\mathcal{L}_2^m$ corresponds to square summable signals and λ_k is the k th eigenvalue. If the input signal of the system F has mean anisotropy with an upper bound a, the anisotropic norm can be defined as

$$\sup_{\overline{\mathbf{A}}(W)\le a} Q(Z,W)=|||F|||_a .$$

For a nonspherical system (i.e., the one whose scaled $\mathcal{H}_2$ norm is smaller than the $\mathcal{H}_\infty$ norm), the anisotropic norm has a remarkable property: either the scaled $\mathcal{H}_2$ or $\mathcal{H}_\infty$ norm can be obtained as the limiting cases:

$$\frac{1}{\sqrt{m}}\|F\|_2\le|||F|||_a\le\|F\|_\infty .$$

Note that the left bound is reached under zero mean anisotropy; the right bound, as mean anisotropy tends to infinity (when the sequence loses randomness).

2. PROBLEM STATEMENT

In this paper, we describe dynamic objects (both the controlled plant and the controller) in the time domain using the state-space representation. Consider a linear discrete time-invariant system with multiplicative noise of the form

$$\begin{aligned}x(k+1)&=Ax(k)+B_1w(k)+B_2u(k),\\ z(k)&=C_1x(k)+D_{12}u(k),\\ y(k)&=C_2x(k)+D_{21}w(k)\end{aligned}\tag{1}$$

with the zero initial condition $x(0)=0$. Here, $x(k)\in\mathbb{R}^{n_x}$ is the state vector; $\{w(k)\}_{k\ge 0}$, $w(k)\in\mathbb{R}^{m_w}$, is a colored sequence with a known upper bound a on mean anisotropy; $u(k)\in\mathbb{R}^{m_u}$ is the control input; $z(k)\in\mathbb{R}^{p_z}$ is the controlled output; $y(k)\in\mathbb{R}^{p_y}$ is the observed output. All matrices in system (1) have compatible dimensions. By assumption, system (1) is controllable. Unlike the system considered in [13], where multiplicative noises were included in the control coefficient, the system matrix A in the current problem statement is represented as

$$A=A_0+\sum_{i=1}^{n}\mu_i(k)A_i,$$

where the matrices A_i are known and have appropriate dimensions. The random variables $\mu_i(k)$, $i=1,\dots,n$, obey the standard Gaussian distribution with zero mean and unit covariance, are mutually independent of each other and of the external disturbance vectors $w(t)$ for all time instants k and t.

The problem is to find matrices A_c, B_c, C_c, and D_c of the state-space realization of a full-order dynamic controller

$$\begin{aligned}\xi(k+1)&=A_c\xi(k)+B_cy(k),\\ u(k)&=C_c\xi(k)+D_cy(k),\end{aligned}\tag{2}$$

where $\xi_i(k)\in\mathbb{R}^{n_x}$ stands for the controller's internal state, under which the anisotropic norm of the closed-loop control system would not exceed a number $\gamma>0$.

3. THE MAIN RESULT

Before proceeding to the main result, we consider a linear discrete time-invariant system F with multiplicative noise of the following form:

$$\begin{aligned}x(k+1)&=(A_0+\sum_{i=1}^{n}\mu_i(k)A_i)x(k)+Bw(k),\\ z(k)&=Cx(k)+Dw(k),\end{aligned}\tag{3}$$

where $x(k)\in\mathbb{R}^{n_x}$ denotes the state vector, $w(k)\in\mathbb{R}^{m_w}$ is a disturbance, and $z(k)\in\mathbb{R}^{p_z}$ means the system output. Real matrices have compatible dimensions. By assumption, the input sequence is random with a given upper bound a on its mean anisotropy. A boundedness condition of the anisotropy norm was derived in [17] in terms of a special system of equations and inequalities. The analysis of systems with multiplicative noise was reduced to a convex optimization problem in [24].

For system (3), the anisotropic norm will be bounded under the following conditions.

Theorem 1 [24]. *Consider system* (3) *and let the mean anisotropy of the external disturbance be bounded above by a given number* $a \ge 0$. *The anisotropic norm of the system will not exceed a given threshold* γ,

$$||| F |||_a \le \gamma,$$

if the system of inequalities

$$\begin{bmatrix} \sum_{i=0}^{n} A_i^{\mathrm{T}} R A_i - R + C^{\mathrm{T}} C & * \\ B^{\mathrm{T}} R A_0 + D^{\mathrm{T}} C & -\eta I_{m_w} + D^{\mathrm{T}} D + B^{\mathrm{T}} R B \end{bmatrix} \prec 0, \quad (4)$$

$$\begin{bmatrix} \eta I_{m_w} - S - D^{\mathrm{T}} D & * \\ RB & R \end{bmatrix} \succ 0, \quad (5)$$

$$\ln \det S \ge 2a + m_w \ln(\eta - \gamma^2)$$

has solutions $R \succ 0,\ S \succ 0,$ *and* $\eta > \gamma^2$.

In inequalities (4) and (5) and further, the expression $[\cdot] \prec 0$ should be understood in the sense of the negative definiteness of an appropriate matrix, and the asterisk $(*)$ denotes a block symmetric with respect to the principal diagonal.

The original system (1) closed with the controller (2) takes the form

$$\begin{aligned} \zeta(k+1) &= (A_0 + \sum_{i=0}^{n} \mu_i(k) A_i)\zeta(k) + Bw(k), \\ z(k) &= C\zeta(k) + Dw(k), \end{aligned} \quad (6)$$

where $\zeta(k) \in \mathbb{R}^{2n_x}$ is the augmented state vector

$$\zeta(k) = \begin{bmatrix} x(k) \\ \xi(k) \end{bmatrix},$$

and the matrices $A_i \in \mathbb{R}^{2n_x \times 2n_x}$, $i = 0, \ldots, n$, $B \in \mathbb{R}^{2n_x \times m_w}$, $C \in \mathbb{R}^{p_z \times 2n_x}$, and $D \in \mathbb{R}^{p_z \times m_w}$ have the following block structure:

$$A_0 = \begin{bmatrix} A_0 + B_2 D_c C_2 & B_2 C_c \\ B_c C_2 & A_c \end{bmatrix},$$

$$A_i = \begin{bmatrix} A_i & 0 \\ 0 & 0 \end{bmatrix},\ B = \begin{bmatrix} B_2 D_c D_{21} \\ B_c D_{21} \end{bmatrix}, \quad (7)$$

$$C = \begin{bmatrix} C_1 + D_{12} D_c C_2 & D_{12} C_c \end{bmatrix},\ D = D_{12} D_c D_{21}.$$

For the closed-loop control system (6), based on the results of [20, 25], we now formulate the following statement regarding the boundedness of the anisotropic norm as a convex optimization problem.

Theorem 2. *Consider system* (3) *with multiplicative noise and let the mean anisotropy of the external disturbance be bounded above by a given number* $a \ge 0$. *For a fixed number* $\gamma > 0$, *the dynamic controller* (2) *ensures the boundedness of the anisotropic norm,* $||| F |||_a < \gamma$, *if the system of inequalities*

$$\begin{bmatrix}
-\Pi_{11} & * & * & * & * & * & * & \ldots & * & * & * \\
-I_{n_x} & -\Phi_{11} & * & * & * & * & * & \ldots & * & * & * \\
0 & 0 & -\eta I_{m_w} & * & * & * & * & \ldots & * & * & * \\
A_0\Pi_{11} + B_2\mathbf{C} & A_0 + B_2\mathbf{D}C_2 & B_1 + B_2\mathbf{D}D_{21} & -\Pi_{11} & * & * & * & \ldots & * & * & * \\
\mathbf{A} & \Phi_{11}A_0 + \mathbf{B}C_2 & \Phi_{11}B_1 + \mathbf{B}D_{21} & -I_{n_x} & -\Phi_{11} & * & * & \ldots & * & * & * \\
A_1\Pi_{11} & A_1 & 0 & 0 & 0 & -\Pi_{11} & * & \ldots & * & * & * \\
0 & 0 & 0 & 0 & 0 & -\Pi_{12}^{\mathrm{T}} & -\Pi_{22} & * & * & * & * \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
A_n\Pi_{11} & A_n & 0 & 0 & 0 & 0 & 0 & \ldots & -\Pi_{11} & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & -\Pi_{12}^{\mathrm{T}} & -\Pi_{22} & * \\
C_1\Pi_{11} + D_{12}\mathbf{C} & C_1 + D_{12}\mathbf{D}C_2 & D_{12}\mathbf{D}D_{21} & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & -I_{p_z}
\end{bmatrix} \prec 0, \quad (8)$$

$$\begin{bmatrix} \Psi - \eta I_{m_w} & * & * & * \\ B_1 + B_2\mathbf{D}D_{21} & -\Pi_{11} & * & * \\ \Phi_{11}B_1 + \mathbf{B}D_{21} & -I_{n_x} & -\Phi_{11} & * \\ D_{12}\mathbf{D}D_{21} & 0 & 0 & -I_{p_z} \end{bmatrix} \prec 0, \qquad (9)$$

$$\eta > \gamma^2, \Pi_{11} \succ 0, \Phi_{11} \succ 0, \begin{bmatrix} \Pi_{11} & I_{n_x} \\ I_{n_x} & \Phi_{11} \end{bmatrix} \succ 0, \qquad (10)$$

$$\ln \det \Psi \geq 2a + m_w \ln(\eta - \gamma^2), \qquad (11)$$

is solvable with respect to the variables $\eta > 0$, $\Psi = \Psi^{\mathrm{T}}$, $\Phi_{11} = \Phi_{11}^{\mathrm{T}}$, $\Pi_{11} = \Pi_{11}^{\mathrm{T}}$, $\Pi_{22} = \Pi_{22}^{\mathrm{T}}$, Π_{12}, $\mathbf{A}, \mathbf{B}, \mathbf{C}$, *and* $\mathbf{D}$. *Moreover, the controller's gain matrices are related to the solution of inequalities* (8)-(11) *as follows:*

$$\begin{aligned} A_c &= \Phi_{12}^{-1}(\mathbf{A} + \Phi_{11}B_2\mathbf{D}C_2\Pi_{11} - \mathbf{B}C_2\Pi_{11} \\ &\quad - \Phi_{11}B_2\mathbf{C} - \Phi_{11}A_0\Pi_{11})\Pi_{12}^{-\mathrm{T}}, \\ B_c &= \Phi_{12}^{-1}(\mathbf{B} - \Phi_{11}B_2D_c), \\ C_c &= (\mathbf{C} - D_cC_2\Phi_{11})\Pi_{12}^{-\mathrm{T}}, \\ D_c &= \mathbf{D} \end{aligned} \qquad (12)$$

where

$$\Phi_{12} = (I_{n_x} - \Phi_{11}\Pi_{11})\Pi_{12}^{-\mathrm{T}},$$

the matrices Φ_{12} *and* Π_{12} *are nonsingular*, *and* $\Pi_{12}^{-\mathrm{T}} = (\Pi_{12}^{\mathrm{T}})^{-1}$.

The proof of this theorem is given in the Appendix.

Remark 1. An important requirement is the coincident dimensions of the state vectors of the plant $x(k)$ and the dynamic controller $\xi(k)$. In this case, the gain matrices of the controller (2) can be found unambiguously. According to [25], under the full column rank of the matrices Φ_{12} and Π_{12}, the controller's gain matrices A_c, B_c, C_c, and D_c exist but are not unique. ♦

Based on Theorem 2, it is easy to formulate another statement, which has already become classical for anisotropic problems in convex optimization terms.

Theorem 3. *The anisotropic norm of system* (6) *is bounded above by the minimum threshold* γ *if the convex optimization problem*

$$\gamma^2 \underset{(8)-(11)}{\rightarrow} \min$$

is solvable with respect to the variables γ^2, η, $\Psi = \Psi^{\mathrm{T}} \succ 0$, $\Phi_{11} = \Phi_{11}^{\mathrm{T}} \succ 0$, $\Pi_{11} = \Pi_{11}^{\mathrm{T}} \succ 0$, $\Pi_{22} = \Pi_{22}^{\mathrm{T}} \succ 0$, Π_{12}, $\mathbf{A},\mathbf{B},\mathbf{C}$, *and* $\mathbf{D}$. *The gain matrices of the dynamic controller* (2) *are given by formulas* (12).

Remark 2. Due to multiplicative noise in system (1), the convex optimization problem contains the matrix Π. This allows one not to check the existence of two matrices satisfying the equation $\Phi_{12}\Pi_{12}^{\mathrm{T}} = I_{n_x} - \Phi_{11}\Pi_{11}$, a necessary condition in the case of systems without multiplicative noise [23]. ♦

4. NUMERICAL SIMULATION

In this section, we analyze the numerical experiment carried out on the aircraft takeoff-landing model [26]. The control is implemented by changing the angle of the aircraft's rear nozzle, thrust through the rear nozzle, and thrust through the front nozzle, which has a fixed position; the aircraft's pitch and position of the center of mass, as well as the rates of their change, are chosen as the state variables. The linear discrete-time model in the state space is described by the matrices

$$A_0 = \begin{bmatrix} 1.0000 & 0.0004 & -0.0536 & -0.0976 \\ -0.0007 & 0.9966 & 0.4938 & -0.0100 \\ 0.0000 & 0.0001 & 0.9980 & 0.0000 \\ 0.0000 & 0.0000 & 0.0100 & 1.0000 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0.0003 & 0.0000 & -0.0001 \\ -0.0002 & 0.0005 & 0.0008 \\ -0.0006 & 0.0000 & 0.0034 \\ 0.0000 & 0.0000 & 0.0001 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 0.0021 & 0.0000 & -0.0003 \\ 0.0000 & 0.0020 & 0.0004 \\ 0.0000 & 0.0000 & 0.0012 \\ 0.0000 & 0.0000 & 0.0000 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 0.7071 & 0 & 0 & 0 \end{bmatrix}, \; D_{12} = [0 \;\; 0 \;\; 0.7071],$$

$$C_2 = \begin{bmatrix} 0 & 0 & 57.2958 & 0 \\ 0 & 0 & 0 & 57.2958 \\ 0.1045 & -0.9945 & 0.1375 & 51.5791 \\ -0.0002 & 0.0045 & 0 & 0 \end{bmatrix},$$

$$D_{21} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0.0212 & 0 & 0 \end{bmatrix}.$$

Multiplicative noise as a single term enters the dynamics equation with the matrix coefficient $A_1 = A_0 \cdot 10^{-2}$. Mean anisotropy is chosen to be 5. The following gain matrices of the dynamic anisotropy-based controller were obtained during the calculations:

$$A_c = \begin{bmatrix} 1.9773 & -2.3361 & -7.8981 & 49.2172 \\ -0.0232 & 1.7152 & -1.1628 & 4.7785 \\ -0.0368 & 0.6143 & -0.1482 & 0.1708 \\ -0.1262 & 0.4600 & 0.2936 & -2.2690 \end{bmatrix},$$

$$B_c = \begin{bmatrix} 0.0078 & 0.5486 & -1.0382 & -0.0013 \\ 0.0024 & -0.8364 & 0.08879 & 0.0003 \\ 0.0115 & -0.2878 & 0.3202 & -0.0187 \\ 0.0017 & -0.1795 & 0.2283 & -0.0028 \end{bmatrix},$$

$$C_c \cdot 10^{-4} = \begin{bmatrix} -0.0640 & -1.9133 & 1.2458 & -5.1640 \\ -0.0311 & -0.0188 & 0.1100 & -0.6393 \\ 0.0000 & -0.0068 & 0.0039 & -0.0149 \end{bmatrix},$$

$$D_c \cdot 10^{-3} = \begin{bmatrix} -0.0250 & 9.3135 & -9.8997 & -0.0111 \\ -0.0067 & -0.2866 & 0.3740 & 0.0037 \\ -0.0001 & 0.0329 & -0.0353 & 0.000 \end{bmatrix}.$$

The convex optimization problem was numerically solved using standard MATLAB tools with additional packages for semidefinite programming problems [27, 28].

Figure 1 shows the Bode diagram for two anisotropy values, 1 (Fig. 1a) and 5 (Fig. 1b). Note that for any mean anisotropy exceeding 10, the simulation gives approximately the same results characteristic of $\mathcal{H}_\infty$ control.

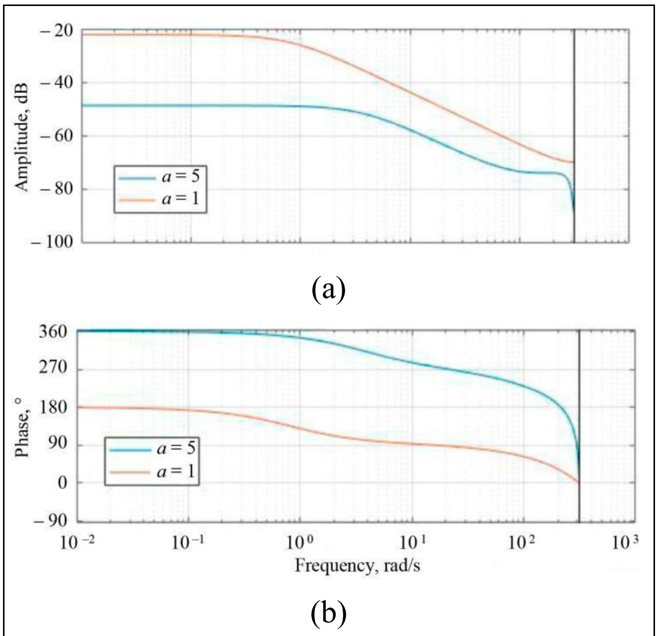

Fig. 1. The Bode diagrams of closed-loop control systems.

The table below presents the upper bounds of the anisotropic norm of the closed-loop control system calculated under different values of mean anisotropy.

The bounds of the anisotropic norm

a	0	1	5	10	15
γ	0.0012	0.2197	0.3087	0.3142	0.3142

Also, it seems interesting to compare the controlled outputs under anisotropy-based control and standard $\mathcal{H}_2$ control. Let mean anisotropy be equal to 5. We calculate the Euclidean norm for the controlled output, plotted in Fig. 2, where the application of anisotropy-based control is indicated by AB. As it turns out, for anisotropy-based control, this norm takes a value of 0.0158; for $\mathcal{H}_2$ approach, a value of 0.0856. In other words, with anisotropy-based control, the quadratic performance criterion can be improved by 72%.

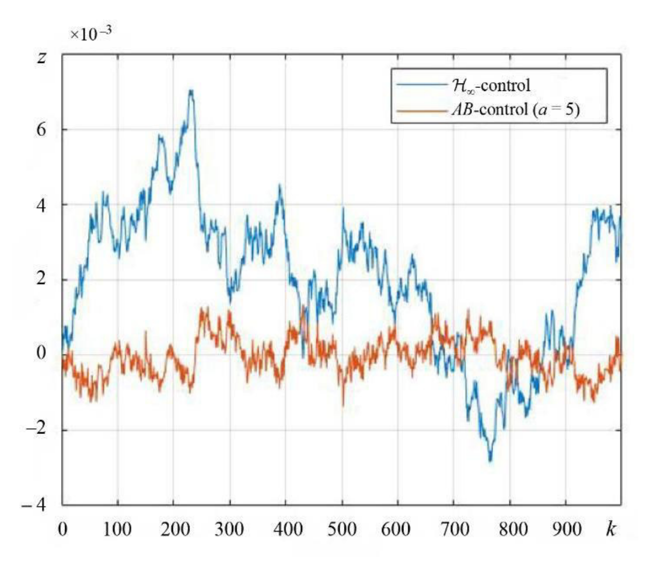

Fig. 2. The controlled outputs of the system with different control.

CONCLUSIONS

This paper has considered an algorithm for calculating the gain matrices of a dynamic anisotropy-based controller in the state space. The bounded real lemma for time-invariant systems has been used as the main tool. By assumption, the dynamic controller has full dimension, which ensures its uniqueness. Sufficient conditions for the existence of a dynamic anisotropy-based controller for the closed-loop system have been established in terms of a special system of nonlinear matrix inequalities. A linearizing inversible change of variables has been applied to reduce the boundedness conditions of the anisotropic norm of the closed-loop system to the solvability condition of the special system of inequalities. A threshold for the upper bound of the anisotropic norm of the closed-loop system has been obtained by solving a convex optimization problem.

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APPENDIX

P r o o f of Theorem 2. According to Theorem 1, system (6) closed by the dynamic controller (2) has a bounded anisotropic norm if the inequalities

$$\begin{bmatrix} \sum_{i=0}^{n} A_i^{\mathrm{T}} \Phi A_i - \Phi + C^{\mathrm{T}} C & * \\ B^{\mathrm{T}} \Phi A_0 + D^{\mathrm{T}} C & -\eta I_{m_w} + D^{\mathrm{T}} D + B^{\mathrm{T}} \Phi B \end{bmatrix} \prec 0, \quad \text{(A1)}$$

$$\begin{bmatrix} \eta I_{m_w} - \Psi - D^{\mathrm{T}} D & * \\ \Phi B & \Phi \end{bmatrix} \succ 0, \quad \text{(A2)}$$

$$\ln\det \Psi \ge 2a + m_w \ln(\eta - \gamma^2) \quad \text{(A3)}$$

have solutions $\eta > 0$, $\Psi = \Psi^{\mathrm{T}}$, and $\Phi = \Phi^{\mathrm{T}}$. The system of inequalities (A1)–(A3) is nonlinear with respect to the system matrices that depend on the controller's matrices (7). To correct this, we apply the Schur complement lemma [29] to inequality (A1):

$$\begin{bmatrix} -\Phi & * & * & * & \ldots & * & * \\ 0 & -\eta I_{m_w} & * & * & \ldots & * & * \\ A_0 & B & -\Phi^{-1} & * & \ldots & * & * \\ A_1 & 0 & 0 & -\Phi^{-1} & \ldots & * & * \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ A_n & 0 & 0 & 0 & \ldots & -\Phi^{-1} & * \\ C & D & 0 & 0 & \ldots & 0 & -I_{p_z} \end{bmatrix} \prec 0. \quad \text{(A4)}$$

Let $\Pi = \Phi^{-1}$ be the new matrix variable. Obviously, $\Phi\Pi = I_{2n_x}$, and the matrices Φ and Π have a block structure:

$$\Phi = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{12}^{\mathrm{T}} & \Phi_{22} \end{bmatrix}, \quad \Pi = \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^{\mathrm{T}} & \Pi_{22} \end{bmatrix}.$$

We introduce the matrices

$$\Phi_1 = \begin{bmatrix} I_{n_x} & \Phi_{11} \\ 0 & \Phi_{12}^{\mathrm{T}} \end{bmatrix}, \quad \Pi_1 = \begin{bmatrix} \Pi_{11} & I_{n_x} \\ \Pi_{12}^{\mathrm{T}} & 0 \end{bmatrix}.$$

It is easy to show that

$$\Pi_1^{\mathrm{T}} \Phi \Pi_1 = \Phi_1^{\mathrm{T}} \Pi_1 = \Phi_1^{\mathrm{T}} \Pi \Phi_1 = \Pi_1^{\mathrm{T}} \Phi_1 = \begin{bmatrix} \Pi_{11} & I_{n_x} \\ I_{n_x} & \Phi_{11} \end{bmatrix}. \quad \text{(A5)}$$

Applying a congruent transformation with the matrix

$$\text{block diag}\,(\Pi_1^{\mathrm{T}},\ I_{m_w},\ \Phi_1^{\mathrm{T}},\ I_{n_x}, \ldots,\ I_{n_x},\ I_{p_z})$$

to inequality (A4) yields the new inequality

$$\begin{bmatrix} -\Pi_1^{\mathrm{T}}\Phi\Pi_1 & * & * & * & \ldots & * & * \\ 0 & -\eta I_{m_w} & * & * & \ldots & * & * \\ \Phi_1^{\mathrm{T}} A_0 \Pi_1 & \Phi_1^{\mathrm{T}} B & -\Phi_1^{\mathrm{T}}\Pi\Phi_1 & * & \ldots & * & * \\ A_1\Pi_1 & 0 & 0 & -\Pi & \ldots & * & * \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ A_n\Pi_1 & 0 & 0 & 0 & \ldots & -\Pi & * \\ C\Pi_1 & D & 0 & 0 & \ldots & 0 & -I_{p_z} \end{bmatrix} \prec 0. \quad \text{(A6)}$$

Inequality (A6) is still nonlinear in some matrix variables. The blocks $-\Pi_1^{\mathrm{T}}\Phi\Pi_1$ and $-\Phi_1^{\mathrm{T}}\Pi\Phi_1$ can be written according to the notation (A5). Consider the third block in the first column; it also has a block structure:

$$\Phi_1^{\mathrm{T}} A_0 \Pi_1 = \begin{bmatrix} A_0\Pi_{11} + B_2\mathbf{C} & A + B_2\mathbf{D}C_2 \\ \mathbf{A} & \Phi_{11}A_0 + \mathbf{B}C_2 \end{bmatrix},$$

where

$$\mathbf{A} = \Phi_{12}A_c\Pi_{12}^{\mathrm{T}} + \Phi_{12}B_cC_2\Pi_{11}$$
$$+ \Phi_{11}B_2C_c\Pi_{12}^{\mathrm{T}} + \Phi_{11}(A_0 + B_2D_cC_2)\Pi_{11},$$
$$\mathbf{B} = \Phi_{12}B_c + \Phi_{11}B_2D_c,$$
$$\mathbf{C} = C_c\Pi_{12}^{\mathrm{T}} + D_cC_2\Phi_{11},$$
$$\mathbf{D} = D_c$$

is a linearizing change of variables similar to the one proposed in [20, 30]. The blocks $\Phi_1^{\mathrm{T}} B$ and $C\Pi_1$ can be represented as follows:

$$\Phi_1^{\mathrm{T}} B = \begin{bmatrix} B_1 + B_2\mathbf{D}D_{21} \\ \Phi_{11}B_1 + \mathbf{B}D_{21} \end{bmatrix},$$

$$C\Pi_1 = [C_1\Pi_{11} + D_{12}\mathbf{D} \quad D_{12}\mathbf{D}D_{21}].$$

Thus, we have arrived at inequality (8). Next, the Schur complement lemma can be applied to inequality (A2) to obtain

$$\begin{bmatrix} \Psi - \eta I_{m_w} & * & * \\ \Phi B & -\Phi & * \\ D & 0 & -I_{p_z} \end{bmatrix} \prec 0.$$

Now we perform a congruent transformation of the last inequality using the matrix

$$\text{block diag}\,(I_{m_w},\ \Pi_1^{\mathrm{T}},\ I_{p_z}).$$

Clearly, this transformation brings to inequality (9). Note that the special inequality (11) remains unchanged.

The inverse change of variables (12) is uniquely defined under the nonsingularity of the matrices Φ_{12} and Π_{12}.

REFERENCES

1. Astrom, K.J., *Introduction to Stochastic Control Theory*, Academic Press, 1970.
2. Zames, G., Feedback and Optimal Sensitivity: Model Reference Transformations, Multiplicative Seminorms, and Approximate Inverses, *IEEE Transactions on Automatic Control*, 1981, vol. 26, no. 2, pp. 301–320.
3. Haddad, W.M., Bernstein, D.S., and Mustafa, D., Mixed-Norm H2/H_∞ Regulation and Estimation: the Discrete-Time Case, *Systems & Control Letters*, 1991, vol. 16, no. 4, pp. 235–247.
4. Zhou, K., Glover, K., Bodenheimer, B., and Doyle, J., Mixed H2 and H∞ Performance Objectives I: Robust Performance Analysis, *IEEE Transactions on Automatic Control*, 1994, vol. 38, pp. 1564–1574.
5. Khargonekar, P.P., Rotea, M.A., and Baeyens, E., Mixed H2/H∞ Filtering, *International Journal of Robust and Nonlinear Control*, 1996, vol. 6, pp. 313–330.
6. Scherer, C.W., *Mixed H2/H_∞ Control*, Springer-Verlag, 1995.
7. Semyonov, A.V., Vladimirov, I.G., and Kurdyukov, A.P., Stochastic Approach to H_∞–optimization, *Proceedings of the 33rd Conference on Decision and Control*, 1994, pp. 2249–2250.
8. Vladimirov, I.G., Kurdjukov, A.P., and Semyonov, A.V., Anisotropy of Signals and the Entropy of Linear Stationary Systems, *Doklady Math.,* 1995, vol. 51, pp. 388–390.
9. Vladimirov, I.G., Diamond, P., and Kloeden, P., Anisotropy–based Robust Performance Analysis of Finite Horizon Linear Discrete Time Varying Systems, *Automation and Remote Control*, 2006, vol. 67, no. 7, pp. 1265–1282.
10. Vladimirov, I.G., Kurdyukov, A.P., and Semyonov, A.V., On Computing the Anisotropic Norm of Linear Discrete-time-invariant Systems, *Proceedings of 13th IFAC World Congr.*, San Francisco, 1996, pp. 179–184.
11. Kustov, A.Yu., State-Space Formulas for Anisotropic Norm of Linear Discrete Time Varying Stochastic System, *Proceedings of 15th Int. Conf. on Electr. Eng., Comp. Science and Autom. Control*, Porto, 2018, pp. 1–6.
12. Yurchenkov, A.V., Kustov, A.Yu, Kurdyukov, A.P. Anisotropy-based Bounded Real Lemma for Discrete-time Systems with Multiplicative Noise, *Doklady Math.*, vol. 93, no. 2, pp. 238–240.
13. Yurchenkov, A.V., Anisotropy-Based Controller Design for Linear Discrete-Time Systems with Multiplicative Noise, *Journal of Computer and Systems Sciences Int.,* 2018, vol. 57, no. 6, pp. 864–873.
14. Won, M. and Ranade, G., Control of Systems with Multiplicative Observation Noise, *Proceedings of the* 62nd IEEE Conference on Decision and Control, Singapore, 2023, pp. 3950–3955.
15. Lu, C., Lu, X., Wang, H., et al., Control for Multiplicative Noise Systems with Packet Dropouts and Multiple Delays, *Proceedings of the* 40th Chinese Control Conference, Shanghai, 2021, pp. 1544–1549.
16. Belov, I.R., Yurchenkov, A.V., and Kustov, A.Yu., Anisotropy-Based Bounded Real Lemma for Multiplicative Noise Systems: the Finite Horizon Case, *Proc. of the 27th Med. Conf. on Control and Automation*, Akko, 2019, pp. 148–152.
17. Kustov, A.Yu., Timin, V.N., and Yurchenkov, A.V., Boundedness Condition for Anisotropic Norm of Linear Discrete Time-invariant Systems with Multiplicative Noise, *Journal of Physics: Conference Series*, 2021, vol. 1864, art. no. 012068.
18. Yurchenkov, A.V., Kustov, A.Yu., and Timin, V.N., The Sensor Network Estimation with Dropouts: Anisotropy-based Approach, *Automatica*, 2023, vol. 151, art. no. 110924.
19. Yurchenkov, A.V., Default Sensor Network Setup based on the Anisotropic Criterion, *Herald of the Bauman Moscow State Tech. Univ. Ser. Nat. Sci.*, 2023, vol. 106, no. 1, pp. 45–63. (In Russian.)
20. Gahinet, P., Explicit Controller Formulas for LMI-based H∞ Synthesis, *Automatica*, 1996, vol. 32, pp. 1007–1014.
21. Diamond, P., Vladimirov, I.G., Kurdyukov, A.P., and Semyonov, A.V., Anisotropy-based Performance Analysis of Linear Discrete Time Invariant Control Systems, *International Journal of Control*, 2001, vol. 74, no. 1, pp. 28–42.
22. Diamond, P., Kloeden, P., and Vladimirov, I.G., Mean Anisotropy of Homogeneous Gaussian Random Fields and Anisotropic Norms of Linear Translation-Invariant Operators on Multidimensional Integer Lattices, *Journal of Applied Mathematics and Stochastic Analysis*, 2003, vol. 16, no.3, pp. 209–231.
23. Vladimirov, I.G., Kurdyukov, A.P., and Semyonov, A.V., The Stochastic Problem of H∞-Optimization, *Doklady Math.*, 1995, vol. 52, pp. 155–157.
24. Yurchenkov, A.V., An Anisotropy-Based Boundedness Criterion for Time-Invariant Systems with Multiplicative Noises, *Control Sciences*, 2022, no. 5, pp. 13–20.
25. Scherer, C.W., Gahinet, P., and Chilali, M., Multiobjective Output-feedback Control via LMI Optimization, *IEEE Transactions on Automatic Control*, 1997, vol. 42, pp. 896–911.
26. Toffner-Clausen, S., *System Identification and Robust Control: A Case Study Approach*, Berlin: Springer-Verlag, 1996.
27. Lofberg, J., YALMIP: A Toolbox for Modeling and Optimization in MATLAB, *Proceedings of the 2004 IEEE International Conference on Robotics and Automation*, New Orleans, 2004, pp. 284–289.
28. Sturm, J.F., Using SeDuMi 1.02, a MATLAB Toolbox for Optimization over Symmetric Cones, *Optimization Methods and Software*, 1999, vol. 11, pp. 625–653.
29. Boyd, S. and Vandenberghe, L., *Convex Optimization*, Cambridge: Cambridge University Press, 2004.
30. Tchaikovsky, M.M., Synthesis of Anisotropic Controllers via Convex Optimization and Semidefinite Programming, *Large-Scale Systems Control*, 2013, no. 42, pp. 100–152. (In Russian.)

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ON COALITIONAL RATIONALITY IN A THREE-PERSON GAME

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Abstract. To determine the solution of any game in mathematical game theory, it is necessary to establish what behavior of the players should be considered optimal. In noncooperative games (games without coalitions), the concept of optimality is related, e.g., to the concepts of Nash and Berge equilibria. Optimality in the theory of cooperative games is characterized by the conditions of individual and collective rationality. This paper considers a three-person cooperative game in normal form. For this game, the concept of coalitional rationality is introduced by embracing the conditions of individual and collective rationality with some combination of the concepts of Nash and Berge equilibria. Sufficient conditions are established under which the game has a coalitional equilibrium of this type. In addition, the existence of such a solution in mixed strategies is proved in the case of continuous payoff functions and compact strategy sets of players.

Keywords: maximin, Pareto maximum, Slater maximum, coalitional rationality, Germeier convolution, mixed strategies.

INTRODUCTION

Consider a three-person game described by an ordered triple

$$\Gamma = \left\langle \{1, 2, 3\}, \{X_i\}_{i=1,2,3}, \{f_i(x)\}_{i=1,2,3} \right\rangle.$$

In the game Γ, $\{1, 2, 3\}$ is the set of players; each player chooses his/her *strategy* $x_i \in X_i \subset \mathbb{R}^{n_i}$ ($i = 1, 2, 3$), which results in a *strategy profile* $x = (x_1, x_2, x_3) \in X = \prod_{i=1}^{3} X_i \subset \mathbb{R}^n$, $n = \sum_{i=1}^{3} n_i$. On the set X of all strategy profiles, a *payoff function* $f_i(x)$ of each player i $(i = 1, 2, 3)$ is defined, and its value is called the *payoff* of player i. The game under study is restricted to three persons as it suffices to illustrate the main idea behind the conceptual solution defined below. Moreover, considering the game of four or more persons would lead to a large variety of coalitional structures and, consequently, to more cumbersome formulas.

Conflicts mathematically modeled, in particular, by the three-person game Γ, are usually investigated from the normative point of view, establishing what behavior of the players should be considered optimal (reasonable, expedient). The key features of optimality in mathematical game theory are intuitive beliefs about *profitability, stability, and fairness* [1]. The concept of Nash equilibrium (NE) [2, 3], dominating in noncooperative games, as well as Berge equilibrium (BE), active equilibrium, and equilibrium in threats and counterthreats, which appeared under the former's direct influence [4], is based on the property of stability. These and some other notions of optimality [5] exist in the theory of noncooperative games. In such games, each player usually pursues his/her individual goals; moreover, each player cannot join other players in a coalition to choose coordinated strategies. The antipode to this setup is cooperative games [6]: any unions of players are allowed in order to "struggle" for common interests, and unlimited negotiations are possible between players to choose and use a joint strategy profile. Of course, by the natural assumption, all agreements will be respected by the players. Optimality in cooperative game theory is characterized by the conditions of *individual* [6] and *collective* [6] *rationality*. Individual rationality means that each player's payoff is not smaller than his/her guaranteed payoff reached by acting independently (i.e., using his/her maximin strategy). Collective rationality is ensured by an appropriate vector maximum (in the Slater, Pareto, Geoffrion, Borwein, or any other sense), arising when all players create the grand coalition.

In this paper, an important notion is the *coalitional structure* of a game (the partition of players into pairwise disjoint subsets). For the three-person game Γ, there exist five possible coalitional structures: $\mathfrak{P}_1 = \{\{1\}, \{2\}, \{3\}\}$, $\mathfrak{P}_2 = \{\{1, 2\}, \{3\}\}$, $\mathfrak{P}_3 = \{\{1, 3\}, \{2\}\}$, $\mathfrak{P}_4 = \{\{1\}, \{2, 3\}\}$, and $\mathfrak{P}_5 = \{\{1, 2, 3\}\}$. Here, the structure $\mathfrak{P}_1$ corresponds to the noncooperative "character" of the game whereas the structure $\mathfrak{P}_5$ to the cooperative one. Let us formulate the conditions of individual rationality for the coalitional structure $\mathfrak{P}_1$. Hereinafter, we adopt the short notation $-i = \{\{1, 2, 3\} \setminus \{i\}\}\ \ \forall i \in \{1, 2, 3\}$.

For a strategy profile $x^* = (x_1^*, x_2^*, x_3^*) \in X$, the condition of individual rationality means

$$\begin{aligned} f_i^0 &= \max_{x_i \in X_i} \min_{x_{-i} \in X_{-i}} f_i(x_i, x_{-i}) \\ &= \min_{x_{-i} \in X_{-i}} f_i(x_i^0, x_{-i}) \le f_i(x^*),\ i = 1, 2, 3. \end{aligned} \tag{1}$$

In other words, under the maximin strategy x_i^0, we have the inequalities

$$f_i^0 \le f_i(x^*),\ i = 1, 2, 3. \tag{2}$$

For the coalitional structure $\mathfrak{P}_5$ in the game Γ, the condition of collective rationality will be ensured by Pareto maximality. More precisely, on the set $X^* \subset X$ of strategy profiles, a strategy profile $x^* \in X^* \subset X$ is *Pareto maximal* in the tri-criteria problem $\Gamma_{X^*} = \langle X^*, \{f_i(x)\}_{i=1,2,3} \rangle$ if $\forall x \in X^*$ the system of inequalities $f_i(x) \ge f_i(x^*),\ i = 1, 2, 3,$ is inconsistent, with at least one inequality being strict. According to Karlin's lemma [7], if

$$\sum_{i=1}^{3} f_i(x) \le \sum_{i=1}^{3} f_i(x^*)\ \ \forall x \in X^*, \tag{3}$$

then the strategy profile x^* is Pareto maximal in the problem Γ_{X^*}.

1. THE CONDITION OF COALITIONAL RATIONALITY

Based on a suitable combination of the concepts of NE and BE, we will formalize this condition for the coalitional structures $\mathfrak{P}_2, \mathfrak{P}_3$, and $\mathfrak{P}_4$.

For the coalitional structure $\mathfrak{P}_2$, the condition of a coalitional equilibrium means the four inequalities

$$f_1(x_1^*, x_2^*, x_3) \le f_1(x^*)\ \forall x_3 \in X_3, \tag{4}$$

$$f_2(x_1^*, x_2^*, x_3) \le f_2(x^*)\ \forall x_3 \in X_3, \tag{5}$$

$$f_1(x_1, x_2, x_3^*) \le f_1(x^*)\ \forall x_j \in X_j,\ j = 1, 2, \tag{6}$$

$$f_2(x_1, x_2, x_3^*) \le f_2(x^*)\ \forall x_j \in X_j,\ j = 1, 2; \tag{7}$$

for the structure $\mathfrak{P}_3$, the four inequalities

$$f_1(x_1^*, x_2, x_3^*) \le f_1(x^*)\ \forall x_2 \in X_2, \tag{8}$$

$$f_3(x_1^*, x_2, x_3^*) \le f_3(x^*)\ \forall x_2 \in X_2, \tag{9}$$

$$f_1(x_1, x_2^*, x_3) \le f_1(x^*)\ \forall x_k \in X_k,\ k = 1, 3, \tag{10}$$

$$f_3(x_1, x_2^*, x_3) \le f_3(x^*)\ \forall x_k \in X_k,\ k = 1, 3; \tag{11}$$

finally, for the structure $\mathfrak{P}_4$, the four inequalities

$$f_2(x_1, x_2^*, x_3^*) \le f_2(x^*)\ \forall x_1 \in X_1, \tag{12}$$

$$f_3(x_1, x_2^*, x_3^*) \le f_3(x^*)\ \forall x_1 \in X_1, \tag{13}$$

$$f_2(x_1^*, x_2, x_3) \le f_2(x^*)\ \forall x_l \in X_l,\ l = 2, 3, \tag{14}$$

$$f_3(x_1^*, x_2, x_3) \le f_3(x^*)\ \forall x_l \in X_l,\ l = 2, 3. \tag{15}$$

A strategy profile $x^* \in X$ satisfying all these 12 requirements will be called *coalitionally rational* in the game Γ. Let X^* denote the set of all such strategy profiles; obviously, $X^* \subset X$.

When determining the optimal solution of the game Γ, we will use not all the sixteen inequalities (the three (2), the one (3), and the twelve (4)–(15)) but only seven of them: they are the implications of the others, see the two lemmas below.

Lemma 1. *If inequalities* (6), (14), *and* (15) *are valid, they imply, respectively,*

$$\begin{aligned} f_i(x^*) \ge f_i^0 &= \max_{x_i \in X_i} \min_{x_{-i} \in X_{-i}} f_i(x_i, x_{-i}) \\ &= \min_{x_{-i} \in X_{-i}} f_i(x_i^0, x_{-i}),\ i = 1, 2, 3. \end{aligned}$$

P r o o f. Indeed, due to inequality (6), we have $f_1(x_1, x_2, x_3^*) \le f_1(x^*)\ \ \forall x_j \in X_j,\ j = 1, 2.$ Given the strategy $x_1 = x_1^0$ of player 1, the latter inequality leads to

$$\begin{aligned} f_1(x^*) &\ge f_1(x_1^0, x_2, x_3^*) \ge \min_{x_2, x_3} f_1(x_1^0, x_2, x_3) \\ &= \max_{x_1} \min_{x_2, x_3} f_1(x_1, x_2, x_3) = f_1^0. \end{aligned}$$

Similar statements are established for players $i = 2, 3$ from inequalities (14) and (15), respectively. ♦

Lemma 2. *The following obvious implications are true:* (10)→(4), (14)→(5), (6)→(8), (15)→(9), (7)→(12), and (11)→(13).

Remark 1. According to Lemmas 1 and 2, when determining the optimal solution of the game Γ based on the conditions of individual, collective, and coalitional rationality, it suffices to use only the seven requirements (3), (6), (7), (10), (11), (14), and (15) instead of all the sixteen ones (2)–(15). ♦

Thus, we arrive at the following notion of an optimal solution of the game Γ with $f=(f_1, f_2, f_3)\in\mathbb{R}^3$.

Definition. A pair $(x^*, f(x^*))\in X\times\mathbb{R}^3$ is called a coalitional equilibrium (CE) of the game Γ if:

– The six equalities hold:

$$\begin{aligned}
&\max_{x_1,x_2} f_j(x_1, x_2, x_3^*) = f_j(x^*),\ j=1, 2,\\
&\max_{x_1,x_3} f_k(x_1, x_2^*, x_3) = f_k(x^*),\ k=1, 3,\\
&\max_{x_2,x_3} f_l(x_1^*, x_2, x_3) = f_l(x^*),\ l=2, 3.
\end{aligned} \tag{16}$$

– The strategy profile $x^*\in X$ is Pareto maximal on the set of all coalitional equilibria X^* of the game Γ.

Remark 2. As an optimal solution of the game Γ, we take the pair composed of a strategy profile x^* and the corresponding payoff vector $f(x^*)=(f_1(x^*), f_2(x^*), f_3(x^*))$. Indeed, the existence of a pair $(x^*, f(x^*))$ provides an immediate answer to the two questions of mathematical game theory:

– What should players do in the game Γ?

– What will they receive as a result?

The answer is: the players should follow the corresponding strategies x_i^* from the strategy profile $x^*=(x_1^*, x_2^*, x_3^*)$.

Remark 3. We enumerate the advantages of CE as a solution of the game Γ:

• According to Lemma 1, applying x^* ensures the conditions of individual rationality.

• The strategy profile x^* brings all players to the highest payoffs (Pareto maximal with respect to the other CE in the game Γ). As we believe, this fact is an analog of the condition of collective rationality from the theory of cooperative games.

• The requirements (4)–(15) mean, e.g., the dual-purpose allocation of the resources of player 1. That is:

– Without forgetting his/her individual interests, player 1 strives to help, as much as possible, player 2 in the union {1, 2} as a member of the coalitional structure $\mathfrak{P}_2$ (see the requirements (6) and (7)).

– Without forgetting his/her interests, player 1 also helps player 3 as a member of the union {1, 3} of the coalitional structure $\mathfrak{P}_3$ (see the requirements (10) and (11)).

As we believe, formalizing these two requirements in the first and second rows of the expression (16) is a modification of the concept of NE to the case of a bi-criteria payoff function of the players; the third row of the expression (16) can be understood as a realization of the concept of BE for the same bi-criteria setup. Similar considerations concern players 2 and 3.

Finally, CE also involves the *principle of stability*: due to condition (16), an arbitrary unilateral deviation of any coalitions (composed of one or two players) from x^* cannot improve the payoff of the deviated coalition in the game Γ as compared to $f_i(x^*)$, $i = 1, 2, 3$.

Remark 4. Once an optimal solution is determined, mathematical game theory recommends settling two issues:

– Does such a solution exist?

– How can it be found? ♦

The answers are provided in the next section.

2. SUFFICIENT CONDITIONS

Let us proceed to the key result of this paper. We introduce the two n-vectors $x=(x_1, x_2, x_3)\in X\subset\mathbb{R}^n$, $n=\sum_{i=1}^{3} n_i$, and $z=(z_1, z_2, z_3)\in X$, as well as the seven scalar functions

$$\begin{aligned}
&\varphi_1(x, z) = f_1(x_1, x_2, z_3) - f_1(z),\\
&\varphi_2(x, z) = f_2(x_1, x_2, z_3) - f_2(z),\\
&\varphi_3(x, z) = f_1(x_1, z_2, x_3) - f_1(z),\\
&\varphi_4(x, z) = f_3(x_1, z_2, x_3) - f_3(z),\\
&\varphi_5(x, z) = f_2(z_1, x_2, x_3) - f_2(z),\\
&\varphi_6(x, z) = f_3(z_1, x_2, x_3) - f_3(z),\\
&\varphi_7(x, z) = \sum_{i=1}^{3} f_i(x) - \sum_{i=1}^{3} f_i(z).
\end{aligned} \tag{17}$$

Using the payoff functions of the players in the game Γ, we construct the Germeier convolution of the seven functions:

$$\varphi(x, z) = \max_{k=1,\dots,7} \varphi_k(x, z), \tag{18}$$

defined on the set $X\times(Z=X)\subset\mathbb{R}^{2n}$, where $X=\prod_{i=1}^{3} X_i$ is the set of all strategy profiles in the game Γ.

A saddle point $(\bar{x}, z^*) \in X \times Z$ of the scalar function $\varphi(x, z)$ (17), (18) in the zero-sum two-person game

$$\Gamma^\alpha = \langle X, Z = X, \varphi(x, z) \rangle \tag{19}$$

is defined by the chain of inequalities

$$\varphi(x, z^*) \le \varphi(\bar{x}, z^*) \le \varphi(\bar{x}, z) \ \forall x, z \in X, \tag{20}$$

with $z^* \in X^*$ representing the minimax strategy, i.e., $\min_{z \in X} \max_{x \in X} \varphi(x, z) = \max_{x \in X} \varphi(x, z^*)$.

Proposition. *If there exists a saddle point $(\bar{x}, z^*)$ in the game Γ^α, then the minimax strategy $z^* \in X$ of this game is a CE of the original game Γ_3.*

P r o o f. With the strategy profile $z = \bar{x}$ substituted into inequalities (20), from formula (17) we obtain $\varphi(\bar{x}, \bar{x}) = 0, k = 1,..., 7,$ for all $\varphi_k(\bar{x}, \bar{x}) = 0, k = 1,..., 7.$ Then, due to inequalities (20) and the transitivity property,

$$\varphi(x, z^*)=\max\{f_1(x_1, x_2, z_3^*) - f_1(z^*),$$
$$f_2(x_1, x_2, z_3^*) - f_2(z^*),\ f_1(x_1, z_2^*, x_3) - f_1(z^*),$$
$$f_3(x_1, z_2^*, x_3) - f_3(z^*),\ f_2(z_1^*, x_2, x_3) - f_2(z^*),$$
$$f_3(z_1^*, x_2, x_3) - f_3(z^*),\ \sum_{i=1}^{3} f_1(x) - \sum_{i=1}^{3} f_i(z^*)\} \le 0$$
$$\forall x_i \in X_i,\ i = 1, 2, 3.$$

Consequently,

$$\begin{aligned} & f_j\left(x_1, x_2, z_3^*\right) \le f_j(z^*) \quad \forall x_j,\ j = 1, 2, \\ & f_k\left(x_1, z_2^*, x_3\right) \le f_k(z^*) \quad \forall x_k,\ k = 1, 3, \\ & f_l\left(z_1^*, x_2, x_3\right) \le f_l(z^*) \quad \forall x_l,\ l = 2, 3, \\ & \sum_{r=1}^{3} f_r(x) \le \sum_{r=1}^{3} f_r(z^*) \quad \forall x \in X^* \subset X. \end{aligned} \tag{21}$$

By the first three inequalities of (21) and the requirement (16), the strategy profile $z^* \in X$ is coalitionally rational in the game Γ. The last inequality of (21) and the inclusion $X^* \subset X$ ensure [7, *p. 71*] the Pareto maximality of the strategy profile x^* in the tri-criteria problem $\Gamma_{X^*} = \left\langle X^*, \{f_i(x)\}_{i=1,2,3} \right\rangle$. ♦

Remark 5. The above proposition provides the following constructive method for calculating a coalitional equilibrium of the game Γ:

– Construct the function $\varphi(x, z)$ by formulas (17) and (18).

– Find the saddle point $(\bar{x}, z)$ of the function $\varphi(x, z)$ from the chain of inequalities (20) [8].

– Find the values of the three functions $f_i(z^*)$, $i = 1, 2, 3$.

Then the pair $(z^*, f(z^*) = (f_1(z^*), f_2(z^*), f_3(z^*))) \in X \times \mathbb{R}^3$, represents a coalitional equilibrium of the game Γ.

Remark 6. If the (N + 1) scalar functions $\varphi_j(x, z),\ j = 1,\ldots,7,$ are continuous on the set $X \times Z$, and the sets $X, Z \in \operatorname{comp} \mathbb{R}^n$, then the function $\varphi(x, z) = \min_{j=1,\ldots,N+1} \varphi_j(z, z)$ is also continuous on the set $X \times Z$.

The proof of an even more general result is available in many textbooks on operations research; for example, see the book [9, *p. 54*]. It has even appeared in textbooks on convex analysis [10, *p. 146*]. ♦

Finally, the following theorem is crucial in this paper.

Theorem (existence in mixed strategies). *If the game Γ includes strategy sets $X_i \in \operatorname{comp} \mathbb{R}^{n_i}$ and payoff functions $f_i(\cdot) \in C(X)$, $i = 1, 2, 3$, then there exists a coalitional equilibrium in mixed strategies in this game.*

CONCLUSIONS

First of all, let us emphasize the *new* results of *cooperative game theory* obtained in this paper.

• The notion of a coalitional equilibrium (CE) has been formalized by considering the interests of any coalition in the game Γ.

• A constructive method for calculating CE has been provided. This method reduces to finding the minimax strategy for a special Germeier convolution, effectively constructed using the payoff functions of the players.

• The existence of CE in mixed strategies has been proved under standard mathematical programming conditions (the continuous payoff functions and compact strategy sets of the players).

As we believe, the *new qualitative results* following from this paper are also important:

– The results can be extended to cooperative games with any finite number of players (more than three). In these games, NE (BE) corresponds to Nash equilibrium (Berge equilibrium), respectively.

– CE ensures the stability of a coalitional structure to an arbitrary unilateral deviation of any coalitions.

– CE is applicable even if coalitional structures change during the game or even if all coalitions remain in force.

– CE can be used to create stable unions (alliances) of players.

And these are not all the advantages of CE.

Note another positive property as well. So far, the theory of cooperative games has been focused on the conditions of individual and collective rationality. Meanwhile, the individual interests of the players correspond to the concept of NE with its "selfish" character; collective rationality matches the concept of BE with its "altruism." However, such "forgetfulness" is not inherent to the human nature of players. These drawbacks of both concepts are leveled by coalitional rationality. Indeed, under the conditions of coalitional rationality, player 1 helps player 2 as a member of the coalition $\{1, 2\}$ of the coalitional structure $\mathfrak{P}_2$ and player 3 as a member of the coalition $\{1, 3\}$ of the coalitional structure $\mathfrak{P}_3$, not forgetting about him-/herself in both roles. And the other players act similarly. Thus, CE fills the gap between NE and BE by adding "care of others" to NE and self-care to BE.

REFERENCES

1. Vorob'ev, N.N., *Game Theory: Lectures for Economists and Systems Scientists*, Springer, 1977.
2. Nash, J., Non-cooperative Games, *The Annals of Mathematics*, 1951, vol. 54, no. 2, pp. 286–295.
3. Nash, J., Equilibrium Points in N-person Games, *Proc. Nat. Acad. Sci.*, 1950, vol. 36, pp. 48–49.
4. Zhukovskiy, V.I. and Chikrii, A.A., *Lineino-kvadratichnye differentsial'nye igry* (Linear-Quadratic Differential Games), Kiev: Naukova Dumka, 1994. (In Russian.)
5. Viale, R., *Routledge Handbook of Bounded Rationality*, London–New York: Routledge, 2021.
6. Zhukovskiy, V.I., *Kooperativnye igry pri neopredelennosti i ikh prilozheniya* (Cooperative Games under Uncertainty and Their Applications), Moscow: Editorial URSS, 2009. (In Russian.)
7. Podinovskii, V.V. and Noghin, V.D., *Pareto-optimal'nye resheniya mnogokriterial'nykh zadach* (Pareto Optimal Solutions of Multicriteria Problems), Moscow: Fizmatlit, 2007. (In Russian.)
8. Vorob'ev, N.N., *Foundations of Game Theory: Noncooperative Games*, Basel–Boston–Berlin: Birkhäuser, 1994.
9. Morozov, V.V., Sukharev, A.G., and Fedorov, V.V., *Issledovanie operatsii v zadachakh i uprazhneniyakh* (Operations Research in Problems and Exercises), Moscow: Vysshaya Shkola, 1968. (In Russian.)
10. Dmitruk, A.V., *Vypuklyi analiz. Elementarnyi vvodnyi kurs* (Convex Analysis: An Elementary Introductory Course), Moscow: MAKS-PRESS, 2012. (In Russian.)

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CONSTRUCTING SCIENTIFIC PUBLICATION PROFILES BASED ON TEXTS AND COAUTHORSHIP CONNECTIONS (IN THE FIELD OF CONTROL THEORY AND ITS APPLICATIONS)

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Abstract. The calculation of scientific publication profiles is crucial in the systematization of scientific knowledge and support for scientific decision-making. This paper proposes a method for forming publication profiles in the field of control theory, based on the integration of text analysis and coauthorship network analysis. We describe a basic algorithm that analyzes publication texts by a thematic classifier as well as its enhanced version that considers network connections within a heuristic approach. The methods are examined using expert assessments and quantitative metrics; according to the examination results, combining textual and network data significantly improves the accuracy of publication profiles. Hypotheses about a relationship between the thematic similarity and network proximity of publications are tested, and the approach proposed is validated accordingly. In addition, directions for further research are identified.

Keywords: publication network, publication profile, control theory, graph neural networks, text analysis.

INTRODUCTION

Thematic analysis of scientific publications is an important tool for justifying scientific decisions and identifying trends in various fields of knowledge [1–6]. One of the most common approaches to text analysis is thematic modeling [7], which is used to calculate scientific publication profiles. However, when abstracts or texts of publications have limited length and/or contain imprecise terms, using only textual information may lead to low accuracy of profiles.

The inclusion of network data, such as coauthorship or citation connections, has already demonstrated its utility in several disciplines: considering the structural relationships between publications can improve the quality of classification and more adequately reflect hidden thematic dependencies [8–10]. In particular, *graph neural networks* (GNNs) [11, 12] have proven to be an effective network analysis tool, as they simultaneously cover both node features and graph topology.

This study aims to develop and evaluate improved methods for constructing publication profiles that combine text and network information analysis. The main results of the work are summarized below:

- A basic algorithm for calculating publication profiles based on a thematic classifier in the field of control theory and its applications is presented.
- “Advanced” algorithms considering network data are developed. In particular, they are a heuristic method that extends the basic profile with connected publications (through a coauthorship or citation) and a method based on GNNs that deeply integrates structural information on the connections between publications.
- The efficiency of these algorithms is evaluated, showing their higher accuracy compared to the ones using only textual information.
- Relationships between the thematic similarity of publications (assessed by their profiles) and network characteristics (e.g., common neighbors in a graph) are investigated. As part of this analysis, several hypotheses are formulated and tested.

The following sections of the paper describe in detail the implementation of the algorithms, the metrics used, and the experimental results.

1. METHODS

1.1. The Basic Algorithm for Profile Calculation

In this paper, for each scientific publication l, we construct a *basic profile* $p(l)$ using the thematic classifier of ISAND, an information system for scientific activity analysis [13]. This classifier is based on the principles outlined in [14] and represents a hierarchical ontology of themes in control theory and its applications. Its fragment is shown in Fig. 1, where long theme names are abbreviated; the full version of the classifier can be found at: https://www.ipu.ru/sites/default/files/page_file/ClassifierCS.xlsx. The original form[1] was used for marking up publications by a group of experts; the results of applying the basic and network algorithms for marking up publications were then compared with the expert markup.

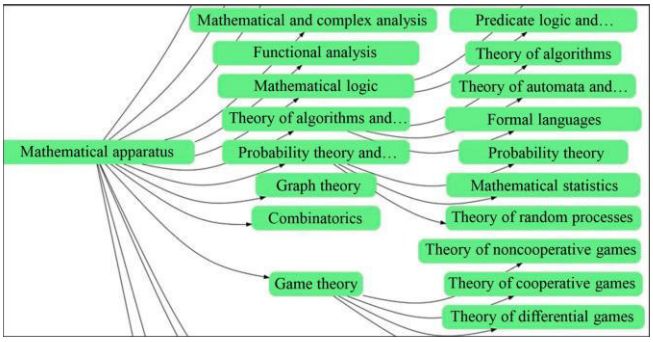

Fig. 1. The thematic classifier (the ISAND ontology).

A publication profile $p(l)$ is a stochastic vector $(p_{l1}, p_{l2}, \ldots, p_{ln})$ in which each component p_{li} is the normalized frequency of terms related to theme i in publication l.

1.2. Advanced Profile Calculation Methods

Consider a graph $G(V, E)$ in which each vertex $l \in V$ corresponds to one publication and each edge $(l, m) \in E$ denotes a coauthorship connection. In other words, publications l and m have a non-empty intersection of their authors, $|K(l) \cap K(m)| > 0$, where $K(l)$ is the set of authors (coauthors) of publication l. In the original graph, each vertex l is initialized by the basic profile vector $p(l)$.

[1] URL: https://docs.google.com/forms/d/e/1FAIpQLSfR47ZQyjI9wrMgqRPP85j_uZCeUI95dNFnMR-2ruCfq3XtIg/viewform

1.2.1. A Heuristic Method

To improve the accuracy of the basic profile of publication l, consider publications associated with l by a coauthorship connection and issued within a fixed period $\delta \in \mathbb{N}$ (time window). The default value is δ = 4 years, but it can be adjusted based on empirical data. In several disciplines, including control theory, a window of 3–5 years is common to assess scientific activity. In particular, according to the methodological recommendations of the Higher Attestation Commission (VAK RF), it is necessary to consider publications issued in the last five years. Thus, δ = 4 years is a reasonable choice from the viewpoint of assessing the scientific activity of researchers.

An *extended profile* $p_e(l)$ is a weighted combination

$$p_e(l) = \alpha p(l) + (1-\alpha)\frac{\sum_{m \in L_\delta(l)} w_{lm} p(m)}{\sum_{m \in L_\delta(l)} w_{lm}},$$

where $L_\delta(l)$ denotes the set of publications connected to l and issued within the time window δ, and $\alpha \in (0, 1]$ is the coefficient regulating the contribution of the original and network profiles. The value of α can be chosen empirically, e.g., based on the results of cross-validation on a delayed sample.

The coefficient $w_{lm} \in [0, 1]$ reflects the contribution of publication m to profile l. In this case, the share of common authors is taken into consideration:

$$w_{lm} = \frac{|K(l) \cap K(m)|}{|K(m)|}.$$

Note that if $\sum_{m \in L_\delta(l)} w_{lm} = 0$ (publication l has no connected publications for the last δ years), the profile $p_e(l)$ coincides with $p(l)$ by definition. Generally speaking, the heuristic method smoothens the "noise" in the basic profiles as well as generates a network profile for publications with missing or uninformative abstracts.

1.2.2. A Method Based on Graph Neural Networks

To further improve the accuracy of profiles, we apply a graph neural network (GNN) trained on the graph $G(V, E)$. Initially, each node $i \in V$ receives a feature vector $\mathbf{h}_i^{(0)} = p(i)$. At the kth layer of the

GNN, the vector $\mathbf{h}_i^{(k)}$ is recalculated considering the neighbors $\mathcal{N}(i)$ by the formula

$$\mathbf{h}_i^{(k)} = \sigma\left(\sum_{j\in\mathcal{N}(i)} \frac{1}{c_{ij}} W^{(k)} \mathbf{h}_j^{(k-1)}\right)$$

with the following notation: $\mathbf{h}_i^{(k)}$ is the new representation (profile) of node *i* in the *k*th layer of the neural network; $W^{(k)}$ is the trained weight matrix (model parameters); σ is a nonlinear activation function (e.g., ReLU); c_{ij} is a normalization coefficient regulating the contribution of neighbor nodes (e.g., $c_{ij} = \sqrt{\deg(i)\deg(j)}$); finally, $\mathcal{N}(i)$ is the set of neighbors of node *i* in the publication graph.

Passing through several layers of the GNN gives the resulting vector $\mathbf{h}_i^{(K)}$, which can be treated as the profile of publication *i* "deeply integrated" in the network topology. With appropriate training (using a quality metric and a target function), this approach can identify and consider complex dependencies between publications, which often improves the accuracy and informativeness of publication profiles.

Well, in this paper we propose the following approaches to extend the basic profile of scientific publications:

- **the heuristic method**, which specifies a linear combination of a publication profile with the profiles of connected papers;
- **the GNN method**, which aggregates features in a more sophisticated way based on a training sample.

Both approaches enrich the basic profiles by considering indirect thematic connections through a coauthorship, which facilitates the analysis of scientific publications (classification and recommendation of relevant papers). The heuristic method is characterized by high interpretability and simple implementation; however, its accuracy may be limited in complex network structures with unobvious connections. In turn, the GNN method identifies complex dependencies between publications, thereby being preferable whenever the in-depth analysis of coauthorship structures is required. However, this method may need large datasets and significant computational resources for effective model training.

2. EXPERIMENTS

To evaluate the effectiveness of these methods for calculating publication profiles in the field of control theory and applications, we used a sample of 20 thousand items (the publication database of the Trapeznikov Institute of Control Sciences, the Russian Academy of Sciences). The dataset included texts (for the construction of textual features) and coauthorship information (for the construction of network features).

Two types of profiles were calculated:

– the basic profiles (by abstract texts only),

– the extended profiles (considering the network structure, i.e., coauthorship connections).

The following criteria were used to evaluate the quality of the resulting profiles in quantitative terms:

- expert assessments: subject matter experts assessed the relevance of the themes assigned to each publication;
- quality metrics: the values of *Precision@k*, completeness, and F_1-measure were calculated. To determine *Precision@k*, the *k* most probable themes from the profile were selected and compared to the reference themes identified by the experts.

The next section presents the results of experiments, i.e., the analysis of the distances between different profiles and the test of hypotheses about a relationship between the thematic similarity and network proximity of publications.

3. RESULTS

With the extended algorithm, which considers network information, the results were significantly improved in accuracy relative to the basic approach. For example, the heuristic method (Fig. 2) achieved *Precision@k* = 37%, whereas the basic method provided only 25% (on the sample marked by experts). This indicates the high utility of network features for the formation of thematic profiles.

In this study, a *graph neural network* (GNN) architecture was developed. It includes three sequential layers of a *graph convolutional network* (GCN) and Dropout regularization layers. To evaluate the quality of the model, we partitioned the sample (several hundred publications) in a proportion of 70%/15%/15% into training, validation, and testing subsamples. The training was carried out for 100 epochs, with the final model parameters selected from the best results achieved on the validation subsample. The average value of *Precision@k* (for *k* = 3) across all runs was 39%, constituting a 2% increase over the heuristic method used previously. Therefore, the accuracy of prediction was improved.

3.1. Analysis of the Distances between Profiles

For a more detailed comparison, we analyzed the distribution of distances between the extended and

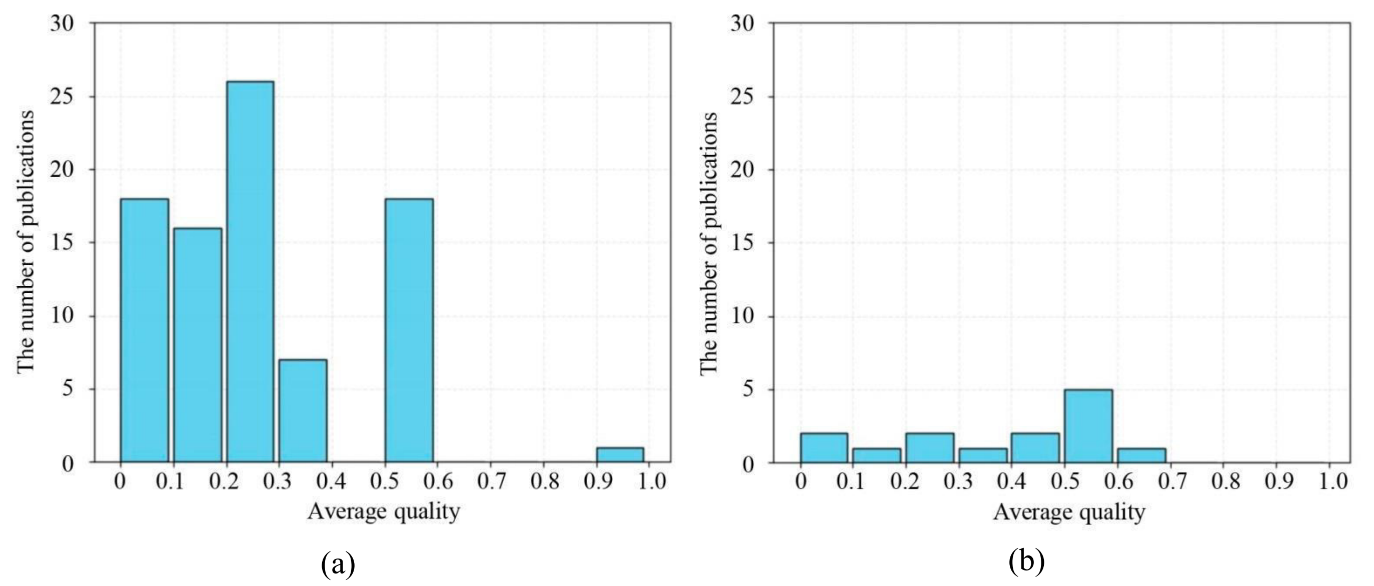

Fig. 2. Quality evaluations of the methods for constructing publication profiles: (a) basic (BaseAlgo, 25% accuracy over abstracts) and (b) network (HAdvAlgo, 37% accuracy over abstracts). The horizontal axis corresponds to the average quality of profiles whereas the vertical axis to the number of publications with such quality.

basic profiles for the entire dataset of publications (Fig. 3). Since the metric $d \in [0, 1]$ (*L1*) was used, two characteristic peaks can be observed on the graph:

• The first peak (for $d = 0$) corresponds to cases when an author has only one publication. In such a situation, the network profile almost coincides with the basic one.

• The second peak (for high values of d) is observed for publications where network information (coauthorship) strongly affects the final profile and/or textual data are too scarce (complicating the adequate calculation of the basic profile).

In particular, if an abstract is very short or contains few relevant terms, the basic profile may be weakly informative. In these cases, the impact of network data turns out to be the most significant, and the distance between the profiles (basic and extended) increases noticeably.

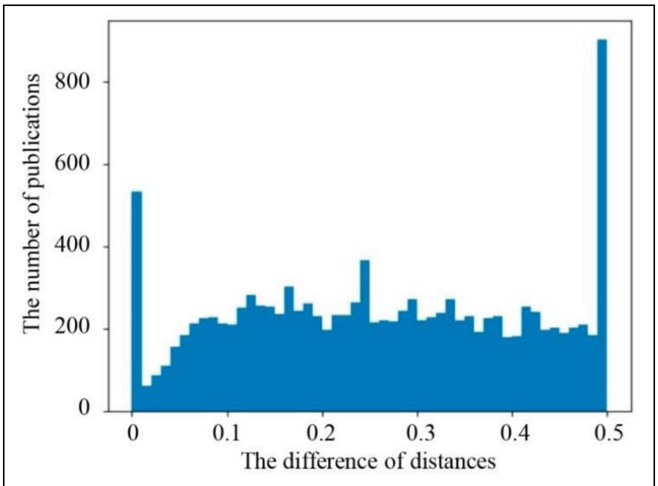

Fig. 3. The distribution of the distances between basic and extended profiles.

3.2. Test of Hypotheses about a Relationship between the Thematic Similarity and Network Proximity of Publications

Hypotheses concerning a relationship between the thematic similarity (by profile) and network proximity (by the coauthorship graph) of publications were also investigated.

Hypothesis 1: *The profiles of randomly selected publications differ from each other.*

The calculations confirmed this hypothesis: the average distance between the profiles of random publications was about 0.9 (with values ranging from 0 to 1), indicating a significant diversity of themes in the field of control theory.

Hypothesis 2. *The closer the content of the abstracts of two randomly selected publications, the closer their profiles will be.*

Tests using vector representations (*embeddings*) showed no significant correlation. Note that different language models were applied to construct them: RuSciBert, SciBert, and Sentence Embeddings. Probably, textual abstracts were too short or heterogeneous to ensure consistency with the thematic profiles extended with network information. A more detailed analysis of the nature of the discrepancies (the number of terms, language variation, etc.) is needed in the future.

Hypothesis 3. *The more terms an abstract has, the higher the correlation between the profiles and vector representations (embeddings) of these abstracts will be.*

This hypothesis was confirmed: the correlation coefficient increased from 0.25 (with five terms) to 0.88 (with eight terms). The result emphasizes the importance of the completeness and accuracy of abstracts in the sense of terms used.

Hypothesis 4. *The similarity of the profiles of two publications depends on:*

- *the existence of a coauthorship connection,*
- *the coincidence of the authors' composition.*

This hypothesis was confirmed: for pairs of publications with common authors, the average distance between the profiles (0.63) turned out to be noticeably smaller than for all other pairs (0.88).

Hypothesis 5. *The smaller the period between publications is, the closer their profiles will be.*

During certain periods, there may be bursts of interest in particular technologies or phenomena in a given research area (e.g., "big data," "machine learning"), which should be reflected in the content of publications. Such bursts may be related to different phases of the popularity cycle of technologies. However, in the field of control theory, this hypothesis was not confirmed. According to the analysis results, for random pairs of publications, there were no significant changes in the level of similarity of their profiles depending on the period. However, for non-random pairs of publications, such dependence takes place. (Pairs of publications connected to each other in the network are called non-random.)

Hypothesis 6. *The similarity of the profiles of two publications depends on the topological strength of their connection.*

The hypothesis was not confirmed: the resulting data (Fig. 4) did not reveal a significant correlation between the number of common neighbors in the network and the distance between profiles. (The distance is zero if the profiles are equal.) The topological strength of a connection means the number of coauthorship connections between publications. (The topological strength of a connection is zero if there exist no coauthorship connections.)

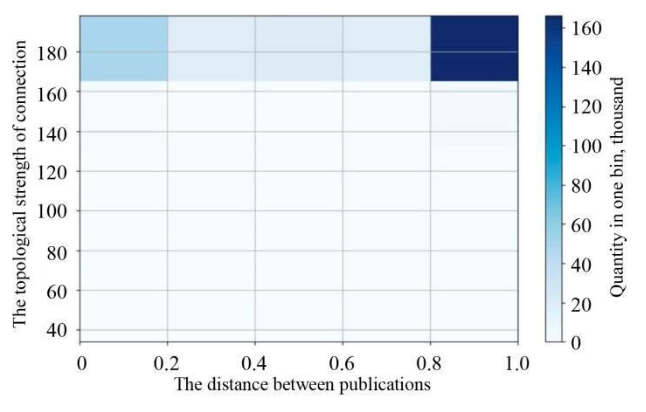

Fig. 4. The dependence of the proximity of publications on the topological strength of the connection between them. A bin is a cell that partitions N into M equal rectangles.

4. DISCUSSION OF THE RESULTS

According to the experimental results, considering network features (coauthorship) improves the accuracy of constructing publication profiles. The heuristic method is valuable for its simplicity and interpretability, which is especially convenient at the stage of initial assessment of profiles and analysis of the subject matter. At the same time, the direct application of this method has demonstrated its disadvantages: if the authors have few papers or the abstracts are too short, the quality of the basic profile will remain low, and even network information does not always compensate for the lack of textual data.

When testing the hypotheses, it has been established that there exists a significant diversity of themes in the field of control theory (Hypothesis 1) and that "textual similarity" (Hypothesis 2) does not necessarily lead to profile similarity, especially for incomplete abstracts. In addition, the more terms a publication contains, the more the similarity of embeddings will determine the similarity of profiles (Hypothesis 3). The similarity of profiles depends on the existence of a connection between publications (Hypothesis 4). However, the test results of the other hypotheses (5 and 6) have shown that proximity in time or a large number of common neighbors do not guarantee the similarity of publication: additional factors and additional research are required here.

Among the limitations of this study, we mention data sparsity (a small number of publications per one author) and the heterogeneous quality of the abstracts used to construct basic publication profiles. It seems promising to investigate further how the use of citation networks, keywords, longer texts (full-text papers), and advanced GNN models (e.g., *graph attention networks*) can improve the accuracy of publication profiles.

CONCLUSIONS

Thus, the hybrid methods proposed in this paper, combining textual and network features, are significantly superior to the basic (textual) approach in constructing scientific publication profiles. Several hypotheses about the thematic similarity and network proximity of publications have been tested. According to the test results, in some cases, network connections turn out to be of a much higher utility for theme identification than the content of short abstracts. The results of this study will be used to develop methods for analyzing scientific publications and systematizing knowledge in the field of control theory and applications.

REFERENCES

1. Kryzhanovskaya, S.Yu., Vlasov, A.V., Eremeev, M.A., and Vorontsov, K.V., Semiautomatic Summarization of Thematic Samples of Scientific Publications: Problems and Approaches, *Tezisy dokladov 20-oi Vserossiiskoi konferentsii s mezhdunarodnym uchastiem "Matematicheskie metody raspoznavaniya obrazov"* (Abstracts of the 20th All-Russian Conference with International Participation "Mathematical Methods of Image Recognition"), Moscow, 2021, pp. 333–338. (In Russian.)
2. Shibayama, S., Yin, D., and Matsumoto, K., Measuring Novelty in Science with Word Embedding, *PLoS ONE*, 2021, no. 7, pp. 1–16.
3. Yuan, W., Liu, P., and Neubig, G., Can We Automate Scientific Reviewing?, *Journal of Artificial Intelligence Research*, 2022, no. 75, pp. 171–212.
4. Cachola, I., Lo, K., Cohan, A., and Weld, D., TLDR: Extreme Summarization of Scientific Documents, *Findings of the Association for Computational Linguistics: EMNLP 2020*, 2020, pp. 4766–4777.
5. Bao, P., Hong, W., and Li, X., Predicting Paper Acceptance via Interpretable Decision Sets, in *Companion Proceedings of the Web Conference 2021 (WWW 21)*, New York: Association for Computing Machinery, 2021, pp. 461–467.
6. Kasanishi, T., Isonuma, M., Mori, J., and Sakata, I., SciReviewGen: A Large-scale Dataset for Automatic Literature Review Generation, *arXiv:2305.15186*, 2023, pp. 1–19. DOI: https://doi.org/10.48550/arXiv.2305.15186
7. Blei, D.M., Ng, A.Y., and Jordan, M.I., Latent Dirichlet Allocation, *Journal of Machine Learning Research*, 2003, no. 3, pp. 993–1022.
8. Hasegawa, T., Arvidsson, H., Tudzarovski, N., et al., Edge-Based Graph Neural Networks for Cell-Graph Modeling and Prediction, *Information Processing in Medical Imaging*, 2023, vol. 13939, pp. 265–277.
9. Xiong, C., Li, W., Liu, Y., and Wang, M., Multi-Dimensional Edge Features Graph Neural Network on Few-Shot Image Classification, *IEEE Signal Processing Letters*, 2021, vol. 28, pp. 573–577.
10. Faber, L., Lu, Y., and Wattenhofer, R., Should Graph Neural Networks Use Features, Edges, or Both?, *arXiv: 2103.06857*, 2021, pp. 1–12. DOI: https://doi.org/10.48550/arXiv.2103.06857
11. Zhou, J., Cui, G., Hu, S., et al., Graph Neural Networks: A Review of Methods and Applications, *AI Open*, 2020, vol. 1, pp. 57–81.
12. Kipf, T.N. and Welling, M., Semi-Supervised Classification with Graph Convolutional Networks, *arXiv:1609.02907*, 2017, pp. 1–14. DOI: https://doi.org/10.48550/arXiv.1609.02907
13. Gubanov, D.A., Kuznetsov, O.P., Sukhoverov, V.S., and Chkhartishvili, A.G., On the Construction of Profiles in the Thematic Space of Control Theory, *Materialy 9-oi Mezhdunarodnoi konferentsii "Znaniya-Ontologii-Teorii"* (Proceedings of the 9th International Conference "Knowledge–Ontology–Theories"), Novosibirsk, 2023, pp. 89–94. (In Russian.)
14. Kuznetsov, O.P. and Sukhoverov, V.S., An Ontological Approach to Determining the Subject Matter of Scientific Text, *Ontology of Designing*, 2016, vol. 6, no. 1, pp. 55–66. (In Russian.)

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32ND INTERNATIONAL CONFERENCE ON PROBLEMS OF COMPLEX SYSTEMS SECURITY CONTROL

In November 2024, the 32nd International Conference on Problems of Complex Systems Security Control took place at the Trapeznikov Institute of Control Sciences, the Russian Academy of Sciences (ICS RAS), Moscow. The conference was dedicated to the memory of Vladimir V. Kul'ba, Dr. Sci. (Eng.), Prof., Honored Scientist of the Russian Federation, and the conference founder. It was held face-to-face, with over 100 participants.

The plenary session was opened by Academician *D.A. Novikov*, Director of ICS RAS. His welcoming speech was dedicated to the memory of Kul'ba, the permanent head of the Conference Program Committee.

I.V. Chernov, Chair of the plenary session, described the main stages of Kul'ba's scientific activities as well as the fields of his fundamental and applied research. As emphasized by the speaker, from 1962 to 2024, Kul'ba headed several research fields at ICS RAS. Under his leadership and with his direct participation, the following R&D results were obtained:

– analysis methods for complex management systems;

– theoretical foundations for using the principles of modularity and type designs in data processing systems design;

– methods and technologies for automating the design of software and information support for open-architecture and real-time systems;

–methodological foundations for improving the effectiveness of organizational control and management in emergencies;

– methods for solving theoretical and applied information security problems for control systems at the organizational and software-hardware levels;

– an information control methodology;

– theoretical and methodological foundations of scenario management.

The speaker paid special attention to the Conference on Problems of Complex Systems Security Control, Kul'ba's favorite brainchild. The conference history goes back to the early 1990s, when, in response to Kul'ba's proposal, the Institute of Control Sciences initiated the International Scientific Conference on Control Problems in Emergencies. The initiative was supported by the RAS Presidium and the State Committee for Civil Defense, Emergencies, and Elimination of Consequences of Natural Disasters under the President of the RSFSR (later, reorganized into the same-name ministry of the Russian Federation). In addition to ICS RAS, the organizers of the conference included the Scientific Council for the State Scientific and Technical Program "Security," the Keldysh Institute of Applied Mathematics RAS, the Institute of Design Automation RAS, the Kharkevich Institute for Information Transmission Problems RAS, and St. Petersburg State University. Starting from 1999 (the 7th conference), Russian State University for the Humanities became one of the conference organizers.

Initially, the main theme of the conference was fundamental and applied research into improving the effectiveness of emergency management. Later, the Organizing Committee of the conference decided to change its name to the current one, following the appearance of new fields in the subject areas under consideration and the related ones, a significant widening of the topics of conference submissions, and the wishes expressed by the majority of regular conference participants. Since 1998 the conference name and the composition of its sections have remained almost unchanged, and the last 32nd Conference is not an exception.

The conference was attended by 104 authors from 33 organizations, who presented 73 papers. The conference program included the following sections:

1. General theoretical and methodological issues of security support;
2. Problems of economic and sociopolitical security support;
3. Problems of information security support;
4. Cybersecurity. Security aspects in social networks;

5. Ecological and technogenic security;

6. Modeling and decision-making for complex systems security control;

7. Automatic systems and means of complex systems security support.

Many papers presented at the conference were devoted to the study and solutions of the problems of ensuring the key components of national security, namely, military-political, scientific, industrial-technological, social, economic, informational, and technogenic. The topicality of this range of problems, representing one of the most complex set of problems in the theory and methodology of organizational control and management and several related scientific disciplines, has significantly increased in recent years. It is connected with various objective geopolitical reasons and the continuing growth of international tension.

V.V. Shumov presented the paper "Analysis of Factors Affecting the Achievement of the Goals of the Special Military Operation." He considered a conceptual formalized model for assessing the level of national security of a state using a power production function, reflecting the dichotomy of the human values of development and self-preservation. The author's formalized representation of the security function is structurally the product of two components: the function of sovereignty (development), based on the Cobb–Douglas power function, and the preservation function. The first function covers a set of geographical, demographic, and socio-technological factors to assess the geopotential of the state; the second function reflects the ability of the state to resist destructive processes (including those inspired from the outside) and develop sustainably. The modeling results were described and analyzed to assess:

(a) Russia's security level, place, and role in global processes against the background of the ongoing geopolitical inversion (change of the world leader);

(b) the security level of the European Union and Ukraine in the context of its regions (as of 2013, i.e., for the period preceding the Euromaidan, which provoked an acute political and economic crisis).

The main goal of the modeling studies was to assess the possible consequences of a significant complication of the military and political situation in the world, associated with the growingly aggressive actions of Russia's geopolitical adversaries aimed at weakening, inflicting a strategic defeat, and ultimately dismembering Russia. According to the analysis results, the response potential to parry existential threats to the Russian state and society exceeds the capabilities of Western countries, and the actions of their ruling elites do not meet the vital interests of their people. Hence, there are objective favorable conditions for achieving the security and sovereignty goals of the Russian Federation.

The paper "On a New Approach to the Design of Complex Organizational and Technical Systems" (*S.V. Chvarkov*, *S.N. Podchufarov*, and *R.M. Kufrik*) was devoted to the problems of increasing the effectiveness of modern weapon systems design. The following essential problems were emphasized in the first part of the paper:

1. the level of initial goal problem formulations (military and operational requirements for the development of complex products and systems), which often diverge from modern realities;

2. the incomplete compliance of the given requirements with the needs of practice, primarily concerning the development of information and control systems (an integral part of modern complexes and weapon systems that largely determines the effectiveness of their combat application).

According to the authors' point of view, the reason is the insufficient level of correctness for the descriptions of particular subject areas (including related ones), due to, on the one hand, the high dynamism of information technology development and, on the other hand, the unsolved problems of passing from a non-formalized (linguistic) description to a formalized (mathematical) representation of control problems solved by complex organizational and technical systems.

The paper provided a detailed analysis of several organizational and technological drawbacks of the current practice of complex systems development, which include:

– definite disproportions in the distribution of funds for the development and modernization of mathematical, software, and hardware support for design processes;

– irrational desire to design expensive or unique, rather than unified, control complexes;

– insufficient attention to the development of prediction models, which serve to identify the evolution of weapon systems and analyze the character of armed struggle considering the asymmetry of using armed forces and nonmilitary means, etc.

Based on the analysis results (including foreign experience), Chvarkov et al. proposed an approach to solving the problems under study, which combines the problems of national security and defense with those of the economic, scientific, technical, industrial, and technological development of the state, considering the available real possibilities and limitations and the level of similar R&D results of probable adversaries.

In the paper “Opportunities for Active Adaptation of the Russian Economy to New Challenges,” *N.I. Komkov*, *V.V. Sutyagin, and N.N. Volodina* considered a set of economic potential development problems as the most important component of the national security. As noted therein, the need to counteract the military-political and sanction pressure of Russia’s geopolitical adversaries as well as attempts to isolate its national economy required rapid adaptation of public administration to the emerging threats. Thanks to this adaptation, contrary to the expectations of ill-wishers, the Russian economy is steadily growing, like the public support level for the country’s top leadership. Nowadays, a set of new problems of ensuring economic growth in the current unfavorable conditions comes to the forefront.

The paper was focused on the design and analysis of an infological model of a full reproduction cycle based on the achievements of scientific and technological progress under the regular change of high technologies formed through innovations. Komkov et al. emphasized that science is the key link of this model: the effective realization of its potential largely determines the basic directions and rates of innovation development of a modern economy.

According to the authors’ retrospective analysis, the low share of high-tech products in the Russian economy since the early 2000s was largely due to the dominance and availability of innovative technologies imported from EU countries and the USA. This factor actually blocked the development of domestic science and reduced the interest of industrial companies in the prospects of their development, which eventually led to a decline in the volume and level of prediction studies on the problems of scientific and technological development. Komkov et al. assessed the possibilities of economic growth in modern conditions and identified key factors directly affecting the processes under consideration. Among them, the most important ones are:

– the state of the innovation sphere and its conjugation with the economy;

– the manageability and coordination of economic development processes and the innovation sphere;

– the ability to adapt the economy to the effective assimilation of progressive innovative solutions and technologies;

– the availability of necessary and sufficient funding, as well as the vigor and purposeful actions of executive authorities to coordinate the interests of economic agents and develop the potential of the innovation sphere and the entire economy. According to the authors, the mechanism of state indicative planning should become a tool for solving innovative development problems. For companies and enterprises, this mechanism forms planned tasks consistent with a stable tax system, regular funding, and the coordination of the Central Bank’s activity with financial structures issuing monetary resources and bonds.

A rather wide group of conference papers were devoted to the results of research into various methodological and applied problems of increasing the effectiveness of national security control processes, namely: “Security Control of Complex Systems in the New Reality” (*G.G. Malinetskii*, *T.S. Akhromeeva*, and *S.A. Toropygina*); “A Complex of Strategic Security Models for the Russia’s Perimeter” (*V.V. Tsyganov*); “Urgent Critical Threats to the Information-Psychological Security of Social Objects” (*E.A. Derbin*); “The Method of Pseudo-Retrospective Manipulation of Consciousness as an Information Warfare Tool” (*A.N. Fomichev*); “An Information Security Model in the Case of Two Disinformation Sources” (*N.G. Kereselidze*); “Synergetic Foundations of System Approach to Complex Systems Security” (*G.G. Malinetskii* and *V.S. Smolin*); “Justification of Hybrid Models for Analyzing the Operating Environment in Descriptive Examples of Assessing the Effectiveness of Complex Systems” (*A.V. Rozhnov*); “Military Security as a Factor of Socioeconomic and Innovative Development of Public Systems” (*O.I. Krivosheev*); “Justification of the Urban Risks Map Project as Applied to the Current Military-Strategic Conditions in the Russian Federation” (*D.E. Fesenko*); “Mathematical Modeling of Economic Security within a Unified Digital Platform of Production Management” (*V.I. Medennikov*); “Means of Laser Space Communication Systems” (*V.V. Leshchenko* and *I.N. Panteleimonov*); “Network Cooperation Prospects in the Innovation System of the Russian Federation in New Conditions” (*N.N. Lanter*); “The Potential of a Long-Term Savings Program as a Tool for Improving the Pension Welfare of Citizens” (*A.E. Abramov*, *A.A. Sorokolad*, and *M.I. Chernova*); “Ecological Sovereignty under the Sustainable Development of Regional Mesosystems” (*R.E. Torgashev*); finally, “On the Sustainability of Investment and Loan Insurance in Microfinance” (*O.B. Bairamov*).

Methodological and applied issues of using scenario and cognitive modeling technologies as an information support tool for preparing and implementing management decisions under uncertainty and risk were the subject of several interesting papers. Among them, note the following: “Directions to Apply the Scenario Approach to the Safety Control of Organizational Systems” (*I.V. Chernov*); “Development of a Forecasting Support System Based on the Integration of Cognitive Analysis, Information Source Monitoring, and Time Series Analysis Methods” (*Z.K. Avdeeva*, *O.A. Volgina*, *E.D. Ermolaev*, and *A.A. Chereshko*); “Study of the Characteristics of Complex Systems Security Control” (*D.A. Kononov*); “Scenario Technologies for Reducing Uncertainty in Security Control” (*V.L. Shultz*, *I.V. Chernov*, and *A.B. Shelkov*); “Analyzing the Sustainability of Territorial Development: Simulation Modeling” (*G.V. Gorelova*); “The Structure, Principles, and Problems of Group Hierarchical Control of Regional Security” (*N.V. Komanich*); “Applicability of Scenario Analysis Methods in the Information Security of the Russian Federation” (*E.D. Ermolaev* and *S.V. Feoktistov*); finally, “The Influence of the Public Control System on the Protest Potential of Society” (*V.R. Feizov*).

Many conference papers dealt with the problems of information security and cybersecurity: their relevance is constantly growing in the era of rapid development of digital technologies.

R.V. Meshcheryakov, *O.O. Evsyutin*, *A.O. Iskhakova*, and *A.V. Dushkin* presented the paper “Ensuring the Information Security of Semistructured Data When Solving Information Protection Problems.” They considered the problems of improving protection mechanisms for data without a fixed format and a clear structure. As noted in the study, the heterogeneity of semistructured data, the complexity of their processing and ensuring their security, and their significant volumes require new methods for assessing information security. (By various estimates, the share of semistructured data in the total volume of corporate information reaches 80–90%.) The authors focused on the analysis of promising trends in the development of information security systems. On the one hand, such systems should provide a sufficient level of infrastructure protection and security of semistructured data considering the specifics of their acquisition, updating, processing, and analysis; on the other, they should meet definite requirements regarding the speed of processing and communication with data sources and end users.

The paper “A Method for Assessing the Information Security Risks of Complex Systems” (*N.F. Volodina*, *A.D. Kozlov*, and *N.L. Noga*) was devoted to the problems of preventing hacker cyberattacks on Russian distributed information systems and resources. The authors emphasized the growing topicality and complexity of ensuring information security due to global digitalization and incessant attacks on Russian information resources by the geopolitical adversaries. At present, the target of hacker attacks is shifting towards destabilizing the socio-political situation in the country and inflicting direct economic damage by organizing personal data leaks and disrupting (and even destroying) critical information infrastructure facilities.

In the paper, a methodology for assessing information security risks based on the mathematical apparatus of fuzzy logic and regression analysis was presented. This methodology allows determining a set of parameters affecting, to the maximum degree, the possible realization of various threats through the identified vulnerabilities in the nodes and other structural components of complex distributed information systems. In practice, the methodology can be applied to predict risk levels under uncertainty and an ambiguous risk dependence on various factors, including subjective ones. This approach increases the effectiveness of measures elaborated and implemented to prevent or reduce damage from malicious attacks on information systems in the most dangerous areas as well as minimizes the cost of measures to protect information resources.

In the paper “Development of an Analyzer Model for Phishing Attacks,” *V.M. Alekseev* and *S.N. Chichkov* considered the problem of increasing the effectiveness of information protection in corporate networks. To solve the problem, they proposed a two-level structure of an information protection system for a corporate fully connected network with different intelligent analyzers to recognize and block computer attacks. At the first (external) level, analyzers control and examine information flows at the corporate network input, detecting and preventing service denial attacks of various types, phishing attacks, attacks on applications, and other attempts to penetrate the network from the outside. At the second level, analyzers monitor activity within the corporate network, viewing traffic between the automated workstations of users and administrators, system servers, and connected mobile devices, as well as network and peripheral equipment, detecting infected devices, preventing the spread of malware, etc.

The authors developed and implemented the analyzers using the methods of statistical analysis, time series, probability theory and statistics, machine learning, optimization of network monitoring parameters and resource allocation, as well as hashing algorithms and graph algorithms to model network interactions. Also, the features of the main methods and approaches to the development of a phishing attack analyzer based on their mathematical interpretation, as well as signature and heuristic analysis technologies, were described in detail.

A large group of conference participants presented solutions of various information security and data protection problems for automated systems: *V.V. Vedischev* and *R.V. Batischev* (“An Optimization Problem Statement for Choosing the Measures and Means of Information Protection for State Information Systems”); *R.E. Asratyan*, *S.S. Vladimirova*, *E.A. Kurako*, and *V.L. Orlov* (“Features of Ensuring Technological Independence in the Development of Systems with Service-Browser Architecture”); *A.D. Domashkin* and *L.N. Loginova* (“A Comparative Analysis of Machine Learning Algorithms for Anomaly Detection in Information Systems”); *M.V. Vedmedeva* and *V.G. Mironova* (“Information Systems Evolution: from Simple Solutions to Complex Infrastructures”); *L.E. Mistrov* (“Foundations of Justifying an Information Security Criterion for Organizational and Technical Systems”); *A.A. Shiroky* (“An Express Risk Assessment Method for a Computer Network with the Star Topology”); *I.A. Andronov* and *V.G. Sidorenko* (“Advantages of Artificial Intelligence Application When Working with Documents in terms of Information Security”); *A.A. Sidorenko* and *Yu.R. Tedeev* (“Increasing the Information Security of Control Channels by Applying Corrective Codes”); *A.Yu. Iskhakov* and *M.V. Mamchenko* (“A User Authentication Algorithm Based on Behavioral Analytics and Machine Learning for Web Resources”); *A.G. Uimin* (“A Continuous-Discrete Biometric Identification System Based on Analysis of the Computer Mouse Data Flow”); *A.A. Salomatin* (“A User Authentication Algorithm Based on Static Characteristics of Computer Hardware”); *A.G. Cheban* and *E.A. Anisimova* (“The Principles of Organization and Design of Secure Videoconference Systems”); *L.N. Loginova* and *A.D. Drozdov* (“Analysis of Information Security Threats When Using Telegram Bots in Business”); *V.P. Kuminov* and *V.G. Sidorenko* (“Analyzing the Cryptographic Resistance of Pseudorandom Number Generators Using Machine Learning”); *D.I. Pravikov* and *V.A. Murashkin* (“Approaches to the Quantitative Assessment of Information Security at an Enterprise of the Fuel and Energy Complex”); *V.O. Sirotyuk* (“Increasing the Security of Digital Intellectual Property Management Systems”); finally, *S.K. Somov* (“Methods for Reducing the Computational Complexity of Optimal Data Array Allocation Algorithms in Distributed Data Processing Systems”).

Also, many conference papers were traditionally devoted to the problems of preventing and eliminating the consequences of natural and man-made emergencies as well as ensuring the safety and reliability of transport systems.

In the first thematic group, note the following papers: “Statistical Observation Forms for Hydrological Situation in Settlements During Floods Caused by Heavy Precipitation” (*V.A. Akimov*, *D.V. Buryak*, and *E.O. Ivanova*); “On Managing the Individual Risk of Death and Health Damage in Emergencies Caused by Catastrophic Floods” (*I.Yu. Oltyan*); “Statistical Observation Forms for Fire-Prevention Situation in Forest Areas” (*V.A. Akimov*, *E.O. Ivanova*, and *M.A. Pulikov*); “Feedback During the Audit of Industrial Safety Control Systems” (*V.A. Tkachenko*); “An Attack on Robotic Systems as a Method of Information-Technical Impact” (*V.A. Zorin*); “Numerical Modeling of a Concentrated Vertical Explosive Impact on a Slab with a Solid Foundation” (*V.K. Musaev*); finally, “Vibroacoustic Diagnosis Methods for Equipment” (*O.B. Skvortsov* and *V.I. Stashenko*).

Several conference participants considered the problems of ensuring the safety of transport systems and objects: *E.A. Kuklev* and *D.M. Mel’nik* (“Intellectual Decision Support in Flight Safety Control of Civil Aircraft Providers Based in Scenario Modeling of Rare Events”); “Ensuring the Safety of Train Traffic under the Coordinate Interval Regulation Method” (*V.G. Novikov*); “A Decision Support System in Technical Re-equipment Problems of the Railway Industry” (*S.V. Makshakov*); “A Distributed Sensor Model with Multi-Fiber Multiplexing to Monitor the Location of Rolling Stock” (*V.M. Alekseev* and *D.N. Khusenov*); “Increasing the Safety of Subway Traffic under Compensated Disturbances” (*L.A. Baranov* and *Yungqiang Zhang*); “Application of Augmented Petri Nets to Model the Automated Scheduling of Subway Trains” (*A.I. Safronov*); “Application of PNETLab to Model Intelligent Water Transport Systems” (*N. D. Ivanova* and *I.F. Mikhalevich*); finally, “The Concept of Developing a Trusted Operation Environment for Autonomous Shipping Objects” (*L.A. Baranov*, *I.F. Mikhalevich*, and *S.S. Sokolov*).

The conference proceedings are published electronically[1] and are also available at the official website: https://iccss2024.ipu.ru/prcdngs.

The 33rd International Conference on Problems of Complex Systems Security Control is planned to be held in November–December 2025 at ICS RAS. The conference schedule will be announced in the information letter of the Organizing Committee, which will be published on the official website (https://iccss2025.ipu.ru/) as well as distributed to po-

[1] *Materialy 32-oi Mezhdunarodnoi konferentsii “Problemy upravleniya bezopasnost’yu slozhnykh sistem”* (Proceedings of the 32nd International Conference on Problems of Complex Systems Security Control), November 13, 2024, Moscow, Kalashnikov, A.O. and Chernov, I.V., Eds., Moscow: Trapeznikov Institute of Control Sciences RAS, 2024. (In Russian.)

tential participants, interested parties, and specialized organizations. Also, please contact the Organizing Committee via phone + 7 495 198-17-20 (ext. 1407) or e-mail iccss@ipu.ru. The Technical Secretary of the conference is *Al'fiya Farissovna Ibragimova*.

Academic Secretary of the Organizing Committee

A.B. Shelkov

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