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STATE ESTIMATION METHODS FOR FUZZY INTEGRAL MODELS. PART I: APPROXIMATION METHODS

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Abstract. The existing and newly proposed methods for estimating the state of integral models with fuzzy uncertainty are reviewed. A fuzzy integral model with the limit transition defined in the Hausdorff metric is introduced. This model is used to formulate the state estimation problem for the models described by fuzzy Fredholm–Volterra integral equations. Several fuzzy methods for solving this problem are considered as follows: the fuzzy Laplace transform, the method of "embedding" models (transforming an original system into a higher dimension system and solving the resulting problem by traditional linear algebra methods), the Taylor estimation of the degenerate kernels under the integral sign that are represented by power polynomials, and the estimation of the nondegenerate kernels by degenerate forms using the Taylor approximation. As shown below, in some cases, the estimation results are related to the solution of fuzzy systems of linear algebraic equations. Test examples are solved for them.

Keywords: fuzzy Riemann integral, fuzzy integral model, fuzzy methods for estimating integral models.

INTRODUCTION

The models described by integral equations, further referred to as the integral models, are widespread in different branches of applied physics, mechanics, economics, and other areas dealing with mathematical descriptions of various objects. In the theory of differential equations, the existence and uniqueness of a solution is proved using the principle of contraction mappings, when an original problem is written as an equivalent integral model [1].

In control theory, integral models often represent control systems with feedback [2]. The integral Wiener–Hopf models are used to describe the perturbations affecting a system, an approach to model uncertainty in the processing of current information from an object [3]. The Fredholm and Volterra integral equations are used in the theory of elasticity, gas dynamics and electrodynamics, and ecology, i.e., in all areas obeying the laws of conservation of mass, momentum, and energy. In all cases mentioned, the unknown variables are under the integral sign.

In real conditions, control systems are subjected to various kinds of perturbations. They are represented by various mathematical models, which are being intensively developed on the theoretical basis and actively used in various applications. Among the most widespread theoretical approaches for these purposes, we mention the theory of intervals [4, 5], the theory of fuzzy sets [6], the theory of possibilities [7], hybrid probability theory, the theory of fuzzy mathematical statistics and fuzzy random processes [8], etc.

This paper describes the uncertainties within the theory of fuzzy sets, which is the most adequate and universal representation for various kinds of perturbations. As is easily demonstrated, the models discussed above follow from the general model of the theory of fuzzy sets. For example, in the paper [9], a fuzzy system of linear equations (FSLE) was solved, and one of the solution's coordinates was obtained in the form of a fuzzy membership function. However, fixing its base, we obtain a solution for this coordinate in the interval form.

Similar reasoning can be adopted to construct solution intervals for fuzzy differential equations. In the theory of possibilities, the membership function is interpreted as a certain probability density that, however, does not satisfy the probability axioms accepted in the traditional statistical theory. Therefore, the theory of possibilities is supposed to describe not mass phenomena but the possibilities of an individual object.

Hybrid probability theory represents the traditional probability space for random variables with traditional probability densities with initial or central moments in the form of fuzzy variables with given membership functions (usually triangles). Concerning traditional stochastic processes, the hybrid theory for fuzzy Markov stochastic processes operates with fuzzy states obtained by enlarging crisp states. This approach reduces the dimension of the transition matrix and, consequently, the corresponding computational difficulties during its inversion.

Generally speaking, the modern theory of fuzzy sets is a kind of core that groups various models of uncertainties.

Based on the foregoing, this paper aims to present various methods, both the existing and newly proposed ones, for estimating integral models under fuzzy uncertainty.

The scientific novelty of this paper consists in new state estimation methods developed by the authors for integral models, such as the method of degenerate kernels, the fuzzy least squares method, and the fuzzy Galerkin method. As shown below, "strong/weak" estimation results can occur when the estimation procedure yields fuzzy systems of linear algebraic equations. The authors first investigated this effect when solving the FSLE and then applied to estimate integral models.

Below, fuzzy integral models in the form of the Fredholm–Volterra equations are introduced, and some methods to solve them are considered.

1. BASIC DEFINITIONS

The basic definitions of the theory of fuzzy sets were given in [6]. Let us introduce the definitions used in this paper. The notations are the following: fuzzy variables (numbers) have the subscript "fuz," e.g., x_{fuz} is a fuzzy variable (element), $y_{\text{fuz}}(\boldsymbol{x})$ is a fuzzy function of many variables, where $\boldsymbol{x} = (x_1, x_2,\ldots, x_n)^{\text{T}}$, $y'_{\text{fuz}\,x_i}(\boldsymbol{x})$ is the fuzzy derivative with respect to the variable x_i, and $\dot{\boldsymbol{x}}_{\text{fuz}}(t)$ is the fuzzy time derivative of a vector $\boldsymbol{x}_{\text{fuz}}$.

The belonging of an element x to some set X ($x \in X$) is formalized by *a membership function* $r(x)$, $r \in [0,1]$, $x = x_{\text{fuz}} \in X$ for a fuzzy element x_{fuz}:

$$r(x) = \begin{cases} \underline{r}(x) \in [0, 1], \\ \overline{r}(x) \in [0, 1], \end{cases}$$

where $r(x)$ is a multivalued function with left $\underline{r}(x)$ and right $\overline{r}(x)$ branches with respect to $r(x) = 1$.

The function $r(x)$ is often written in the level representation – the inverse mapping $r^{-1}(x) = x(r) = (\underline{x}(r),\ \overline{x}(r) \mid r \in [0, 1])$. A collective $\{x_{\text{fuz}}\}$ defines a fuzzy set X_{fuz}. For x_{fuz}, the chain of equivalent representations is sometimes used: $x_{\text{fuz}} \Leftrightarrow r(x)$, $r \in [0, 1]$ $\Leftrightarrow (\underline{r}(x),\ \overline{r}(x) \mid \underline{r},\ \overline{r} \in [0, 1] \Leftrightarrow (\underline{x}(r),\ \overline{x}(r) \mid r \in [0, 1])$, etc.

Fuzzy function (*mapping*) $y_{\text{fuz}}(x)$ of fuzzy variables. Let E be the set of all fuzzy variables with a given membership function $r(x)$, $r \in [0, 1]$, $x \in R$. Then $y_{\text{fuz}}(x): R \to E$ defines a fuzzy-valued function, and the following parametric representation holds:

$$y_{\text{fuz}}(x) = y(x, r) = (\underline{y}(x, r),\ \overline{y}(x, r) \mid r \in [0, 1]).$$

The Banach space of fuzzy variables is introduced using the conventional approach of functional analysis [10]. A collection $\{x_{\text{fuz}}\}$ with the addition and multiplication operations and the existence of an inverse element forms a vector (linear) space E. In the space E, the following metric and norm are defined:

$$d(x_{\text{fuz}\,i}, x_{\text{fuz}\,j}) = \sup_{r\in[0,1]} \times$$

$$\times\{\max[|\underline{x}_i(r) - \overline{x}_j(r)|, |\overline{x}_i(r) - \underline{x}_j(r)|]\},$$

$$\|x_{\text{fuz}\,i} - x_{\text{fuz}\,j}\| = d(x_{\text{fuz}\,i}, x_{\text{fuz}\,j}).$$

A fuzzy Cauchy sequence is a sequence of the form

$$\{x_{\text{fuz}\,n}\}: \{d(x_{\text{fuz}\,n}, x_{\text{fuz}\,m}) \to 0\}_{n,m\to\infty},$$

and the space E is complete if

$$x_{\text{fuz}\,n} \underset{n\to\infty}{\to} x_{\text{fuz}},\ x_{\text{f}} \in E.$$

These definitions lead to the Banach space of fuzzy variables (E, d). The pair (E, d) forms a complete metric space.

Fuzzy continuity at a point is defined using the local limit at this point, which is treated in the Hausdorff metric. Fuzzy continuity on an interval is defined as fuzzy continuity for all values of the interval.

According to the general approach, *the fuzzy derivative* of a function with respect to its crisp variable is found by defining the following operations for some fuzzy function described above: subtraction or the existence of an opposite element, multiplication by a constant, passage to the limit in a given metric. This paper uses two types of fuzzy derivatives: the Seikkala derivative $y'^{S}_{\text{fuz}}(x)$ and the Buckley–Feuring derivative $y'^{BF}_{\text{fuz}}(x)$. The following statement holds: if the fuzzy derivatives exist and are continuous at a point $x = x_*$, then they are equal to each other at this point.

A fuzzy integral is understood in the fuzzy Riemann sense [11].

Consider a fuzzy mapping $f_{\text{fuz}}:[a,b]\subset R\to E$, where E is a fuzzy set. If for each partition $P=\{t_0,\dots,t_n\}\in[a,b]$ and $\forall\xi_i\in[t_{i-1},t_i]$, $i=\overline{1,n}$, there exists the representation $R_p=\sum_{i=1}^{n} f_{\text{fuz}}(\xi_i)(t_i-t_{i-1})$ and $\Delta=\max\left\{\left|t_i-t_{i-1}\right|, i=\overline{1,n}\right\}$, then the fuzzy Riemann integral of $f_{\text{fuz}}(t)$ is given by

$$\int_a^b f_{\text{fuz}}(t)dt=\lim_{\Delta\to 0} R_p, \tag{1}$$

where the limit is defined in the Hausdorff metric $d(u,v)$: for $u, v\in E\Rightarrow d(u,v)=\sup\{\max[|\underline{u}(r)-\underline{v}(r)|, |\overline{u}(r)-\overline{v}(r)|]\}$, where $r\in[0,1]\subset R$, and $\underline{u},\overline{u},\underline{v},\overline{v}$ are the nonparametric representations of the fuzzy variables u, v.

Under (1), a function $f_{\text{fuz}}(t)=f(t,r)=(\underline{f}(t,r), \overline{f}(t,r)\,|\, r\in[0,1])$ continuous in the Hausdorff satisfies the relations

$$\underline{\int_a^b f(t,r)dt}=\int_a^b \underline{f}(t,r)dt \text{ and } \overline{\int_a^b f(t,r)dt}=\int_a^b \overline{f}(t,r)dt,$$

$r\in[0,1]\subset R$, where underline and overline indicate the lower and upper value objects, respectively.

If a fuzzy variable $z_{\text{fuz}}(t)$, $t\in[a, b]\subset R$, is under the fuzzy integral sign, it satisfies the fuzzy integral equation

$$z_{\text{fuz}}(t)+\int_a^b K(t,\tau)z_{\text{fuz}}(\tau)d\tau=f(t).$$

By analogy with the traditional classification, there are *fuzzy integral models* described by the Fredholm–Volterra equations of the first and second kinds:

$\int_{t_1}^{t_2} K(t,\tau)z_{\text{fuz}}(\tau)d\tau=u_{\text{fuz}}(t)$ is a fuzzy integral model described by the Fredholm equation of the first kind, where $t\in[t_1,t_2]\subset R$, and $K(t, \tau)$ is a crisp or fuzzy kernel;

$z_{\text{fuz}}(t)-\lambda\int_{t_1}^{t_2} K(t,\tau)z_{\text{fuz}}(\tau)d\tau=u_{\text{fuz}}(t)$ is a fuzzy integral model described by the Fredholm equation of the second kind, where $\lambda\in K$ is a parameter.

The limits of integration can be finite or infinite. The variables satisfy the inequality $t_1\le t,\tau\le t_2$, whereas the kernel $K(t, \tau)$ and the free term $u_{\text{fuz}}(t)$ must be continuous or satisfy the Fredholm conditions.

In the general case, the fuzzy Fredholm equations of the first and second kinds imply the fuzzy Volterra equations of the first and second kinds. The Volterra equations differ from the Fredholm equations by a variable limit of integration:

$\int_{t_1}^{t} K(t,\tau)z_{\text{fuz}}(\tau)d\tau=u_{\text{fuz}}(t)$, $t_1\le t\le t_2$, is a fuzzy integral model described by the Volterra equation of the first kind, where $K(t, \tau)$ is a crisp or fuzzy kernel;

$z_{\text{fuz}}(t)-\lambda\int_{t_1}^{t} K(t,\tau)z_{\text{fuz}}(\tau)d\tau=u_{\text{fuz}}(t)$ is a fuzzy integral model described by the Volterra equation of the second kind.

The Volterra integral equation can be considered a special case of the Fredholm equation with a properly completed kernel. The Volterra equations have several important properties that are not inherent in the Fredholm equations and cannot be derived from them. In view of this aspect, we will use only the general properties of the Fredholm and Volterra equations below.

Sufficient conditions for the existence of a unique solution of fuzzy Fredholm–Volterra integral equations of the second kind were given in [12–14]. For the sake of definiteness, consider a fuzzy Volterra equation of the second kind. For the existence of a fuzzy solution, the method of successive fuzzy approximations is used under the assumption that fuzzy approximations are defined in the rectangle $\Pi=[\tau, t]$, on which they have fuzzy continuity and a bounded Seikkala derivative. Then the sequence of fuzzy approximations converges in the Hausdorff metric to a fuzzy solution. Moreover, due to the boundedness of the derivative, convergence in t follows: the sequence of fuzzy approximations converges uniformly to the desired fuzzy variable, which is taken as a fuzzy solution of the original fuzzy integral equation. The uniqueness of a fuzzy solution is proved by contradiction.

The fuzzy Fredholm–Volterra equations of the first and second kinds (see above) can be represented in a short (operator) form [15]:

$$\lambda(Kz_{\text{fuz}})(t)z_{\text{fuz}}(t)=u_{\text{fuz}}(t) \text{ and}$$
$$[I-\lambda(Kz_{\text{fuz}})(t)]z_{\text{fuz}}(t)=u_{\text{fuz}}(t), \tag{2a}$$

where

$$(Kz_{\text{fuz}})(t)=\int_{t_1}^{t_2} K(t,\tau)z_{\text{fuz}}(\tau)d\tau \tag{2b}$$

is the operator for the fuzzy Fredholm equations,

$$(Kz_{\text{fuz}})(t) = \int_{t_1}^{t} K(t,\tau) z_{\text{fuz}}(\tau) d\tau \qquad (2c)$$

is the operator for the fuzzy Volterra equations of the first and second kinds, and I is an identity operator.

2. PROBLEM STATEMENT

There is a fuzzy model described by the integral equation (2a), (2b), or (2c). It is required to consider different fuzzy estimation methods for its state.

3. FUZZY ESTIMATION METHODS

3.1. Estimation by fuzzy Laplace transform

The definition of the fuzzy Laplace transform and its properties were described in detail in the papers [16, 17]. Also, some examples of applying this transformation to find solutions of various fuzzy Volterra integral equations of the convolution type with crisp and fuzzy kernels were considered therein. The problem was generalized to the case of a fuzzy partial differential component in a fuzzy integral equation. As a result, the fuzzy Laplace transform method was extended to the case of fuzzy linear second-order partial differential equations of the parabolic and hyperbolic types.

3.2. Estimation by embedding

Consider a fuzzy model described by the Fredholm integral equation of the second kind:

$$x_{\text{fuz}}(s) = f_{\text{fuz}}(s) + \int_a^b K(s,\tau) x_{\text{fuz}}(\tau) d\tau. \qquad (3)$$

The existence of a unique solution, the conditions imposed on the functions f_{fuz} and $K(s,\tau)$, the definition of a solution for the equation with fuzzy parameters, the space of functions to find a solution, and the conditions under which equation (3) exists were thoroughly considered in [13, 14].

The exact fuzzy solution of equation (4) is constructed in the form of the infinite series [13]

$$x_{\text{fuz}}(s) = \sum_{i=1}^{\infty} a_{\text{fuz}\,i} h_i(s), \qquad (4)$$

where $\{ h_i(\cdot) \}$ is a sequence of functions in the space $L_2(a,b)$, and $a_{\text{fuz}\,i}$ are fuzzy coefficients.

An approximate solution of equation (4) can be represented as the finite series

$$x_{\text{fuz}}(s) \simeq \tilde{x}_{\text{fuz}\,n}(s) = \sum_{i=1}^{n} \tilde{a}_{\text{fuz}\,i} h_i(s),$$

where $\tilde{a}_{\text{fuz}\,i}$ are the fuzzy coefficients for estimation, and $h_i(s)$ are known functions. To find them, we substitute the expression for $\tilde{x}_{\text{fuz}\,n}(s)$ into equation (3) instead of x_{fuz}. Proceeding in this way, we obtain an equation of the form (3), and the solution is

$$\sum_{i=1}^{n} \tilde{a}_{\text{fuz}\,i} h_i(s) = f_{\text{fuz}}(s) + \sum_{i=1}^{n} \tilde{a}_{\text{fuz}\,i} \int_a^b K(s_j,\tau) h_i(\tau) d\tau. \qquad (5)$$

Equation (5) contains n unknown fuzzy variables $\tilde{a}_{\text{fuz}\,1},\ldots,\tilde{a}_{\text{fuz}\,n}$. To calculate them, we need n equations and therefore use n points $s_1,\ldots,s_n \in [a,b]$. The resulting fuzzy system of linear equations for the coefficients $\tilde{a}_{\text{fuz}\,i}$ is

$$\sum_{i=1}^{n} h_i(s_j) \tilde{a}_{\text{fuz}\,i} = f_{\text{fuz}\,i}(s_j) + \sum_{i=1}^{n} \left(\int_a^b K(s_j, \tau) h_i(\tau) d\tau \right) \tilde{a}_{\text{fuz}\,i},\ j = \overline{1,n}.$$

In the matrix form, it can be written as

$$A\tilde{a}_{\text{fuz}} = f_{\text{fuz}} + B\tilde{a}_{\text{fuz}}, \qquad (6)$$

where $A = (a_{ij})$ and $B = (b_{ij})$ are matrices with the crisp elements $a_{ij} = h_i(s_j)$ and $b_{ij} = \int_a^b K(s_j,\tau) h_i(\tau) d\tau$, $i, j = \overline{1,n}$; $\tilde{a}_{\text{fuz}} = (\tilde{a}_{\text{fuz}\,1},\ldots,\tilde{a}_{\text{fuz}\,n})^T$ and $f_{\text{fuz}} = (f_{\text{fuz}}(s_1),\ldots,f_{\text{fuz}}(s_n))^{\text{T}}$ are vectors with fuzzy components.

The matrix equation (6) reduces to the standard form

$$\tilde{A}a_{\text{fuz}} = f_{\text{fuz}},\ \tilde{A} = A - B, \qquad (7)$$

and the resulting system is solved by the method of embedding [9, 18].

According to this method, equation (7) is transformed to the extended (embedded) system:

$$S_{(2n\times 2n)} \cdot X_{\text{fuz}\,(2n\times 2n)} = Y_{\text{fuz}\,(2n\times 2n)},$$

where $X_{\text{fuz}} = (\underline{\tilde{a}}_1,\ldots,\underline{\tilde{a}}_n \,|\, \bar{\tilde{a}}_1,\ldots,\bar{\tilde{a}}_n)^{\text{T}}$ and $Y_{\text{fuz}} = (\underline{f}_1,\ldots,\underline{f}_n \,|\, \bar{f}_1,\ldots,\bar{f}_n)^{\text{T}}$.

The matrix S has a block structure: $S = \begin{pmatrix} S_1 & S_2 \\ S_2 & S_1 \end{pmatrix}$. The matrix S_1 is obtained from the matrix ($A - B$) by zeroizing all negative elements. To construct the matrix S_2, we should replace all negative elements in the matrix ($A - B$) by their absolute values and all other elements by zeros:

$$s_{ij} = a_{ij} - b_{ij},\ s_{i+n,j+n} = a_{ij} - b_{ij},\ a_{ij} - b_{ij} > 0;$$

$$s_{i,j+n} = -\left(a_{ij} - b_{ij}\right),\ s_{i+n,j} = -\left(a_{ij} - b_{ij}\right),$$
$$a_{ij} - b_{ij} < 0 .$$

If $|S| \neq 0$ (S is nonsingular), then

$X_{\text{fuz}} = S^{-1}Y_{\text{fuz}}$,

where $S^{-1} = \begin{pmatrix} U & V \\ V & U \end{pmatrix}$, $U = 0.5[(S_1 + S_2)^{-1} + (S_1 - S_2)^{-1}]$, and $V = 0.5[(S_1 + S_2)^{-1} - (S_1 - S_2)^{-1}]$.

The case of a singular matrix S was considered in detail in the papers [19, 20].

The accuracy of the approximate solution can be estimated as follows.

The residual r_n and error ε_n vectors are determined via the Hausdorff metric:

$$r_n = D(f_{\text{fuz}}, L\tilde{x}_{\text{fuz}}) = (d(f_{\text{fuz}}, L\tilde{x}_{\text{fuz}\,1}), ..., d(f_{\text{fuz}}, L\tilde{x}_{\text{fuz}\,n}))^{\text{T}},$$
$$\varepsilon_n = D(\tilde{x}_{\text{fuz}}, x_{\text{fuz}}) = (d(\tilde{x}_{\text{fuz}}, x_{\text{fuz}\,1}), ..., d(\tilde{x}_{\text{fuz}}, x_{\text{fuz}\,n}))^{\text{T}},$$

where

$L = I - K$ and $K = (Kx_{\text{fuz}})(s) = \int_a^b K(s, \tau)x_{\text{fuz}}(\tau)d\tau.$

The following upper bound on the approximate solution accuracy was derived in [21]:

$$\|\varepsilon_n\| \leq \|r_n\| \cdot [1 - \|K\|]^{-1} \text{ for } \|K\| < 1.$$

Example 1. Consider an integral equation of the form

$$x_{\text{fuz}}(s) = f_{\text{fuz}}(s) + \int_{-1}^{1} (s+1)x_{\text{fuz}}(\tau)d\tau,$$

where $a = -1$, $b = 1$, and

$$f_{\text{fuz}}(s) = f(s, r) = (\underline{f}(s, r) = s^3(r^2 + r),\ \overline{f}(s, r) = s^3(4 - r^3 - r) | r \in [0, 1]),\ -1 \leq s,\ \tau \leq 1.$$

It is required to estimate the state by the method of embedding.

Solution. We choose $h_1(s) = 1$ and $h_2(s) = s^3$, assuming that $s_1 = -1$ and $s_2 = 1$. Then the elements of equation (5) take the following form:

$$\underline{f}(s_1) = -(r^2 + r),\ \underline{f}(s_2) = (r^2 + r),\ \overline{f}(s_1) = -(4 - r^3 - r),\ \overline{f}(s_2) = (4 - r^3 - r);$$

$$A = \begin{pmatrix} h_1(s_1) & h_2(s_1) \\ h_1(s_2) & h_2(s_2) \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix};$$

$$B = \begin{pmatrix} b_{11} = \int_{-1}^{1} (-1+1)1d\tau & b_{12} = \int_{-1}^{1} (-1+1)(-1)^3 d\tau \\ b_{21} = \int_{-1}^{1} (1+1)1d\tau & b_{22} = \int_{-1}^{1} (1-1)1^3 d\tau \end{pmatrix},$$

$$B = \begin{bmatrix} 0 & 0 \\ 4 & 0 \end{bmatrix} \Rightarrow \tilde{A} = A - B = \begin{bmatrix} 1 & -1 \\ -3 & 1 \end{bmatrix} \Rightarrow$$

$$\Rightarrow S = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 3 & 0 & 0 & 0 \end{bmatrix} \Rightarrow$$

$$\Rightarrow (S_1 + S_2)^{-1} = \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix}^{-1} = 0.5\begin{bmatrix} 1 & -1 \\ -3 & 1 \end{bmatrix},$$

$$(S_1 - S_2)^{-1} = \begin{bmatrix} 1 & -1 \\ -3 & 1 \end{bmatrix}^{-1} = 0.5\begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix} \Rightarrow$$

$$\Rightarrow \begin{bmatrix} \underline{a}_1 \\ \underline{a}_2 \\ -\overline{a}_1 \\ -\overline{a}_2 \end{bmatrix} = \begin{bmatrix} -0.5 & 0 & 0 & 0.5 \\ 0 & -0.5 & 1.5 & 0 \\ 0 & 0.5 & -0.5 & 0 \\ 1.5 & 0 & 0 & -0.5 \end{bmatrix} \begin{bmatrix} -(r^2 + r) \\ (r^2 + r) \\ -(4 - r^3 - r) \\ (4 - r^3 - r) \end{bmatrix};$$

$$V = 0.5[(S_1 + S_2)^{-1} - (S_1 - S_2)^{-1}] = \begin{bmatrix} 0 & 0.5 \\ 1.5 & 0 \end{bmatrix}.$$

Therefore,

$$\begin{bmatrix} \underline{a}_1 \\ \underline{a}_2 \\ -\overline{a}_1 \\ -\overline{a}_2 \end{bmatrix} = \begin{bmatrix} -0.5 & 0 & 0 & 0.5 \\ 0 & -0.5 & 1.5 & 0 \\ 0 & 0.5 & -0.5 & 0 \\ 1.5 & 0 & 0 & -0.5 \end{bmatrix} \begin{bmatrix} -(r^2 + r) \\ (r^2 + r) \\ -(4 - r^3 - r) \\ (4 - r^3 - r) \end{bmatrix},$$

$$\begin{cases} a_{\text{fuz}\,1} = (\underline{a}_1(r), \overline{a}_1(r)) = (0.25r^3 + 0.25r^2 + 0.5r - 1, \\ -0.25r^3 - 0.25r^2 - 0.5r + 1 | r \in [0,1]), \\ a_{\text{fuz}\,2} = (\underline{a}_2(r), \overline{a}_2(r)) = (0.25r^3 + 0.75r^2 + 0.5r + 1, \\ -0.75r^3 + 0.25r^2 - 0.5r + 3 | r \in [0,1]). \end{cases} \tag{8}$$

The approximate estimate of the state is

$$x_{\text{fuz}}(s) \simeq x_{\text{fuz}\,n=2}(s) = a_{\text{fuz}\,1}h_1(s) + a_{\text{fuz}\,2}h_2(s)\Big|_{\substack{h_1(\cdot)=1 \\ h_2(\cdot)=s^3}} = a_{\text{fuz}\,1} + a_{\text{fuz}\,2}s^3,$$

where $a_{\text{fuz}\,1}$ and $a_{\text{fuz}\,2}$ are given by (8).

This estimate can be strong or weak; see the method proposed by the authors in the paper [19, 20]. ♦

3.3. Taylor estimation

In the general form, this method is often considered for a fuzzy system of integral equations [21]. For the sake of simplicity, we will implement a particular case of this system described by a single Fredholm equation of the second kind:

$$x_{\text{fuz}}(s) = f_{\text{fuz}}(s) + \int_a^b K(s, \tau)x_{\text{fuz}}(\tau)d\tau, \tag{9}$$

where $a \leq s, \tau \leq b$; $K(s, \tau)$ is a given crisp kernel differentiable by both variables on the interval $[a, b] \subset R$; $x_{\text{fuz}}(s)$ is a fuzzy unknown found from equation (9).

Let $f_{\text{fuz}}(s)$ and $x_{\text{fuz}}(s)$ have the parametric representations

$$f_{\text{fuz}}(s) = f(s,r) = (\underline{f}(s,r), \overline{f}(s,r) | r \in [0,1]),$$

$$x_{\text{fuz}}(s) = x(s,r) = (\underline{x}(s,r), \overline{x}(s,r) | r \in [0,1]).$$

Then equation (9) can be written in the parametric form

$$\begin{cases} \underline{x}(s, r) = \underline{f}(s, r) + \int_a^b \underline{U}(\tau, r) d\tau, \\ \overline{x}(s, r) = \overline{f}(s, r) + \int_a^b \overline{U}(\tau, r) d\tau, \end{cases} \quad (10)$$

$r \in [0,1]$,

where

$$\underline{U}(\tau, r) = \begin{cases} K(s, \tau)\underline{x}(\tau, r), \ K(s,\tau) \ge 0, \\ K(s, \tau)\overline{x}(\tau, r), \ K(s,\tau) < 0; \end{cases}$$

$$\overline{U}(\tau, r) = \begin{cases} K(s, \tau)\overline{x}(\tau, r), \ K(s,\tau) \ge 0, \\ K(s, \tau)\underline{x}(\tau, r), \ K(s,\tau) < 0. \end{cases}$$

Assume that the following inequalities hold on the interval $[a, b] \subset R$:

$$\begin{cases} K(s, \tau) \ge 0, \ a \le \tau \le c, \\ K(s, \tau) < 0, \ c < \tau \le b. \end{cases}$$

Then the system of equations (10) can be reduced to

$$\begin{cases} \underline{x}(s, r) = \underline{f}(s, r) + \int_a^c K(s, \tau)\underline{x} \times \\ \times(\tau, r) d\tau + \int_c^b K(s, \tau)\overline{x}(\tau, r) d\tau, \\ \overline{x}(s, r) = \overline{f}(s, r) + \int_a^c K(s, \tau)\overline{x} \times \\ \times(\tau, r) d\tau + \int_c^b K(s, \tau)\underline{x}(\tau, r) d\tau. \end{cases} \quad (11)$$

Now we expand the integrand functions $\underline{x}(\tau, r)$, $\overline{x}(\tau, r)$ in (11) into the Taylor polynomials of degree n. For a fixed point $\tau = z$, we obtain

$$\begin{cases} \underline{x}(s, r) = \underline{f}(s, r) + \int_a^c K(s,\tau) \sum_{i=0}^{N} \frac{1}{i!} \underline{x}_\tau^{(i)}(\tau, r)(\tau - z)^i d\tau + \\ \quad + \int_c^b K(s, \tau) \sum_{i=0}^{N} \frac{1}{i!} \overline{x}_\tau^{(i)}(\tau, r)(\tau - z)^i d\tau. \\ \overline{x}(s,r) = \overline{f}(s,r) + \int_a^c K(s, \tau) \sum_{i=0}^{N} \frac{1}{i!} \overline{x}_\tau^{(i)}(\tau, r)(\tau - z)^i d\tau + \\ \quad + \int_c^b K(s, \tau) \sum_{i=0}^{N} \frac{1}{i!} \underline{x}_\tau^{(i)}(\tau, r)(\tau - z)^i d\tau, \end{cases} \quad (12)$$

$$\underline{x}_\tau^{(i)}(\tau,r) = \left.\frac{\partial^i \underline{x}(\tau,r)}{\partial \tau^i}\right|_{\tau=z} \text{ and } \overline{x}_\tau^{(i)}(\tau,r) = \left.\frac{\partial^i \overline{x}(\tau,r)}{\partial \tau^i}\right|_{\tau=z}.$$

Differentiating both of equations (12) $p = \overline{0, n}$ times with respect to the variable s yields:

$$\begin{aligned} &\underline{x}_s^{(p)}(s,r) = \underline{f}_s^{(p)}(s,r) + \sum_{i=0}^{N} \frac{1}{i!} \int_a^c K_s^{(p)}(s,\tau) \underline{x}_\tau^{(i)}(\tau,r) \times \\ &\times(\tau - z)^i d\tau + \sum_{i=0}^{N} \frac{1}{i!} \int_c^b K_s^{(p)}(s,\tau) \overline{x}_\tau^{(i)}(\tau,r)(\tau - z)^i d\tau, \\ &\overline{x}_s^{(p)}(s,r) = \overline{f}_s^{(p)}(s,r) +) \sum_{i=0}^{N} \frac{1}{i!} \int_a^c K_s^{(p)}(s, \tau \overline{x}_\tau^{(i)}(\tau,r) \times \\ &\times(\tau - z)^i d\tau + \sum_{i=0}^{N} \frac{1}{i!} \int_c^b K_s^{(p)}(s,\tau) \underline{x}_\tau^{(i)}(\tau,r)(\tau - z)^i d\tau, \end{aligned} \quad (13)$$

where $\underline{x}_s^{(p)}(s,r) = (\frac{\partial^p \underline{x}(s,r)}{\partial s^p})$, $\overline{x}_s^{(p)}(s,r) = (\frac{\partial^p \overline{x}(s,r)}{\partial s^p})$, and $K_s^{(p)}(s, \tau) = \left(\frac{\partial^p K(s, \tau)}{\partial s^p}\right)$, $p = \overline{0, n}$.

Interchanging the integral and sum signs, we write the system of equations (13) as

$$\begin{aligned} &\underline{x}_s^{(p)}(s,r) = \underline{f}_s^{(p)}(s,r) + \sum_{i=0}^{N} \frac{1}{i!} \int_a^c K_s^{(p)}(s,\tau) \underline{x}_\tau^{(i)}(\tau, r) \times \\ &\times(\tau - z)^i d\tau + \sum_{i=0}^{N} \frac{1}{i!} \int_c^b K_s^{(p)}(s,\tau) \overline{x}_\tau^{(i)}(\tau, r)(\tau - z)^i d\tau, \\ &\overline{x}_s^{(p)}(s,r) = \overline{f}_s^{(p)}(s,r) +) \sum_{i=0}^{N} \frac{1}{i!} \int_a^c K_s^{(p)}(s, \tau) \overline{x}_\tau^{(i)}(\tau, r) \times \\ &\times(\tau - z)^i d\tau + \sum_{i=0}^{N} \frac{1}{i!} \int_c^b K_s^{(p)}(s,\tau) \underline{x}_\tau^{(i)}(\tau, r)(\tau - z)^i d\tau. \end{aligned} \quad (14)$$

Denoting

$$S_{pi}^1 = \frac{1}{i!} \int_a^c K_s^{(p)}(s, \tau)(\tau - z)^i d\tau,$$

$$S_{pi}^2 = \frac{1}{i!} \int_c^b K_s^{(p)}(s, \tau)(\tau - z)^i d\tau,$$

we reduce equations (14) to

$$\begin{aligned} &\underline{x}_s^{(p)}(s, r) = \underline{f}_s^{(p)}(s, r) + \sum_{i=0}^{N} S_{pi}^1(s, \tau) \underline{x}_\tau^{(i)} \times \\ &\quad \times(\tau, r) + \sum_{i=0}^{N} S_{pi}^2 \overline{x}_\tau^{(i)}(\tau, r), \\ &\overline{x}_s^{(p)}(s, r) = \overline{f}_s^{(p)}(s, r) + \sum_{i=0}^{N} S_{pi}^1(s, \tau) \overline{x}_\tau^{(i)} \times \\ &\quad \times(\tau, r) + \sum_{i=0}^{N} S_{pi}^2 \underline{x}_\tau^{(i)}(\tau, r), \end{aligned} \quad p = \overline{0, n}. \quad (15)$$

Introducing the vectors

$$\underline{X}(s, r) = (\underline{x}_s^{(0)}, ..., \underline{x}_s^{(n)})^{\mathrm{T}}, \ \overline{X}(s, r) = (\overline{x}_s^{(0)}, ..., \overline{x}_s^{(n)})^{\mathrm{T}},$$

$\underline{F}(s, r) = (\underline{f}_s^{(0)}, ..., \underline{f}_s^{(n)})^{\mathrm{T}}, \bar{F}(s, r) = (\bar{f}_s^{(0)}, ..., \bar{f}_s^{(n)})^{\mathrm{T}}$

and the matrices

$$S^1 = \left(S_{pi}^1\right),\ S^2 = \left(S_{pi}^2\right),\ \left(p, i\right) = \overline{0, n},$$

for $s, z = a^* \in [a, b]$ we finally write (15) in the matrix form

$$X = F + \left(S^1 + S^2\right) X,\ X = (\underline{X}, \bar{X})^{\mathrm{T}},\ F = (\underline{F}, \bar{F})^{\mathrm{T}}. \quad (16)$$

The solution is

$$S = \begin{pmatrix} S^1 & S^2 \\ S^2 & S^1 \end{pmatrix} \Rightarrow (S - I)X = -F \Rightarrow X^* = $$
$$= -(S - I)^{-1} F, \left|S - I\right| \neq 0.$$

The convergence of the solution X^* to the exact solution $\tilde{X}\left(X^* \underset{n \to \infty}{\to} \tilde{X}\right)$ was proved in [22].

Example 2. Consider an integral equation of the form

$$x_{\text{fuz}}(s) = f_{\text{fuz}}(s) + \int_0^2 s^2 (1 + \tau) x_{\text{fuz}}(\tau) d\tau, \quad (17)$$

where

$$f_{\text{fuz}}(s) = f(s, r) = \left(\underline{f}(s, r) = sr, \bar{f}(s, r) = \left(\frac{14}{3}\right) s^2 (r - 2) \right)$$

and $r \in [0, 1] \subset R_1$.

It is required to determine $x_{\text{fuz}}(s) = x(s, r) = \left(\underline{x}(s, r), \bar{x}(s, r)\right)$.

To find the solution, we write equation (17) in the parametric form:

$$\begin{aligned} \underline{x}(s, r) &= sr + \int_0^2 s^2 (1 + \tau) \underline{x}(\tau, r) d\tau, \\ \bar{x}(s, r) &= \left(\frac{14}{3}\right) s^2 (r - 2) \int_0^2 s^2 (1 + \tau) \bar{x}(\tau, r) d\tau. \end{aligned} \quad (18)$$

In these expressions, the kernel is $K(s, \tau) = s^2 (1 + \tau) \geq 0\ \forall \tau \in [0, 2]$, where the interval [0,2] defines the limits of integration in equation (18). Therefore, this interval does not contain the partition point c present in the system of equations (12).

We expand the unknown integrand functions $\underline{x}(\tau, r), \bar{x}(\tau, r)$ in (17) into the Taylor polynomials of degree n, letting $n = 1$ for simplicity. For $\tau = z \in [0, 2]$, we obtain:

$$\underline{x}(\tau, r) = 1 + \left(\frac{\partial \underline{x}}{\partial \tau}\right)\Bigg|_{\tau = z} (\tau - z),$$

$$\bar{x}(\tau, r) = 1 + \left(\frac{\partial \bar{x}}{\partial \tau}\right)\Bigg|_{\tau = z} (\tau - z),\ 0 \leq \tau,\ z \leq 2,\ r \in [0, 1].$$

Since $n = 1$, each of equations (18) should be differentiated with respect to s, $p = 0$ and $p = 1$ times:

$$p = 0 \Rightarrow \frac{\partial^0}{\partial s^0} \Rightarrow$$

$$\Rightarrow \begin{cases} \underline{x}(s, r) = sr + \int_0^2 s^2 (1 + \tau) \underline{x}(\tau, r) dr, \\ \bar{x}(s, r) = \dfrac{14}{3} s^2 (r - 2) + \int_0^2 s^2 (1 + \tau) \bar{x}(\tau, r) d\tau, \end{cases}$$

$$p = 1 \Rightarrow \frac{\partial^1}{\partial s^1} \Rightarrow$$

$$\Rightarrow \begin{cases} \dot{\underline{x}}_s(s, r) = r + \int_0^2 2s(1 + \tau) \underline{x}(\tau, r) dr, \\ \dot{\bar{x}}_s(s, r) = \dfrac{28}{3} s(r - 2) + \int_0^2 s^2 (1 + \tau) \bar{x}(\tau, r) d\tau. \end{cases}$$

Next, we consider the vectors X and F and the elements of the matrix S:

$$X = (\underline{x}\left| s = a^*, r), \ \dot{\underline{x}}_s (s = a^*, r); \right.$$
$$(\bar{x}\left| s = a^*, r), \ \dot{\bar{x}}_s (s = a^*, r)^{\mathrm{T}} \right.$$

is the vector of fuzzy variables to be determined; $a^* \in [a, b]$;

$$F = \left(\underline{f}(s = a^*, r) = a^* r,\ \dot{\underline{f}}_s = 1 \cdot r;\ \bar{f}(s = a^*, \right.$$
$$\left. r) = \frac{14}{3} a^{*2} (r - 2),\ \dot{\bar{f}}_s = \frac{28}{3} a^* (r - 2) \right)^{\mathrm{T}}$$

is a given vector of fuzzy variables.

The elements S_{pi}^2 of the matrix S^2 are 0 since the interval [0,2] does not contain the partition point c. Therefore, $S^2 = \left(S_{pi}^2 = 0\right)$. The elements S_{pi}^1 of the matrix S^1 are

$$S_{pi}^1 = \frac{1}{i!} \int_0^2 K_s^{(p)}(s, \tau)(\tau - z)^i d\tau =$$

$$\frac{1}{i!} \int_0^2 [(s = a^* = 0)^2 (1 + \tau)]_s^{(p)} (\tau - z = a^* = 0)^i d\tau = 0,$$

$$p, i = 0, 1.$$

Hence, $S^1 = \left(S_{pi}^1\right) = 0$, and consequently,

$$S = \begin{pmatrix} S^1 = 0 & S^2 = 0 \\ S^2 = 0 & S^1 = 0 \end{pmatrix}.$$

Then equation (16) gives

$$S - I\big|_{s = 0} = -I.$$

As a result, the matrix equation

$$\left(S-I\right)\Big|_{S=0}\cdot X=-F\Rightarrow IX=F$$

takes the form

$$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}\underline{x}\\\dot{\underline{x}}_s\\\overline{x}\\\dot{\overline{x}}\end{pmatrix}=\begin{pmatrix}ar\\1-r\\\left(\frac{14}{3}\right)a^2(r-2)\\\left(\frac{28}{3}\right)a(r-2)\end{pmatrix}\Rightarrow x_{\text{fuz}}=x(s,\,r)=$$

$$=\left(\underline{x}(s,\,r)\Big|_{a=s}=sr,\,\overline{x}(s,\,r)=\left(\frac{14}{3}\right)s^2(r-2)\Big|r\in[0,1]\right).$$

In the general case, it is necessary to define the fuzzy set of integral equations [22, 23]

$$x_{\text{fuz}\,i}(s)=f_{\text{fuz}\,i}(s)+\sum_{j=1}^{m}\int_a^b K_{ij}(s,\,\tau)x_{\text{fuz}\,i}(\tau)d\tau\,,\quad i=\overline{1,m}\,. \tag{19}$$

In this system, $a\le s,\,\tau\le b$, and $K_{ij}(s,\,\tau),\,i,\,j=$, $=\overline{1,\,m}$, are given crisp kernels differentiable by both variables on the interval $[a,b]$; $f_{\text{fuz}\,i}$ are given fuzzy functions; $x_{\text{fuz}\,i}(s)=(x_{\text{fuz}\,1}(s),\,\ldots,\,x_{\text{fuz}\,m}(s))^{\text{T}}$ is the fuzzy vector to be determined. The fuzzy variables $f_{\text{fuz}\,i}(s)$ and $x_{\text{fuz}\,i}(s)$ are written in the parametric form

$$f_{\text{fuz}\,i}(s)=f_i(s,r)=(\underline{f}(s,r),\,\overline{f}(s,r)|r\in[0,1]),$$

$$x_{\text{fuz}\,i}(s)=x_i(s,\,r)=(\underline{x}_i(s,r),\,\overline{x}_i(s,\,r)|r\in[0,1]),\ i=\overline{1,\,m}.$$

Next, the sequential transformations described above (see the one-dimensional case) are used: the interval of integration $[a,b]$ is partitioned using the points c_{ij}, $i,\,j=\overline{1,m}$; the unknown variables $\underline{x}_i(s,r),\overline{x}_i(s,r)$ are expanded into the Taylor polynomials of degree n at an arbitrary point $\tau=z\in[a,\,b]\subset R$; each of equations (19), written in the parametric form, is differentiated $p=\overline{0,n}$ times with respect to s; the symbols $\int$ and Σ are interchanged; the corresponding notations are introduced for the vectors and matrices involved.

These transformations yield fuzzy systems of linear equations of the form (16):

$$SX=F\,, \tag{20}$$

where $X(\cdot)=\left(\underline{x}_1^{(n)}(\cdot),\,\overline{x}_1^{(n)}(\cdot),\ldots,\underline{x}_m^{(n)}(\cdot),\,\overline{x}_m^{(n)}(\cdot)\right)^{\text{T}}$ is the unknown vector of fuzzy variables; $(\cdot)$ indicates $(s=a,r)$; (n) is the number of the derivative and the degree of the Taylor polynomial; $\underline{x}_k^{(n)}(\cdot)=$ $=\left(\underline{x}_k(\cdot),\ldots,\,\underline{x}_k^{(n)}(\cdot)\right)^{\text{T}}$; $\overline{x}_k^{(n)}(\cdot)=\left(\overline{x}_k(\cdot),\ldots,\overline{x}_k^{(n)}(\cdot)\right)^{\text{T}}$, $k=\overline{1,\,m}$, are the components of the vector X;

$F(\cdot)=\left(\underline{f}_1^{(n)}(\cdot),\,\overline{f}_1^{(n)}(\cdot),\ldots,\,\underline{f}_m^{(n)}(\cdot),\,\overline{f}_m^{(n)}(\cdot)\right)^{\text{T}}$ is a given vector of fuzzy variables;

$\underline{f}_k^{(n)}=-\left(\underline{f}_k(\cdot),\ldots,\underline{f}_k^{(n)}(\cdot)\right)^{\text{T}}$; $\overline{f}_k^{(n)}=-\left(\overline{f}_k(\cdot),\ldots,\overline{f}_k^{(n)}(\cdot)\right)^{\text{T}}$;

$S=\begin{pmatrix}S^{(1,1)}&\cdots&S^{(1,m)}\\\vdots&\ddots&\vdots\\S^{(m,1)}&\cdots&S^{(m,m)}\end{pmatrix}$ is a matrix with matrix elements

$$W^{(i,j)}=\begin{pmatrix}S_{11}^{(ij)}&S_{12}^{(ij)}\\S_{21}^{(ij)}&S_{22}^{(ij)}\end{pmatrix};\ S_{11}^{(ij)}=$$

$$=S_{22}^{(ij)}=\begin{pmatrix}S_{00}^{(ij)}-1&\cdots&S_{0n}^{(ij)}\\\vdots&\ddots&\vdots\\S_{n0}^{(ij)}&\cdots&S_{nn}^{(ij)}-1\end{pmatrix};$$

$$S_{12}^{(ij)}=S_{21}^{(ij)}=\begin{pmatrix}S_{00}^{*(ij)}&\cdots&S_{0n}^{*(ij)}\\\vdots&\ddots&\vdots\\S_{n0}^{*(ij)}&\cdots&S_{nn}^{*(ij)}\end{pmatrix}.\ \blacklozenge$$

Example 3. Let:

$$\underline{f}_1(s,\,r)=s\cdot r-\frac{27}{4}s^2(r^3-2)-\frac{14}{3}s^2r-\frac{1}{4}s^2r(r^4+2);$$

$$\overline{f}_1(s,\,r)=\frac{14}{3}s^2(r-2)+\frac{3}{4}s^2(r^3-2)-$$

$$-s(r-2)+\frac{9}{4}s^2r(r^4+2);$$

$$\underline{f}_2(s,\,r)=s(s^5+2r)-14.1(s-2)^2(r^3-2)-$$

$$-\frac{8}{3}(s^2+1)r-0.3(s-2)^3r(r^4+2);$$

$$\overline{f}_2(s,\,r)=\frac{8}{3}(s^2+1)(r-2)-s(3r^3-6)+$$

$$+0.9(s-2)^2(r^3-2)+4.7(s-2)r(r^4+2).$$

The collection of kernels is

$$K_{11}(s,\,\tau)=s^2(1+\tau);\ K_{12}(s,\,\tau)=$$

$$=s^2(1-\tau^2),\ K_{21}(s,\,\tau)=(1+s^2)\tau,$$

and $K_{22}(s,\,\tau)=(s-2)(1-\tau^3)$, where $0\le s$ and $\tau\le 2$.

Solution. Choosing the point $z=0$ for the Taylor expansion, we obtain:

$$W^{(1,1)}=\begin{pmatrix}-1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix};W^{(1,2)}=(0)_{i,j}\,,i,\,j=\overline{1,4}\,;$$

$$W^{(2,1)} = \begin{pmatrix} 2 & \frac{8}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & \frac{8}{3} \\ 0 & 0 & 0 & 0 \end{pmatrix};$$

$$W^{(2,1)} = \begin{pmatrix} 2 & 1.2 & -11 & -\frac{94}{5} \\ -3 & -2.2 & 11 & \frac{94}{5} \\ -11 & -\frac{94}{5} & 2 & 1.2 \\ 11 & \frac{94}{5} & -3 & -2.2 \end{pmatrix}.$$

Solving the fuzzy system of linear equations (20) gives

$$X = S^{-1}F,$$

where

$$X = X(a=0, r) = (\underline{x}_1,\ \underline{x}_1^*,\ \overline{x}_1,\ \overline{x}_1^*,\ \underline{x}_2,\ \underline{x}_2^*,\ \overline{x}_2,\ \overline{x}_2^*)^{\mathrm{T}};$$

$$S^{-1}F = (0,\ r,\ 0,\ 2-r,\ 0,\ r^5+2r,\ 0,\ 6-r^3)^{\mathrm{T}}.\ ♦$$

3.4. Estimation by method of degenerate kernels

Let the equation kernel be a finite sum in which each term is the product of some function of τ by some function of s. In this case, equation (19) with the kernel $K(s,\tau) = \sum_{i=1}^{n} a_i(s)b_i(\tau)$ is a fuzzy Fredholm integral equation with a crisp nondegenerate kernel [22, 23]. Like before (see subsection 3.2), the following assumptions are made to ensure the existence of a unique fuzzy solution by the method of successive approximation: $a_i(s)$ is defined, piecewise continuous in the Hausdorff sense, and bounded by the first Seikkala derivative for $s \in [a, b] \subset R$; $b_i(\tau)$ satisfies the same constraint for $\tau \in [0, t] \subset R$.

We modify the well-known method for solving traditional (crisp) equations with degenerate kernels to solve the corresponding fuzzy equation [24].

Consider a fuzzy integral equation (19) with the kernel

$$K(s,\tau) = \sum_{i=1}^{n} a_i(s)b_i(\tau).$$

Assume that the following inequalities hold on the interval of integration $[a, b]$:

$$(i): \begin{cases} \sum_{i=1}^{n} a_i(s)b_i(\tau) \ge 0, \\ a \le \tau \le b; \end{cases} \quad (ii): \begin{cases} \sum_{i=1}^{n} a_i(s)b_i(\tau) < 0, \\ a \le \tau \le b; \end{cases}$$

$$(iii): \begin{cases} \sum_{i=1}^{n} a_i(s)b_i(\tau) \ge 0, \\ a \le \tau \le c, \\ \sum_{i=1}^{n} a_i(s)b_i(\tau) < 0, \\ c \le \tau \le b. \end{cases}$$

In case (i), the system of equations (19) satisfies the relations

$$x_{\text{fuz}}(s) = f_{\text{fuz}}(s) + \int_a^b K(s,\tau)x_{\text{fuz}}(\tau)d\tau \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \underline{x}(s,r) = \underline{f}(s,r) + \sum_{i=1}^{n} a_i(s)\underline{x}_i, \\ \overline{x}(s,r) = \overline{f}(s,r) + \sum_{i=1}^{n} a_i(s)\overline{x}_i, \\ \underline{x}_i = \int_a^b b_i(\tau)\underline{x}(\tau)d\tau,\ \overline{x}_i = \\ \quad = \int_a^b b_i(\tau)\overline{x}(\tau)d\tau. \end{cases} \tag{21}$$

Multiplying the expressions (21) by $b_i(\tau)$ and integrating them on the interval $[a, b]$, we obtain:

$$\underline{x}_i = \underline{f}_i + \sum_{i=1}^{n} a_{ij}^1 \underline{x}_i,\ \underline{f}_i = \int_a^b \underline{f}(\tau)b_i(\tau)d\tau,\ a_{ij}^1 = \int_a^b b_i(\tau)a_j(\tau)d\tau,$$

$$\overline{x}_i = \overline{f}_i a_{ij}^1 \overline{x}_i,\ \overline{f}_i = \int_a^b \overline{f}(\tau)b_i(\tau)d\tau,\ i = \overline{1,n}.$$

These relations lead to the fuzzy system of linear equations

$$X_{\text{fuz}} = AX_{\text{fuz}} + F_{\text{fuz}},$$

where

$$X_{\text{fuz}} = (\underline{X}\,|\,\overline{X})^{\mathrm{T}};\ \underline{X} = (\underline{x}_1,\ldots,\underline{x}_n);\ \overline{X} = (\overline{x}_1,\ldots,\overline{x}_n);$$

$$F_{\text{fuz}} = (\underline{F}\,|\,\overline{F})^{\mathrm{T}},$$

$$F_{\text{fuz}} = (\underline{F}\,|\,\overline{F});\ \underline{F} = (\underline{f}_1,\ldots,\underline{f}_n),\ \overline{F} = (\overline{f}_1,\ldots,\overline{f}_n);$$

$$A = \begin{pmatrix} A^1 & 0 \\ 0 & A^1 \end{pmatrix};\ A^1 = (a_{ij}^1),$$

$$a_{ij}^1 = \int_a^b b_i(\tau)a_j(\tau)d\tau,\ i,\ j = \overline{1,n}.$$

It can be written in the traditional fuzzy calculus form [19, 20]:

$$(I-A)X_{\text{fuz}} = F_{\text{fuz}}, \text{ where } I \text{ denotes an identity matrix.} \tag{22}$$

The case $|I - A| = 0$ was studied in the papers [19, 20].

Example 4. Consider an integral equation of the form (21):

$$x_{\text{fuz}}(s) = f_{\text{fuz}}(s) + \int_0^{0.5} s\cdot\tau\cdot x_{\text{fuz}}(\tau)d\tau\,,\ s\in[0,\ 0.5]\,,$$

where

$$f_{\text{fuz}}(s) = f(s,\,r) = (\underline{f}(s,\,r),\ \overline{f}(s,r)\big|\,r\in[0,1]);$$

$$K(s,\,\tau) = s\,\tau \geq 0,\ a_i(s) = s,\ b_i(\tau) = \tau.$$

The solution is found from the fuzzy matrix equation (22) with the matrix elements

$$X_{\text{fuz}} = X(s,\,r) = (\underline{x}_1\big|\,\overline{x}_1)^{\mathrm{T}};\ F_{\text{fuz}} = F(s,\,r) =$$

$$= \left(\underline{f}_1 = \int_0^{0.5} \underline{f}(\tau)\tau d\tau \overline{f}_1 = \int_0^{0.5} \overline{f}(\tau)\tau d\tau \right)^{\mathrm{T}};$$

$$a_{ij}^1 = \int_0^{0.5} \tau^2 d\tau = \frac{\tau^3}{3}\bigg|_0^{0.5} = \frac{1}{24};\ I - A = 1 - a_{11}^1 = 1 - \frac{1}{24} = \frac{23}{24}.$$

Hence,

$$\underline{x}_1 = \frac{24}{23}\int_0^{0.5} \underline{f}(\tau)\,\tau\,d\tau\,,\ \overline{x}_1 = \frac{24}{23}\int_0^{0.5} \overline{f}(\tau)\,\tau\,d\tau\,,$$

and the solution has the form

$$x_{\text{fuz}}(s) = (\underline{x}(s,r), \overline{x}(s,r)\,|\,r\in[0,1])\,,$$

where

$$\underline{x}(s,r) = \underline{f}(s,r) + s\underline{x}_1\,,\ \overline{x}(s,r) = \overline{f}(s,r) + s\overline{x}_1\,.$$

In case (ii), the calculations similar to case (i) yield

$$(\overline{X}\big|\,\underline{X})^{\mathrm{T}} = (-I\underline{X}\,\big|-I\overline{X})^{\mathrm{T}}\,,$$

due to the multiplication rule of fuzzy variables x,

$$kx = \begin{cases} (k\underline{x},\ k\overline{x}),\ k\geq 0,\ k\in R, \\ (k\overline{x},\ k\underline{x}),\ k<0. \end{cases}$$

After trivial transformations, we finally obtain:

$$\begin{pmatrix} \underline{X} \\ \overline{X} \end{pmatrix} = \begin{pmatrix} A_1 & 0 \\ 0 & A_1 \end{pmatrix}\begin{pmatrix} \underline{X} \\ \overline{X} \end{pmatrix} + \begin{pmatrix} \underline{F} \\ \overline{F} \end{pmatrix} \Leftrightarrow \begin{pmatrix} \underline{X} \\ \overline{X} \end{pmatrix} =$$

$$= \begin{pmatrix} A_1 & 0 \\ 0 & A_1 \end{pmatrix}\begin{pmatrix} -I\underline{X} \\ -I\overline{X} \end{pmatrix} + \begin{pmatrix} -I\underline{F} \\ -I\overline{F} \end{pmatrix} \Leftrightarrow$$

$$\Rightarrow (I + A)X_{\text{fuz}} = -F_{\text{fuz}}.\ \blacklozenge$$

3.5. Estimation by method of nondegenerate kernel approximated by degenerate one

Consider the relation (21), and let the kernel be $K(s,\,\tau) = K(s\,\cdot\,\tau)$. According to the Taylor expansion, for $(s\cdot\tau)\simeq 0$ we obtain

$$K(s\cdot\tau)\sum_{i=1}^{n} e_i(s\cdot\tau)^i = \sum_{i=1}^{n} a_i(s)b_i(\tau).$$

Hence, the equation

$$x_{\text{fuz}}(s) = f_{\text{fuz}}(s) + \int_a^b K(s,\,\tau)x_{\text{fuz}}(\tau)d\tau$$

is solved using the method described in subsection 3.4.

Example 5. Consider the integral equation

$$x_{\text{fuz}}(s) = f_{\text{fuz}}(s) + \int_0^{0.5} \sin(s\cdot\tau)x_{\text{fuz}}(\tau)d\tau\,.$$

Applying the Taylor approximation of the kernel $K(s\cdot\tau)\simeq s\cdot\tau,$ we can use the results of Example 4.

Under the approximation

$$K(s\cdot\tau)\cdot s\cdot\tau - \frac{1}{3}(s\cdot\tau)^3 = a_1\,b_1\Big|_{\substack{a_1=3\\ b_1=\tau}} + a_2\,b_2\Big|_{\substack{a_2=3^3\\ b_2=-\frac{1}{3}\tau^3}}\,,$$

this equation can be also solved using the method from subsection 3.4. ♦

CONCLUSIONS

Based on the definition of a fuzzy Riemann integral, the problem of estimating the states of models described by fuzzy Fredholm–Volterra integral equations has been formulated under the assumed existence of their unique solutions.

Various state estimation methods for fuzzy integral equations have been considered, namely, the fuzzy Laplace transform, the method of "embedding" models, the Taylor estimation of the degenerate kernels, and the estimation of the nondegenerate kernels by degenerate forms. Test examples have been solved for them. As shown above, in some cases, the estimation results are related to the solution of fuzzy systems of linear algebraic equations.

In part II of the survey, other state estimation methods for linear and nonlinear fuzzy integral models will be considered, namely, the least squares method and its modifications, the Galerkin and Chebyshev methods, and sinc functions.

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INFORMATION COMMUNITIES IN SOCIAL NETWORKS. PART I: FROM CONCEPT TO MATHEMATICAL MODELS[1]

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Abstract. This survey covers the literature related to information communities in mutually complementary areas: the formation of information communities in social networks and some applied aspects of identifying and analyzing information communities in social networks. First, mathematical models describing the formation of information communities under uncertainty are considered. Among these models, the most relevant ones are the mathematical models of opinion/belief dynamics reflecting any changes in the beliefs of nodes under the influence of other network nodes and significant effects (in particular, the preservation of differences in beliefs and the divergence of beliefs) that lead to the formation of information communities. In part I of the survey, the concept of an information community is first presented. Then information processing and decision-making by an agent in a social network under external uncertainty are outlined. The factors influencing the formation of information communities in the network are highlighted, and the basic models of Bayesian agents and their extensions are investigated.

Keywords: social networks, information community, formation of information communities, analysis of information communities, belief formation.

INTRODUCTION

The Internet and online social networks have opened up great opportunities for the efficient production, distribution, and consumption of information in society and, therefore, opportunities for rational discussion of various issues and the formation of balanced opinions on them. However, as it turned out, the availability and diversity of information sources and the corresponding alternative points of view do not automatically improve the quality of the information received and the competence of people in socially important issues. On the contrary, the ideas on many issues in society diverge, and separate communities are formed with different or even exact antipodes of opinions on the same issues. This phenomenon can be explained as follows: social network participants are not completely rational agents effectively aggregating information on issues of interest to them since social and psychological factors significantly influence the processing of information by individuals.

In many application areas, an important problem is identifying and studying *information communities* in social networks (the sets of individuals with similar and stable ideas on a certain issue). For example, social and political scientists believe that the formation of isolated communities (information bubbles and echo chambers) poses a threat to society. Empirical research shows a rich variety of information communities in society (for example, polarized communities of Republicans and Democrats in the United States). In such studies, various aspects were analyzed: the exposure of a user to alternative information depending on the preferences of his contacts in the network and online social network algorithms [1], the interaction of communities with different beliefs [2], the informational roles of users [3], etc. Statistical methods, machine learning methods (for example, correlation and cluster

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analysis methods), and social network analysis methods based on the phenomenon of homophily[2] are often used [1–5]. Such methods require preliminary data processing and subsequent interpretation of the results. In other words, a researcher must have an idea of the opinion dynamics in social networks and the existence of information communities in them. Features of information processing by an individual are explored in cognitive science, psychology, and social psychology; for example, see the book [8]. Formal microlevel models of boundedly rational agents are developed to describe the belief dynamics in networks taking these features into account. For example, we refer to [9–13]. In practice, applying these models to identify communities is not easy due to the simplifications and assumptions accepted, the complexity of identifying the model parameters, and the absence of a clearly formulated concept of information communities.

This survey aims to consider the formation models of information communities in social networks (which have microeconomic, cognitive, or socio-psychological foundations) and methods for their identification. The survey is divided into three parts, and part I has the following structure. In Section 1, we define an information community. In Section 2, we briefly describe the process of information processing and decision-making by an agent in a social network under an external uncertainty; also, we highlight the factors affecting the belief dynamics in a social network and, consequently, the formation of information communities in the network. In Section 3, we briefly discuss formal models of the belief dynamics with Bayesian agents leading to the formation of information communities.

1. CONCEPT OF INFORMATION COMMUNITY

Community is a rather vague concept often used informally. Here are some of the definitions available. Community is “an association of humans, peoples, or states with common interests or goals” [14]. A community can be viewed:

• as an association of individuals, i.e.,

– as a group of people with common characteristics or interests, living together within a larger society,

– as a set of individuals with common interests, distributed throughout society,

– as an association of people or nations with a common history or common social, economic, and political interests;

• as a society as a whole [15].

Examples are scientific communities and language communities.

According to these definitions, the characteristics of all individuals within a community are common. This effect is closely related to *homophily*, the inclination of individuals to form relations based on common characteristics [6, 7]. From this point of view, there is a direct connection with the definition adopted in the theory of complex networks, where a community is a set of nodes connected with each other rather than with the nodes of other communities [16]. As such characteristics, we will be concerned with the beliefs of individuals (private beliefs) about some issues (problems). Therefore, we will understand the information *community* as a set of individuals—social network members—united by common stable beliefs[3] about given issues; see the formal definition of a community introduced in the paper [17].

For describing and explaining the formation dynamics of information communities in social networks, appropriate models of the dynamics of private beliefs are needed. Let us distinguish between two types of significant dynamic processes in social networks (Fig. 1): the process of changing the beliefs of network individuals (the network state) within a fixed topology and the process of changing the network topology when the network state “affects” the topology (like-minded people begin to interact more with each other). It is usually assumed that the topology dynamics occur in slow time, whereas the state dynamics in fast time. The most interesting and complex situations are when both processes influence each other, thereby affecting the formation of information communities in a social network.

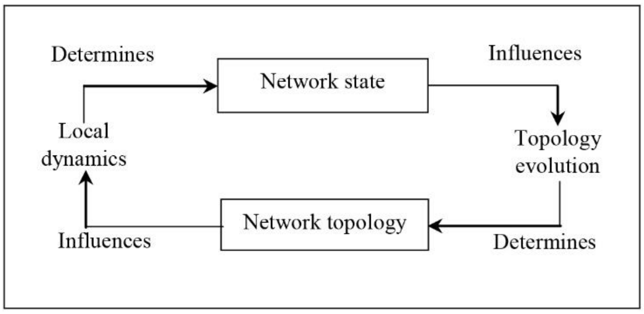

Fig. 1. Mutual influence of network state and topology.

[2] Homophily is actively studied in sociology. In particular, the evidence of and reasons for this phenomenon are considered; for example, see [6, 7]. This paper does not cover the results obtained in sociology.

[3] The terms “belief” and “opinion” are considered synonyms.

This survey considers dynamic models of beliefs in networks with a fixed structure (in some cases, with changing weights of links), where individuals try to eliminate the uncertainty regarding a given issue via social interactions. The common final beliefs of individuals are the condition for forming information communities in the network. These models are discussed further in parts I and II of the survey.

2. FORMING BELIEFS IN SOCIAL NETWORK

Participants of social networks exchange information to eliminate the uncertainty on some issue, forming their beliefs. (In mathematical models, issues are usually formalized by the values of some parameters.) In control of socio-economic systems, a common assumption is that rational individuals (agents) have beliefs about the state of the world $\theta \in \Theta$ (also called the state of nature). The agent's individual preferences are defined on the set of activity results, which particularly depend on the agent's actions and the state of the world. Under the hypothesis of rational behavior, each agent chooses an action yielding the best result for him. The information he possesses regarding the state of the world is essential here. A rational agent seeks to eliminate the existing uncertainty and make decisions under complete information (the hypothesis of determinism) [18]. This paper primarily considers the elimination of an external objective uncertainty (the uncertain state of the world). It is assumed that a rational formation about the state of the world while interacting with his neighbors (whose actions reveal their private information), changing his beliefs according to some updating mechanisms (or information processing rules); see Fig. 2. Rational agents calculate their posterior beliefs by Bayes' rule.

However, individuals are not completely rational. As noted by psychologists [19–21], individuals have *bounded rationality* due to various cognitive limitations (primarily, limited memory and limited computational capabilities) and mental characteristics; see mental components in Fig. 2 and their detailed description in [22]. Moreover, individuals make systematic errors that affect information processing (cognitive biases). Therefore, heuristic updating methods can be considered here, which are based on empirical laws and demonstrate the socio-psychological effects observed in practice. In particular, the social influence on the private beliefs was described in the classical DeGroot model [9]: an agent updates his belief based on the information about the beliefs of his trusted environment in a social network. In meaningfully richer models (for example, those presented in [10–12]), the strength of the influence of neighbors depends on how much their beliefs agree with the agent's belief: the individual's inclination to confirm his point of view is taken into account. This effect can lead to the emergence of communities in which the agents support the same beliefs.

Generally speaking, the dynamics of private beliefs in a social network are influenced by the following factors (Fig. 2):

- *The state of the world* $\theta \in \Theta$ regarding which individuals form their beliefs (for example, the shape of the Earth or a currency exchange rate for tomorrow).
- *The individual's belief about the state of the world*. The belief can be defined, in particular, using some point estimate or distribution of subjective probabilities on the set Θ. The individual's beliefs are limited by memory and may depend on his beliefs about other issues.
- *The updating mechanism for beliefs*. Control of socio-economic systems rests on the assumption that the agents are rational and act according to Bayes' rule. However, boundedly rational individuals can apply heuristic rules.
- *The individual's action*, which reflects his beliefs. Actions from discrete sets are usually less informative than those from continuous ones due to an insufficiently good reflection of the agent's beliefs.
- *The individual's preferences*, defined on the set of his activity results, or a preferences-induced objective function that depends on the individuals' actions and the state of the world.
- *The social network structure*. Obviously, the network structure influences the formation of the private beliefs. Here are some examples of network effects: disconnected networks rarely lead to the coordinated beliefs of individuals; individuals with an advantageous position in the network structure usually have a significant impact on the opinions of others, etc.

Each of these factors concerns the mental characteristics of individuals and determines various information effects in a social network. The models of belief dynamics describe the following information effects:

– *the emergence of a true or false consensus of beliefs in the network* and, consequently, the formation of a global information community (see the definitions in Section 1);

– *the emergence of some disagreement in the network* and, consequently, the formation of various information communities in the network.

The mathematical models of belief dynamics for network agents (see below) incorporate the factors listed above and demonstrate these information effects and, therefore, the possibility of forming information communities in the network. A primary approach is to divide the models, according to the intellectual capabilities of network agents and the updating method for their beliefs, into the models with rational Bayesian agents and the models with boundedly rational agents guided by heuristic belief updating rules.

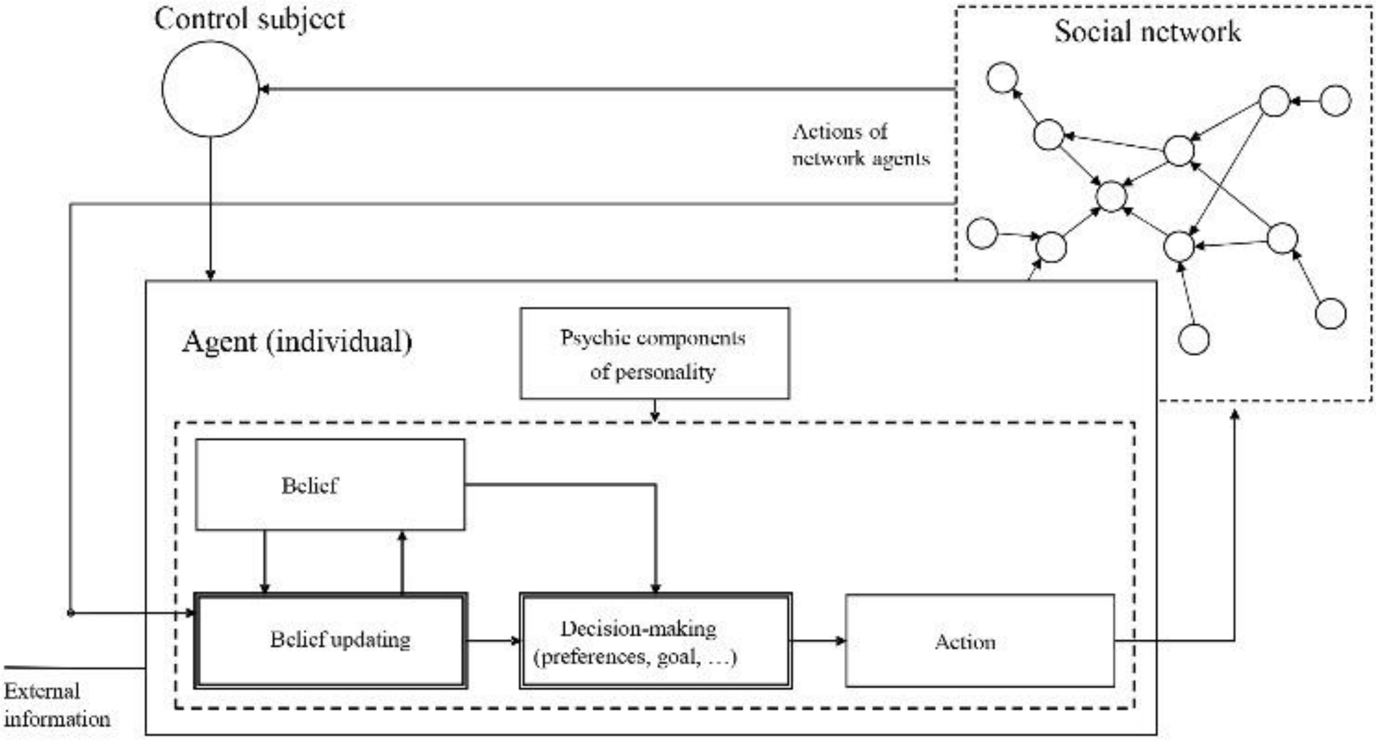

Fig. 2. Information processing and decision-making by agent in social network.

3. FORMING INFORMATION COMMUNITIES IN MODELS WITH BAYESIAN AGENTS

3.1. Forming Private Beliefs

In models with Bayesian agents, the main question is whether the agents can form true beliefs about the state of the world in a network. For the state of the world, the set of admissible values Θ is given, like the set of agents having probability distribution-based beliefs about the state of the world. The agent's learning occurs by processing his available information about the state of the world: *a private signal* and, possibly, the actions of his neighbors. In the latter case (the agent receives information about the actions of his neighbors), learning is called social. For being informative, a signal s must depend on the state of the world θ. However, generally speaking, it does not completely reveal the state of the world, representing a random variable. Information processing obeys Bayes' rule: the incoming information is used to update the individual's prior beliefs and form his posterior beliefs:

$$f(\theta|s) = \frac{\phi(s\,|\,\theta)\, f(\theta)}{\int \phi(s|\theta)\, f(\theta)\, d\theta},$$

where $f(\theta)$ denotes the prior density function of θ; $\phi(s|\theta)$ and $f(\theta|s)$ are the conditional density functions of the parameters s and θ given θ and s, respectively.

In classical learning models, all agents know the model structure: the prior probabilities of the admissible states of nature and the private signals (their conditional distributions given different states of nature). This information is *common knowledge*:

1) Each agent knows this structure.

2) All agents know fact 1);

3) All agents know fact 2), and so on, ad infinitum.

However, each agent knows neither the realizations of the state of the world nor the realizations of the other agents' private signals. The common knowledge assumption is quite strong, being weakened in several studies; for example, see the papers [23, 24].

Further, we discuss two basic updating models for the agent's beliefs, in which particular assumptions about the agent's awareness structure are introduced, and there is no information interaction between different agents.

3.2. Basic Updating Models for Agent's Beliefs

Consider two basic updating models for the agent's beliefs. They can be briefly described as follows. In the elementary binary model, the state of the world takes two values (the state is discrete), and each agent receives a binary signal about the state of the world; in the Gaussian model, the state of the world and private signals are realizations of Gaussian random variables.

In *the binary model*, the set of admissible states is $\theta \in \{\theta_0, \theta_1\}$, where $\theta_0 < \theta_1$; in the elementary statement, $\theta \in \{0, 1\}$. The probability distribution is characterized by one number–the probability of state 1. Private signals take the value 1 or 0 with the probabilities $P(s = 1|\theta = 1) = q$ and $P(s = 0 \mid \theta = 0) = q'$. A private signal is called *symmetric* if $q = q'$. In this case, the parameter q is called the signal accuracy. (A conventional assumption is $q > ½$.)

For the binary model, Bayes' rule can be written as the likelihood ratio

$$\frac{P(\theta=1\mid s)}{P(\theta=0\mid s)}=\frac{P(s\mid\theta=1)}{P(s\mid\theta=0)}\cdot\frac{P(\theta=1)}{P(\theta=0)}.$$

In the binary model, the signal leads to a bounded change of beliefs. If μ is the subjective belief about state 1, its variance is $\mu(1 - \mu)$. This means that new information can increase the variance and decrease the confidence in the resulting estimate. For a sequence of signals $\{s_t\}$ with the same accuracy q, Bayes' rule is applied sequentially. As $t \to \infty$, the agent's belief $\mu_t \to \theta$, and the variance of the estimate tends to 0.

In *the Gaussian model*, the state of the world is a realization of a Gaussian random variable or vector. In the simple case, $\theta \sim N(m, \sigma_\theta^2)$. The distribution *accuracy* is denoted by $\rho_\theta = 1/\sigma_\theta^2$. The private signal $s = \theta + \epsilon$ obeys the Gaussian distribution, where the noise is $\epsilon \sim N(0, 1/\rho_\epsilon)$. After receiving signal s, the updated distribution θ remains Gaussian $N(m', 1/\rho')$ with the parameters

$$\rho' = \rho + \rho_\epsilon,$$

$$m' = \alpha s + (1-\alpha)m, \text{ where } \alpha = \rho_\epsilon/\rho'.$$

Consequently, in the elementary Gaussian learning model, observations lead to an increase in the accuracy of the agent's beliefs (decrease in the variance); the posterior expectation of θ is the weighted sum of the signal and the prior expectation (with weights reflecting the accuracy).

Thus, in the basic learning models, an agent receives a sequence of informative signals and gradually reaches a true estimate for the state of the world. Let us now consider the formation of various information communities in these models.

3.3. Forming Different Information Communities

The question arises: can Bayesian agents[4] reach different beliefs if they receive the same information (the same sequence of signals) about the state of the world?

Cognitive limitations

Agents can reach different beliefs if *their prior beliefs differ* and *their memory is limited*. In [25], some of the agent's signals on the state of the world were supposed to be ambiguous and interpreted differently depending on their current beliefs. In particular, at a time instant t, an agent can receive informative signals a or b about the state of the world, or an ambiguous signal ab, which has to be interpreted and memorized as a or b due to the agent's memory limit. An agent forms a belief λ about the state of the world by Bayes' rule, interpreting the incoming signal ab as a if $\lambda_{t-1} > 1/2$, or as b if $\lambda_{t-1} < ½$. (Thereby, the agent shows the inclination to confirm his point of view.) Let the agents have different prior beliefs (for example, the first agent considers state A to be more likely, and the second agent, state B). If the probability of ambiguous signals is significant, then the agents will reach opposite beliefs about the state of the world with a positive probability.

Cognitive biases

In the paper [26], *the effect of an agent's inclination to confirm his point of view* was described within the binary model. The following assumption was introduced to model the inclination: if an agent receives a signal contradicting his belief about the state of the world, he incorrectly interprets (perceives) this signal with a probability $q > 0$ as confirming his belief. At the same time, he is unaware of the signal misinterpretation and acts like a typical Bayesian agent. As was established therein, under the agent's inclination to confirm his point of view (expressed by the parameter q), he can eventually reach the false belief, despite an infinite sequence of informative signals perceived by him. Accordingly, individual probabilities q can lead to some disagreement among agents in society.

Complex model of beliefs: additional factors and questions

As was demonstrated in [27], in some cases, the intensification of disagreement among individuals observing the same information is rational if they make *different assumptions about additional factors affecting the relationship between the parameters under consideration: the state of the world and the received*

[4] Although this subsection deals with situations with two agents, the considerations are applicable to any set of agents of two types.

signal (i.e., the private beliefs about the problem situation are richer compared to the individuals in classical learning models). This aspect was touched upon in [28], where the role of beliefs about an "auxiliary" issue (not directly related to the "main" issue but affecting the interpretation of signals associated with it) was considered. These beliefs may cause the polarization of beliefs about the main issue. With strained interpretation, due to the specifics of the agent's utility function, this class of models includes the model [29], in which the state of the world $\theta = (\alpha, \beta)$ is a realization of the random variable $\tilde{\theta} = (\tilde{\alpha}, \tilde{\beta})$, $\tilde{\alpha}, \tilde{\beta} \in \{0, 1\}$, and agents with different private signals on α, receiving general signals on β, reach different beliefs about the optimal actions.

Different prior beliefs

The paper [30] considered the polarization of Bayesian individuals' beliefs in a collective choice problem that depends on the state of the world and requires a decision (the choice of some policy). Depending on their beliefs about the state of the world, voters support one or another alternative. Then they observe the degree of success of their choice (the result of the chosen policy) and correct their beliefs about the state of the world. Polarization is excluded if and only if the conditional density of the choice result (given the state of the world and the chosen policy) has *the Monotone Likelihood Ratio Property* (MLRP). Otherwise, polarization cannot be ruled out even under small *differences in the prior beliefs*: the corresponding examples were provided for discrete and continuous indicators of the success of the chosen policy.

Different prior beliefs about conditional signal distributions

Agents can also reach some disagreement if their prior beliefs about the state of the world and their *beliefs about the conditional signal distributions* are different. Let us discuss this aspect in detail.

In [31], a learning model with two Bayesian agents (1 and 2) was considered. The agents observe a sequence of signals $\{s_t\}_{t=0}^{n}$, $s_t \in \{a, b\}$, from an environment. The state of the world is described by the parameter $\theta \in \{A, B\}$ (the true state is A), and the prior belief of agent i about the probability that $\theta = A$ is given by the parameter $\pi^i \in (0,1)$. The agents suppose that for a given parameter θ, the incoming signals are independent and identically distributed: $P(s_t = a \mid \theta = A) = p_A$ and $P(s_t = b \mid \theta = B) = p_B$. Usually, these probabilities are considered known. In reality, however, there may exist an uncertainty of the probability p_θ ($\theta \in \{A, B\}$): for each agent i, this uncertainty is described by his distribution of subjective probabilities F_θ^i.

Consider an infinite sequence of signals $s \equiv \{s_t\}_{t=1}^{\infty}$, and let S be the set of all such sequences. The posterior belief of agent i about the parameter θ given the observed sequence of signals $\{s_t\}_{t=1}^{n}$ is

$$\phi_n^i(s) \equiv P^i\left(\theta = A \middle| \{s_t\}_{t=1}^{n}\right).$$

Recall that the signals are independent and identically distributed. Hence, the posterior probability depends on the number of signals $s_t = a$ by a time instant n:

$$r_n(s) \equiv \#\{t \le n \mid s_t = a\}.$$

According to the strong law of large numbers, $r_n(s)/n$ converges with probability 1 to some frequency $\rho(s) \in [0, 1]$ for all agents. Defining the set $\bar{S} \equiv \{s \in S: \exists \lim_{n\to\infty} r_n(s)/n\}$, we write

$$\phi_n^i(s) = \frac{1}{1 + \frac{1-\pi^i}{\pi^i} \frac{P^i(r_n \mid \theta = B)}{P^i(r_n \mid \theta = A)}},$$

where $P^i(r_n|\theta)$ is the probability of observing exactly r_n signals $s_t = a$ in the sequence of the first n signals given F_θ^i.

As it turned out [31], if for each F_θ^i we have the probability $F_\theta^i(\hat{p}_\theta) = 1$ for some $\hat{p}_\theta > 1/2$ and $F_\theta^i(p) = 0$ for all $p < \hat{p}_\theta$, then: $P^i\left(\lim_{n\to\infty} \phi_n^i(s) = 1 \middle| \theta = A\right) = 1$ (*asymptotic learning*) and $P^i\left(\lim_{n\to\infty} \left|\phi_n^1(s) - \phi_n^2(s)\right| = 0\right) = 1$ (*asymptotic agreement*) for each $i = 1, 2$. Thus, if the individuals know the conditional distributions of signals (which are the same for them), they will learn the true state of the world from observations (almost surely as $n \to \infty$) and reach a consensus regarding the state θ in the case of observing the same sequence of signals. If the limiting frequency of the signal a is $\hat{p}_A$, then the individual believes that $\theta = A$; if this frequency is $1 - \hat{p}_B$, then he believes that $\theta = B$. The probability of all other cases for the agent is 0. If for sufficiently large $n < \infty$, the individuals observe ρ (the frequency of the signal a) different from $\hat{p}_A$ and $(1 - \hat{p}_B)$, they will associate this deviation with sampling variation. However, as the sample grows ($n \to \infty$), it becomes difficult to explain by the sample variation the signal frequency differing from $\hat{p}_A$ and $(1 - \hat{p}_B)$. Therefore, a natural approach is when the individuals are allowed to specify positive (albeit small) probabilities for all admissible values of p_θ. This assumption leads to various consequences; see below.

Theorem 1. *Assume that for each agent i and each value of the parameter* θ, *the probability distribution* F_θ^i *has a continuous, nonzero, and finite density function* f_θ^i *on the interval* [0, 1].

Then for $s \in \bar{S}$:

(a) *There is no asymptotic learning, i.e.,* $P^i\left(\lim_{n\to\infty} \phi_n^i(s) \neq 1 | \theta = A\right) = 1$.

(b) *There is no asymptotic agreement between two agents, i.e.,* $P^i\left(\lim_{n\to\infty} \left|\phi_n^1(s) - \phi_n^2(s)\right| \neq 0\right) = 1$ *whenever* $\pi^1 \neq \pi^2$ *and* $F_\theta^1 = F_\theta^2$ *for each value* $\theta \in \{A, B\}$ [31, 32].

of p_θ. This assumption leads to various consequences; see below.

In fact, learning under such uncertainty can *intensify the disagreement between two Bayesian agents after receiving the same infinite sequence of signals*. This effect is impossible within the standard model; see the next theorem.

Theorem 2. *Assume that for each agent i and each value of the parameter* θ, *the probability distribution* F_θ^i *has a continuous, nonzero, and finite density function* f_θ^i *on the interval* [0, 1]. *In addition, assume that there exists a number* $\epsilon > 0$ *such that* $\left|R^1(\rho) - R^2(\rho)\right| > \epsilon$ *for each frequency* $\rho \in [0, 1]$, where $R^1(\rho) \equiv f_B^i(1-\rho) / f_A^i(\rho)$ *is the likelihood ratio. Then, there exists an open set of prior beliefs* π^1 *and* π^2 *such that for all signals* $s \in \bar{S}$, $\lim_{n\to\infty} \left|\phi_n^1(s) - \phi_n^2(s)\right| > \left|\pi^1 - \pi^2\right|$; *particularly,* $P\left(\lim_{n\to\infty} \left|\phi_n^1(s) - \phi_n^2(s)\right| > \left|\pi^1 - \pi^2\right|\right) = 1$ [31].

Thus, even small differences in the prior beliefs of agents lead to different interpretations of the signals. If the initial discrepancy is small, then the disagreement between the agents will intensify after almost any sequence of signals.

There is no network interaction among the agents in the models of learning and formation of information communities discussed above. In the general case, individuals—members of society—interact with each other within a social network. Hence, the actions of neighbors in the network can provide an agent with additional information about the state of the world. This type of interaction will be discussed in part II of the survey.

CONCLUSIONS

In part I of the survey, the concept of an information community has been outlined, and relevant models for forming the beliefs of individuals who seek to eliminate uncertainty about a given issue(s), eventually forming information communities, have been considered. Approaches to model the updating of private beliefs and the influence of various factors on the achievement of true beliefs and the formation of one or several different information communities in the network have been described. In a society of Bayesian agents, a true belief about the issue is often reached; for the emergence of various information communities, it is necessary to weaken the rationality requirement for individuals and/or introduce assumptions about different awareness of individuals.

Part II of the survey will consider the formation of information communities in network models with Bayesian agents and with naive (“heuristic”) individuals. Finally, part III of the survey will be devoted to empirical studies on the existence of information communities in real social networks and their features.

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DIFFERENTIAL GAMES OF PURSUIT WITH SEVERAL PURSUERS AND ONE EVADER[1]

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Аннотация. A differential game of several players is considered as follows. One player (attacker) penetrates some space, and several other players (pursuers) appear simultaneously to intercept the attacker. Upon detecting the pursuers, the attacker tries to evade them. The dynamics of each player are described by a time-invariant linear system of a general type with scalar control. A quadratic functional is introduced, and the differential game is treated as an optimal control problem. Two subproblems are solved as follows. The first subproblem is to construct a strategy for pursuing the attacker by several players having complete equal information about the game. The second subproblem is to construct such a strategy under incomplete information about the attacker actively opposing the pursuers. The simulation results are presented. The zero-sum differential game solution can be used for studying the final stage of pursuit, in which several pursuers and one evader participate.

Ключевые слова: differential games, linear dynamics, optimal feedback control, Nash equilibrium, Lyapunov functions, Riccati equation.

INTRODUCTION

The theory of differential games as a branch of mathematical control theory is closely related to the mathematical theory of optimal processes, game theory, calculus of variations, and the theory of differential equations. Problems of the theory of differential games stem from many topical applications, such as the pursuit of one controlled object by another, bringing a controlled object into a given state under unknown disturbances, and military or economic problems, to name a few. The formation of the theory of differential games is associated with R.P. Isaacs [1, 2], J.V. Breakwell [3], L.S. Pontryagin [4, 5], E.F. Mishchenko [6], B.N. Pshenichny [7], and many other foreign and Soviet scientists. Since the late 1970s, an independent area in the applied theory of differential games has appeared, dealing with the problems of pursuit, evasion, and target defense [8–19]. In the works by L.S. Pontryagin and E.F. Mishchenko [4–6], sufficient conditions for completing pursuit in linear differential games were established. In the research of N.N. Krasovskii, A.I. Subbotin [8], their students and colleagues, positional differential games were studied; for this class of games, the problems of approach and evasion were formulated, and control procedures implemented on a computer were proposed. The development of differential games theory with application to conflict-controlled systems by the 1990s was summarized by L.A. Petrosyan in his book [9]. The theory of differential games as applied to pursuit problems significantly evolved thanks to A.A. Melikyan, L.S. Vishnevetsky, N.V. Ovakimyan [10–13], and V.S. Patsko and S.S. Kumkov [14, 15]. At the 18th and 19th IFAC World Congresses, there were separate sections devoted to the theory of differential games and the practice of applying this theory to control problems in conflict states [15–19].

This paper considers a differential game with several players. One player (attacker) penetrates some space, and several other players (pursuers) appear simultaneously to intercept the attacker. Upon detecting the pursuers, the attacker tries to evade them. The dynamics of each player are described by a time-

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invariant linear system of a general type with scalar control. Note that this formulation of the game-theoretic problem is quite popular. For example, in the papers [19, 20], distributed game strategies for similar problems were developed and analyzed. The proposed solutions were based on the integration of cooperative control theory and differential game theory. As demonstrated therein, the proposed non-zero-sum game strategies are the Nash solution in terms of functionals (performance criteria) introduced to assess the players' actions. In this paper, a quadratic performance criterion is introduced, and the differential game is treated as an optimal control problem [21], i.e., a zero-sum differential game. Two subproblems are solved as follows. The first subproblem is to construct a strategy for pursuing the attacker by several players who have complete equal information about the game. The second subproblem is to construct such a strategy under incomplete information about the attacker who is actively opposing the pursuers. The simulation results are presented. The zero-sum differential game solution can be used for studying the final stage of pursuit, in which several pursuers and one evader participate.

This paper is organized as follows. Section 1 formulates the problem in which there are several pursuers and one attacker. The pursuers try to intercept the attacker, and the attacker tries to evade them. Each player can detect other players within its radius of sensitivity. Therefore, the game is a game with distributed information. Assumptions are made to exclude the cases when the attacker observes no pursuers or each pursuer observes no objects within its radius of sensitivity.

A common performance criterion is introduced in the zero-sum game to assess the actions of the pursuers and the attacker evading them. The pursuers seek to minimize this criterion, whereas the evading attacker to maximize it.

Section 2 considers the classical differential game with global information. The outcome of this game is based on optimal control theory. A theorem on the existence of solutions of the zero-sum differential game is proved. Also, Section 2 considers the differential game with distributed information.

Section 3 deals with a situation when the evading attacker artificially jams the pursuers to gain an advantage in the game. This means that the pursuers will receive information about the evader's position with some noise. Hence, the controls constructed by the pursuers will contain this noise. Thus, the trajectories along which the pursuers will intercept the evader are suboptimal. In addition, the attacker constructs its strategy for all pursuers detected, trying to escape the center of mass of all pursuers. Since their positions are subjected to noise, the attacker's trajectory will also contain a noise component.

Section 4 presents the simulation results for the differential game of pursuit in various statements considered in the previous sections.

1. PROBLEM STATEMENT

In the problem under consideration, the number of players is (n+ 1), namely, n pursuers and one attacker evading the pursuers. Each player can detect other players in its radius of sensitivity. Thus, the game is a game with distributed information. Let us make some assumptions.

Assumption 2.1. The observation between any pursuer–attacker pair is mutual, whereas the observation between two pursuers is not necessarily mutual.

Assumption 2.2. There exists at least one pursuer–attacker pair in which each member observes the other member, and each pursuer observes at least one other pursuer.

Without these assumptions, the following undesirable cases are possible in the problem: the attacker observes no pursuers, or each pursuer observes neither the attacker nor the other pursuers.

Suppose that the differential game of pursuit takes place in the m-dimensional Euclidean space. The positions of the players can be written as the vectors $y(t)=\left[y_1(t),y_2(t),...,y_m(t)\right]^{\mathrm{T}}$, $y(t)\in R^m$, for the attacker and $x_j(t)=\left[x_{j1}(t),x_{j2}(t),...,x_{jm}(t)\right]^{\mathrm{T}}$, $x_j(t)\in R^m$, for pursuer j = 1, 2,…, n, respectively. We introduce a vector $z_j(t)\in R^m$ of the form

$$z_j(t)=x_j(t)-y(t),\ \ j=1,2,3...,n,$$

which specifies the distance between the attacker and pursuer j. This vector determines the radius of sensitivity for each player.

Denoting $x^{\mathrm{T}}=\left[x_1^{\mathrm{T}},x_2^{\mathrm{T}},\dots,x_n^{\mathrm{T}}\right]$ and $z^{\mathrm{T}}=\left[z_1^{\mathrm{T}},z_2^{\mathrm{T}},\dots,z_n^{\mathrm{T}}\right]$, we compactly write the distance as

$$z(t)=x(t)-\mathbf{1}_n\otimes y(t),$$

where $\mathbf{1}_n$ is the unitary vector of dimensions $n\times 1$, and the symbol $\otimes$ indicates the Kronecker product. In the problems considered below, $t\in\left[t_0,t_f\right]$.

Assumption 2.3. Let us formulate the objectives of different players in this differential game. Consider a positive number $\varepsilon<1$:

– If at some instant t_1, $t_0 \le t_1 \le t_f$, the condition $\|z_j(t_1)\|^2 \le \varepsilon$ holds due to the actions of one or several pursuers, then the game ends because the attacker is intercepted. This outcome is the pursuers' objective in the game.

– If for any t, where $t_0 \le t \le t_f$, we have $\|z(t)\|^2 > \varepsilon$, i.e., the condition of interception is not valid, then at $t = t_f$ the game ends upon reaching the prescribed duration. This outcome is the attacker's objective in the game.

Let the game dynamics be described by an ordinary linear differential equation [9, 10] of the form

$$\frac{d}{dt}z(t) = u_p(t) - \mathbf{1}_n \otimes u_e(t), \qquad (1)$$

where $u_p(t) = \frac{d}{dt}x(t)$ and $u_e(t) = \frac{d}{dt}y(t)$ are the velocities of the pursuers and attacker, respectively.

In the non-zero-sum game for the system (1), we can introduce two performance criteria [19]:
The group of n pursuers strives to minimize the first criterion

$$J_p(z(\cdot),\ u_p(\cdot)) = \frac{1}{2}k_{pf}z^{\mathrm{T}}(t_f)z(t_f) + \frac{1}{2}\int_{t_0}^{t_f}\begin{bmatrix} z(t) \\ u_p(t) \\ \mathbf{1}_n \otimes u_e(t)\end{bmatrix}^{\mathrm{T}} \times \begin{bmatrix} q_p I & 0 & 0 \\ 0 & r_p I & 0 \\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix} z(t) \\ u_p(t) \\ \mathbf{1}_n \otimes u_e(t)\end{bmatrix}dt. \qquad (2)$$

The evading attacker seeks to maximize the second criterion

$$J_e(z(\cdot),\ u_e(\cdot)) = -\frac{1}{2}k_{ef}z^{\mathrm{T}}(t_f)z(t_f) + \frac{1}{2}\int_{t_0}^{t_f}\begin{bmatrix} z(t) \\ u_p(t) \\ \mathbf{1}_n \otimes u_e(t)\end{bmatrix}^{\mathrm{T}} \times \begin{bmatrix} -q_e I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & r_e I\end{bmatrix}\begin{bmatrix} z(t) \\ u_p(t) \\ \mathbf{1}_n \otimes u_e(t)\end{bmatrix}dt, \qquad (3)$$

where k_{pf}, k_{ef}, q_p, q_e, r_p, and r_e are positive parameters.

The first summand in the criterion (2) characterizes a finite value of the differential game, and the parameter ε determines the instant of successful interception, i.e., the fulfillment of the condition $\|z(t_1)\|^2 \le \varepsilon,\ t_0 \le t_1 \le t_f$. Hence, the non-execution of interception should be highly estimated by the pursuers. With this aspect in mind, in the case $\varepsilon < 1$, the parameter k_{pf} can be chosen as $k_{pf} = 1/\varepsilon$. For the evading attacker, the first summand in the criterion (3), which estimates the value of its game at the terminal instant, should be small. In other words, the parameter k_{ef} can be chosen as $k_{ef} = \varepsilon$.

According to these performance criteria, the pursuers strive to minimize the weighted distances between them and the evading attacker under the minimum energy costs. In contrast, the evading attacker seeks to maximize the weighted distances between it and the pursuers under the minimum energy costs.

Unlike [19, 20], this paper considers the zero-sum differential game. There is a common performance criterion minimized by the n pursuers and maximized by the evading attacker. Treating the differential game as an optimal control problem [21], we combine the criteria (2) and (3) as follows:

$$J_\Sigma(z(\cdot),\, u_p(\cdot),\, u_e(\cdot)) = J_p(z(\cdot),\, u_p(\cdot)) - J_e(z(\cdot),\, u_e(\cdot)) = \\ = \frac{1}{2}z^{\mathrm{T}}(t_f)Fz(t_f) + \frac{1}{2}\int_{t_0}^{t_f}\Big\{z^{\mathrm{T}}(t)Qz(t) + u_p^{\mathrm{T}}(t)Ru_p(t) + \\ -\big(\mathbf{1}_n \otimes u_e(t)\big)^{\mathrm{T}} P\big(\mathbf{1}_n \otimes u_e(t)\big)\Big\}dt, \qquad (4)$$

where $F = \left[k_{pf} + k_{ef}\right]I_n$, $Q = \left[q_p + q_e\right]I_n$, $R = r_p I_n$, and $P = r_e I_n$, the parameters k_{pf}, k_{ef}, q_p, q_e, r_p, and r_e are positive, and I_n is an identity matrix of dimensions $n \times n$.

The positive definiteness of the matrices F, Q, R, and P ensures the existence of optimal controls in this differential game [22]. As shown below, choosing the parameters r_p and r_e so that $r_p < nr_e$ corresponds to the case of "strong" pursuers. (In other words, the pursuers excel the evader by their dynamical capabilities.)

For the mathematical description of different situations (stages) in the game with distributed information, by analogy with the paper [19], we introduce the "sensitivity matrix"

$$S(t) = \begin{bmatrix} 1 & s_{01}(t) & s_{02}(t) & \dots & s_{0n}(t) \\ s_{10}(t) & 1 & s_{12}(t) & \dots & s_{1n}(t) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{n0}(t) & s_{n1}(t) & s_{n2}(t) & \dots & 1 \end{bmatrix}, \qquad (5)$$

where the subscript "0" indicates the evading attacker, and the subscripts from "1" to "n" the corresponding pursuers. For players i, j and an instant t, the parameter $s_{ij}(t)$, $t \in \left[t_0, t_f\right]$, $i,\ j = 0, 1, 2, \dots, n$, $0 \le s_{ij}(t) \le 1$, is the degree of significance of the information about the latter player's state used by the former player for accomplishing its objective in the differential game.

In the case $0 < s_{ij}(t) \le 1$, player i observes player j; otherwise ($s_{ij}(t) = 0$), does not. Since each player always observes itself, the diagonal elements of the matrix (5) are constant and equal to 1. Thus, at separate stages, the players' information to accomplish their objective in the differential game may change, reflected via appropriate controls of the attacker and pursuers. This paper does not consider any methods for determining the degree of significance, i.e., the problem of finding the parameters of the matrix $S(t)$ depending on the game conditions. The criterion (4) for the zero-sum differential game with distributed information will be presented in Section 2.2.

In the game with global information, the sensitivity matrix is constant, and its elements are equal to 1.

2. CLASSICAL DIFFERENTIAL GAME AND GAME WITH DISTRIBUTED INFORMATION

2.1. Classical differential game

The classical differential game is a game with global information. It rests on the theory of optimal control: the problem is to design controls $u_p^0(t)$ and $u_e^0(t)$ for which

$$J_\Sigma(z,\ u_p^0(t),\ u_e(t)) \le J_\Sigma(z,\ u_p^0(t),\ u_e^0(t)) \le J_\Sigma(z,\ u_p(t),\ u_e^0(t)).$$

For the classical differential game with several pursuers and linear feedback controls, we have the following result.

Theorem 3.1. *Consider a differential game with n pursuers and one evading attacker with the dynamics* (1) *and the performance criterion* (4). *This game has a value under the condition* $r_p < nr_e$ *if the strategies of players are given by*

$$u_p^0(t) = -\frac{1}{r_p}K(t)z(t),\ \ u_e^0(t) = = -\frac{1}{nr_e}(\mathbf{1}_n^{\mathrm{T}} \otimes I_m)K(t)z(t), \qquad (6)$$

where

$$\frac{d}{dt}K(t) = -K(t)\left[-\frac{1}{r_p}I_n + \frac{1}{nr_e}(\mathbf{1}_n \otimes \mathbf{1}_n^{\mathrm{T}} \otimes I_m)\right]\times \times K(t) - \left[q_p + q_e\right]I_n,\ K(t_f) = \left[k_{pf} + k_{ef}\right]I_n. \qquad (7)$$

This assertion is proved in the Appendix.

From equation (7) it follows that the matrix $K(t)$ is symmetric. A positive definite matrix $K(t)$ is selected from two possible solutions of equation (7). The positive definite property is established when determining the conditions for the existence of optimal controls in the classical differential game. For this purpose, we introduce the Lyapunov function with a positive definite symmetric matrix $K(t)$:

$$V(z(t)) = z^{\mathrm{T}}(t)K(t)z(t).$$

According to the Lyapunov theorem, the matrix equation (7) has a stable solution if

$$\frac{d}{dt}V(z(t)) = z^{\mathrm{T}}(t)\left\{\frac{d}{dt}K(t)\right\}z(t) + \left\{\frac{d}{dt}z(t)\right\}^{\mathrm{T}}K(t)z(t) + + z^{\mathrm{T}}(t)K(t)\left\{\frac{d}{dt}z(t)\right\} \le -z^{\mathrm{T}}(t)\left[q_p + q_e\right]z(t). \qquad (8)$$

Equation (1) with the controls (6) takes the form

$$\frac{d}{dt}z(t) = \left[-\frac{1}{r_p}I_n + \frac{1}{nr_e}(\mathbf{1}_n^{\mathrm{T}} \otimes I_m)\right]K(t)z(t). \qquad (9)$$

Due to equation (9), we write inequality (8) as

$$z^{\mathrm{T}}(t)\left[\frac{d}{dt}K(t) + K(t)\left[-\frac{1}{r_p}I_n + \frac{1}{nr_e}(\mathbf{1}_n \otimes \mathbf{1}_n^{\mathrm{T}} \otimes I_m)\right]^{\mathrm{T}} \times \times K(t) + \left[q_p + q_e\right]I_n\right]z(t) - - z^{\mathrm{T}}(t)K(t)\left[\frac{1}{r_p}I_n - \frac{1}{nr_e}(\mathbf{1}_n \otimes \mathbf{1}_n^{\mathrm{T}} \otimes I_m)\right]K(t)z(t) \le 0.$$

In view of (7), we obtain the following condition for the existence of optimal controls in the differential game:

$$z^{\mathrm{T}}(t)K(t)\left[\frac{1}{r_p}I_n - \frac{1}{nr_e}(\mathbf{1}_n \otimes \mathbf{1}_n^{\mathrm{T}} \otimes I_m)\right]K(t)z(t) \ge 0.$$

Obviously, this condition will hold if the bracketed matrix is positive definite, i.e.,

$$\frac{1}{r_p} > \frac{1}{nr_e}. \qquad (10)$$

This inequality can be satisfied by tuning the parameters r_p and r_e, or the penalty matrices $R = r_pI_n$ and $P = nr_eI_n$.

Let us formulate this result as follows.

Theorem 3.2. *The differential game* (1), (4) *has a value if the penalty matrices* R *and* P *of the performance criterion* (4) *satisfy the relation* $R \prec P$.

Note that under condition (10), the performance criterion with the controls (6) achieves the saddle point, i.e.,

$$J_\Sigma(z(\cdot),\ u_p^0(\cdot),\ u_e(\cdot)) \le J_\Sigma(z^0(\cdot),\ u_p^0(\cdot),\ u_e^0(\cdot)) \le J_\Sigma(z(\cdot),\ u_p(\cdot),\ u_e^0(\cdot)).$$

Condition (10) leads to a logical conclusion: the more the pursuers are, the more successful their game outcome will be.

Theorem 3.3. *Consider a differential game with n pursuers and one evading attacker with the dynamics* (1) *and the performance criterion* (4). *Let* $J_{\Sigma}^{0}(t,z(\cdot))$ *denote the minimax value achieved by* $J_{\Sigma}(z(\cdot),u_p(\cdot),u_e(\cdot))$ *under the feedback optimal controls. This value is*

$$J_{\Sigma}^{0}(t,z(t))=\frac{1}{2}z^{\mathrm{T}}(t)K(t)z(t),\ t_0\le t\le t_f,$$

where $K(t)$ *is a symmetric positive definite matrix satisfying equation* (7) *with the right-end boundary condition.*

This assertion is proved in the Appendix.

In the case $n=1$ (only one pursuer), the controls take the form

$$u_p^0(t)=-\frac{k_p(t)}{r_p}z(t),\ \ u_e^0(t)=-\frac{k_e(t)}{r_e}z(t),$$

where the parameters $k_p(t)$ and $k_e(t)$ satisfy equations (7) with $K(t)=\left[k_p(t)+k_e(t)\right]I_m$ and $n=1$:

$$\frac{d}{dt}k_p(t)-\left[\frac{r_e-r_p}{r_e r_p}\right]k_p^2(t)-\frac{2}{r_p}k_p(t)k_e(t)+q_p=0,$$
$$k_p(t_f)=k_{pf},\qquad (11)$$

$$\frac{d}{dt}k_e(t)-\left[\frac{r_e-r_p}{r_e r_p}\right]k_e^2(t)+\frac{2}{r_e}k_p(t)k_e(t)+q_p=0,$$
$$k_e(t_f)=k_{ef}.\qquad (12)$$

2.2. Differential game with distributed information

The main idea of constructing strategies in the differential game with distributed information is that each player makes a decision based on the available information at a given time instant. The dynamics of the information available to the players (the attacker and pursuers) to form their controls are described by the sensitivity matrix (5).

In general form, the distance between pursuer j, the attacker, and the other pursuers is specified by the vector

$$\tilde{z}_{pj}(t)=x_j(t)-\sum_{i=1}^{n}d_{ij}(t)x_i(t)-f_j(t)y(t).\qquad (13)$$

If the evading attacker observes the actions of several pursuers, then the following information can be available to it:

$$\tilde{z}_e(t)=\sum_{i=1}^{n}e_i(t)x_i(t)-y(t).\qquad (14)$$

Like in the paper [20], the coefficients $d_{ij}(t)$, $f_j(t)$, and $e_i(t)$ in (13) and (14) are composed of the elements of the matrix (5) characterizing the mutual observations of the players:

$$d_{ij}(t)=\left[1-s_{0j}(t)\right]\frac{s_{ij}(t)}{\sum_{l=1}^{n}s_{jl}(t)},$$

$$f_j(t)=s_{0j}(t),\ \ e_j(t)=\frac{s_{0j}(t)}{\sum_{i}^{n}s_{0i}(t)}.$$

The pursuers' strategies are

$$u_{pj}^0(t)=-\frac{k_p(t)}{r_p}\tilde{z}_{pj}(t)=$$
$$=-\frac{k_p(t)}{r_p}\left[x_j(t)-\sum_{i=1}^{n}d_{ij}(t)x_i(t)-f_j(t)y(t)\right]\qquad (15)$$

for j = 1, 2,..., n.

The evading attacker forms its control using the available information (14):

$$u_e^0(t)=-\frac{k_e(t)}{r_e}\tilde{z}_e(t)=-\frac{k_e(t)}{r_e}\left[\sum_{i=1}^{n}e_i(t)x_i(t)-y(t)\right].\ (16)$$

The parameters $k_p(t)$ and $k_e(t)$ in (15) and (16) satisfy equations (11) and (12) with $K(t)=\left[k_p(t)+k_e(t)\right]I_n$ and $n=1$.

Note that these control formulas have been obtained for the system dynamics (1) with the quadratic performance criteria (2) and (3).

We write the expressions (15) and (16) compactly using the Kronecker product:

$$u_p^0(t)=-R^{-1}K(t)\{x(t)-\left[D(t)\otimes I_m\right]x(t)-$$
$$-F(t)\otimes y(t)\},\qquad (17)$$

$$u_e^0(t)=-P^{-1}(\mathbf{1}_n^{\mathrm{T}}\otimes I_m)K(t)\times$$
$$\times\left\{\left[E^{\mathrm{T}}(t)\otimes I_m\right]x(t)-y(t)\right\}.\qquad (18)$$

Here $K(t)$ are solutions of equations (7), $E(t)=\left[e_1(t)\ldots e_n(t)\right]^{\mathrm{T}}$, $F(t)=\left[f_1(t)\ldots f_n(t)\right]^{\mathrm{T}}$, and $D(t)=\left[d_{ij}(t)\right]\in R^{n\times n}$.

Substituting the optimal controls (15) and (16) into the criterion (4) and performing some transformations, we obtain

$$J_{\Sigma}^{0}\left(z(t),\,x(t),\ y(t)\right)=\frac{1}{2}z\left(t_f\right)Fz\left(t_f\right)+$$
$$+\frac{1}{2}\int_{t_0}^{t_f}\left\{x^{\mathrm{T}}(t)\cdot H\left(D(t),E(t)\right)\cdot x(t)\right\}dt+$$
$$+\frac{1}{2}\int_{t_0}^{t_f}\left\{x^{\mathrm{T}}(t)\cdot L\left(D(t),\,E(t)\right)\cdot y(t)+\right.$$
$$\left.+y^{\mathrm{T}}(t)\cdot W\left(F(t)\right)\cdot y(t)\right\}dt,\qquad (19)$$

where

$$H\left(D(t),\,E(t),\;K(t)\right)=Q+K(t)R^{-1}K(t)-K(t)\times$$
$$\times R^{-1}K(t)D(t)\otimes I_m-\left[D(t)\otimes I_m\right]^{\mathrm{T}}K(t)R^{-1}K(t)+$$
$$+\left[D(t)\otimes I_m\right]^{\mathrm{T}}K(t)\times^{-1}K(t)D(t)\otimes I_m-$$
$$-n\left[F(t)\otimes I_m\right]^{\mathrm{T}}\left[\left(1_n^{\mathrm{T}}\otimes I_m\right)K(t)\right]^{\mathrm{T}}P^{-1}\times$$
$$\times\left[\left(1_n^{\mathrm{T}}\otimes I_m\right)K(t)\right]F(t)\otimes I_m,$$
$$L\left(D(t),\,E(t),\;F(t)\right)=2\left[D(t)\otimes I_m\right]^{\mathrm{T}}\times$$
$$\times K(t)R^{-1}K(t)\left[F(t)\otimes I_m\right]-2K(t)R^{-1}K(t)\times$$
$$\times\left[F(t)\otimes I_m\right]-2\left(\mathbf{1}_n^{\mathrm{T}}\otimes I_m\right)^{\mathrm{T}}Q+2n\left[E^{\mathrm{T}}(t)\otimes I_m\right]^{\mathrm{T}}\times$$
$$\times\left[\left(\mathbf{1}_n^{\mathrm{T}}\otimes I_m\right)K(t)\right]^{\mathrm{T}}P^{-1}\left[\left(\mathbf{1}_n^{\mathrm{T}}\otimes I_m\right)K(t)\right],$$
$$W\left(F(t),K(t)\right)=nQ+\left[F(t)\otimes I_m\right]^{\mathrm{T}}\times$$
$$\times K(t)R^{-1}K(t)F(t)\otimes I_m-$$
$$-n\left[\left(\mathbf{1}_n^{T}\otimes I_m\right)K(t)\right]^{\mathrm{T}}P^{-1}\left(\mathbf{1}_n^{T}\otimes I_m\right)K(t).$$

Clearly, the integrand of the criterion (19) considers both the positions of the pursuers and evader and the mutual disposition of the pursuers and evader. Note that under the optimal controls, the value of this functional depends on the number of players and the elements $d_{ij}(t)$, $f_i(t)$, and $e_i(t)$ of the matrices $D(t)$, $F(t)$, and $E(t)$, respectively. In other words, the value depends on what information is available to the players during the game and the type of availability.

In the game with one pursuer, one evading attacker, and global information (the classical differential game; see subsection 2.1), the parameters are $n=1$, $D(t)=0$, and $E(t)=1$. As a result, the controls (17) and (18) become the same as in (6). In this case, the criterion takes the form

$$J_{\Sigma}^{0}(z(t))=\frac{1}{2}z(t_f)Fz(t_f)+\frac{1}{2}\int_{t_0}^{t_f}z^{\mathrm{T}}(t)\times$$
$$\times\left\{Q+K(t)R^{-1}K(t)-K(t)P^{-1}K(t)\right\}z(t)dt.$$

According to Theorem 3.3, $J_{\Sigma}^{0}(t,z(t))$ is given by

$$J_{\Sigma}^{0}(t,z(t))=\frac{1}{2}z^{\mathrm{T}}(t)K(t)z(t),\ t_0\le t\le t_f.$$

Consider a particular case of the differential game with distributed information and the binary sensitivity matrix (5). The element $s_{ij}(t)$ of this binary matrix indicates whether player *i* observes player *j* at a time instant *t* or not: if $s_{ij}(t)=1$, then player *i* observes player *j*; otherwise ($s_{ij}(t)=0$), player *i* does not observe player *j*. Since each player observes itself, the diagonal elements of the matrix (5) are constant and equal to 1.

Case 1. Let pursuer *j* not observe the evading attacker, i.e., $s_{0j}=0,\ f_j=0$. From the expression (17) we therefore have

$$u_{pj}^{0}(t)=-\frac{k_p(t)}{r_p}\left[x_j(t)-\sum_{i=1}^{n}d_{ij}(t)x_i(t)\right].$$

This means that pursuer *j* will follow the nearest observable pursuers.

Case 2. Let pursuer *j* observe the evading attacker, i.e., $s_{0j}=1,\ f_j=1$, and let this player have no information about the other pursuers, i.e., $d_{ij}=0$. From the expression (16) we therefore have

$$u_{pj}^{0}(t)=-\frac{k_p(t)}{r_p}\left[x_j(t)-y(t)\right].$$

This means that pursuer *j* will try to intercept the evading attacker independently.

Consider the evading attacker's strategy (18), noting the following: if the evader observes several pursuers in its radius of sensitivity, then its control will be intended to "escape" the center of mass of all the pursuers detected.

3. DIFFERENTIAL GAME WITH NOISE

Consider a situation when the evading attacker artificially jams the pursuers to gain an advantage in the differential game of pursuit. This means that the pursuers will receive information about the evader's position with some noise. Hence, the controls constructed by the pursuers will contain this noise. Thus, the trajectories along which the pursuers will intercept the evader are suboptimal. In addition, the attacker constructs its strategy for all pursuers it detects, trying to escape the center of mass of all pursuers. Since their positions are subjected to noise, the attacker's trajectory will also contain a noise component. Note that the evading attacker will not be affected by the noise it creates, and its control strategy still depends only on the pursuers' positions.

The controls of the pursuer and attacker in the differential game with global information are given by

$$u_p(t)=-\frac{k_p(t)}{r_p}z(t),\ \ u_e(t)=\frac{d}{dt}y(t).$$

Let $n(t)$ be the noise created by the evading attacker, representing the white noise with the mean $M\left[n(t)\right]=0$ and variance $M\left[n^{\mathrm{T}}(t)n(\tau)\right]=$ $=N(t)\delta(t-\tau)$. Under the new conditions in the differential game, the pursuers will detect the evading attacker along the trajectory $y^{*}(t)=y(t)+n(t)$. Note that the presence of noise may affect the condition of interception (the pursuers' objective in the game); see Assumption 2.3. We therefore introduce a new condition of interception:

$$E\left[\left\|z_j^*(t_1)\right\|^2\right] \le \varepsilon,$$

where $E\left[\left\|z_j^*(t_1)\right\|^2\right]$ is the root-mean-square distance between the attacker and pursuer j, or equivalently, $E\left[\left\|z_j(t)-n(t)\right\|^2\right] \le \varepsilon$. Due to the white noise created by the attacker, we have $E\left[n^{\mathrm{T}}(t)z_j(t)\right]=0$, and the condition of interception takes the form

$$E\left[\left\|z(t)\right\|^2\right] \le \varepsilon - N.$$

If $N > \varepsilon$, the objective of interception cannot be accomplished.

We write the control strategies in the classical differential game:

$$u_p(t) = -\frac{k_p(t)}{r_p} z^*(t) = -\frac{k_p(t)}{r_p}\left[z(t)-n(t)\right].$$

Also, we write the control strategies in the differential game with distributed information:

$$u_e(t) = -\frac{k_e(t)}{r_e}\left[\sum_{i=1}^{n} e_i(t)x_i(t) - y(t)\right],$$

$$u_{pj}(t) = -\frac{k_p(t)}{r_p}\left[x_j(t) - \sum_{i=1}^{n} d_{ij}(t)x_i(t) - f_j(t)y^*(t)\right].$$

If a pursuer does not detect the evading attacker (therefore constructing its control strategy based on the pursuers detecting the attacker), its trajectory will still have a noisy component due to the noisy trajectories of the latter pursuers.

4. EXAMPLE

4.1. Classical differential game

Let us simulate a differential game in which each player has complete information (game with global information). Assume that there are one attacker and three pursuers. Then the game dynamics with the constructed controls are described by an ordinary linear differential equation of the form

$$\frac{d}{dt}x_1(t) = -\frac{k_p(t)}{r_p}(x_1(t)-y(t)),\ x_1(t_0) = [-3,\ 0]^{\mathrm{T}},$$

$$\frac{d}{dt}x_2(t) = -\frac{k_p(t)}{r_p}(x_2(t)-y(t)),\ x_2(t_0) = [3,\ 0]^{\mathrm{T}},$$

$$\frac{d}{dt}x_3(t) = -\frac{k_p(t)}{r_p}(x_3(t)-y(t)),\ x_3(t_0) = [4,\ 1]^{\mathrm{T}},$$

$$\frac{d}{dt}y(t) = -\frac{k_e(t)}{r_e}\left\{\frac{1}{3}\left[x_1(t)+x_2(t)+x_3(t)\right]-y(t)\right\},$$

$$y(t_0) = [0,\ 3]^{\mathrm{T}}.$$

Here the parameters $k_p(t)$ and $k_e(t)$ satisfy the equations

$$\frac{d}{dt}k_p(t) = -q_p + \frac{1}{r_p}k_p^2(t) - \frac{2}{r_e}k_p(t)k_e(t),$$

$$k_p(t_f) = k_{pf} = 1/\varepsilon,$$

$$\frac{d}{dt}k_e(t) = -q_e - \frac{1}{r_e}k_e^2(t) + \frac{2}{r_p}k_p(t)k_e(t),$$

$$k_e(t_f) = k_{ef} = \varepsilon.$$

For the performance criterion (4), we choose the parameters $r_p = 1$, $r_e = 2$, $q_p = 1$, $q_e = 2$, $k_{pf} = 20$, and $k_{ef} = 0.05$. For the condition of interception, we choose the parameter $\varepsilon = 0.04$. Let the game of pursuit occur for $t \in [0, 4]$.

The variations of the parameters $k_p(t)$ and $k_e(t)$ over time are shown in Fig. 1.

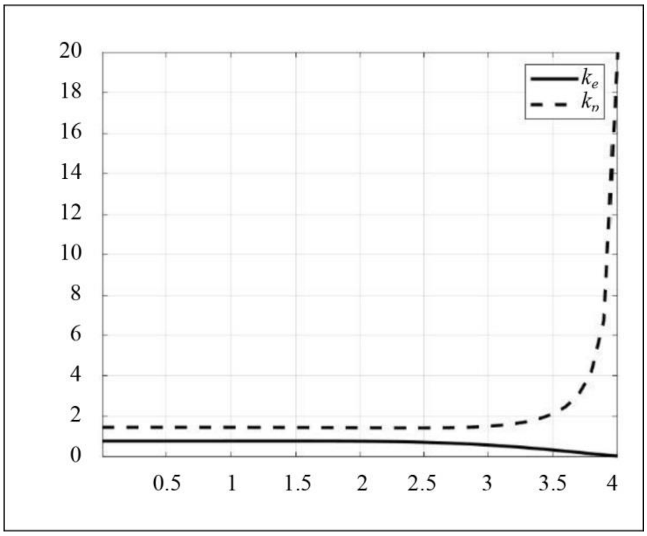

Fig. 1. Variations of parameters $k_p(t)$ and $k_e(t)$.

The graphs of the transient processes in the problems are presented below. Figure 2 shows the trajectories of the pursuers and evading attacker in the classical game with and without noise. In both of the games, the attacker has been intercepted: the condition $\left\|z(t_1)\right\| \le \varepsilon,\ t_1 < 4s$, has been satisfied at $t_1 = 3.58$ (Fig. 2*a*) and $t_1 = 3$ (Fig. 2*b*), where time is measured in conditional machine units.

The classical differential game with centered noise has been simulated using the original model with the same initial conditions.

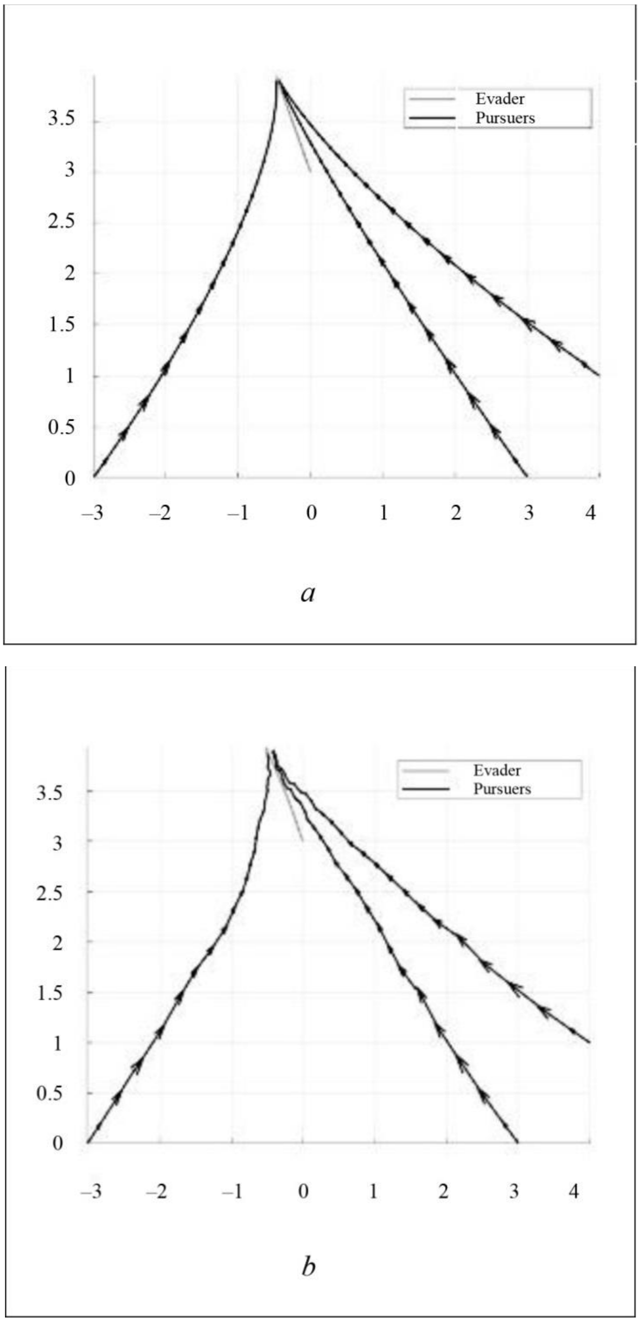

Fig. 2. Transient processes in simple game of pursuit:
(*a*) without noise (game ends at 3.58) and (*b*) with noise (game ends at 3).

4.2. Differential game with distributed information

Suppose that the initial position of the players is the same as in subsection 4.1. We simulate the differential game in which each player has limited information about the other players. Let the sensitivity matrix change three times during the game, which can be expressed as follows:

$$S_1 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \; S_2 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \; S_3 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

In the first period, only one pursuer detects the attacker, and the two other pursuers, detecting the first pursuer only, follow it. In the next period, two pursuers detect the evading attacker and try to intercept it, while the remaining pursuer follows them. In the final period, each of the players detects the others, and the differential game turns into a game with global information (classical differential game).

Like in subsection 4.1, we adopt the differential game with distributed strategies as the basis model and add noise. All operations to obtain a solution are performed by analogy with the previous subsection. Figure 3 shows the trajectories of the pursuers and evader in the game with distributed strategies, without noise and with noise.

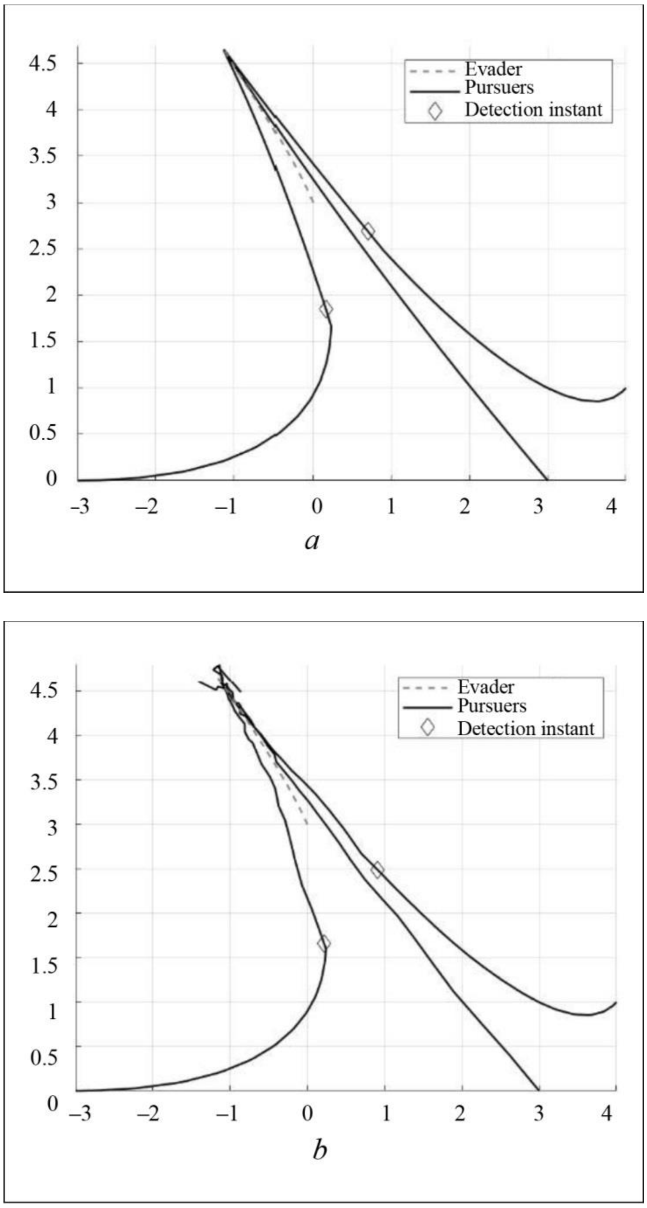

Fig. 3. Transient processes in game of pursuit with distributed information:
(*a*) without noise (game ends at 3.88) and (*b*) with noise (game ends at 4).

The graphs in Fig. 3 show the instants when different pursuers join the pursuit of the evading attacker (when the attacker enters their zones of sensitivity, i.e., at the instants of detection). Figure 3*a* corresponds to the successful interception of the attacker, i.e., the condition $\|z_j(t_1)\|^2 \le \varepsilon,\ t_1 < 4\ s$, is satisfied for $t_1 = 3.88$. Figure 3*b* corresponds to the unsuccessful interception of the attacker, i.e., the condition $\|z(t)\|^2 > \varepsilon$ holds for all $t_0 \le t \le t_f$, and the game ends upon reaching the prescribed duration $t = t_f = 4$.

CONCLUSIONS

This paper has considered a differential game of pursuit with several players. One player (attacker) penetrates some space, and several other players (pursuers) appear simultaneously to intercept the attacker. Upon detecting the pursuers, the attacker tries to evade them. The dynamics of each player are described by a time-invariant linear system of a general type. The strategies of the pursuers and evading attacker have been constructed within two subproblems: (1) all players have complete information about the state of all game participants, and (2) the pursuers have incomplete information about the evading attacker actively opposing them. The distributed strategies and some particular cases of the differential game of pursuit have also been considered. The main idea of constructing strategies for this game is that each player makes a decision based only on the available information at a given time. The simulation results have been provided to illustrate the theory.

APPENDIX

Proof of Theorem 3.1. Let us write the system's Hamiltonian

$$H(z, u_p, u_e, \lambda) = \frac{1}{2}\Big\{z^{\mathrm{T}}(t)\big[q_p + q_e\big]z(t) + r_p u_p^{\mathrm{T}}(t)u_p(t) -$$

$$-nr_e\left(\mathbf{1}_n \otimes u_e(t)\right)^{\mathrm{T}}\left(\mathbf{1}_n \otimes u_e(t)\right)\Big\} + \lambda^{\mathrm{T}}(t)\big[u_p(t) - \mathbf{1}_n \otimes u_e(t)\big].$$

Here, $\lambda(t)$ is the conjugate variable [22], which satisfies the equation

$$\frac{d}{dt}\lambda(t) = -\left\{\frac{\partial H(z, u_p, u_e, \lambda)}{\partial z}\right\}^{\mathrm{T}} = -\big[q_p + q_e\big]z(t) \quad \text{(A.1)}$$

with the boundary condition

$$\lambda(t_f) = \frac{1}{2}\frac{\partial z^{\mathrm{T}}(t_f)\big[k_{pf} + k_{ef}\big]z(t_f)}{\partial z} = \big[k_{pf} + k_{ef}\big]z(t_f)\,.$$

The optimal controls are the stationary points of the Hamiltonian:

$$\frac{\partial H(z, u_p, u_e, \lambda)}{\partial u_p} = 0, \quad \frac{\partial^2 H(z, u_p, u_e, \lambda)}{\partial u_p^2} = r_p > 0, \quad \text{(A.2)}$$

$$\frac{\partial H(z, u_p, u_e, \lambda)}{\partial u_e} = 0, \quad \frac{\partial^2 H(z, u_p, u_e, \lambda)}{\partial u_e^2} = -nr_{ep} < 0\,. \quad \text{(A.3)}$$

Conditions (A.2) and (A.3) determine the optimal controls

$$u_p^0(t) = -\frac{1}{r_p}\lambda(t), \quad u_e^0(t) = -\frac{1}{n\,r_e}(\mathbf{1}_n^{\mathrm{T}} \otimes \lambda(t))\,. \quad \text{(A.4)}$$

Therefore, the variable $\lambda(t)$ is the solution of the two-point boundary value problem (the Euler–Lagrange equations)

$$\frac{d}{dt}z(t) = \left[-\frac{1}{r_p}I_m + \frac{1}{n\,r_e}(\mathbf{1}_n^{\mathrm{T}} \otimes I_m)\right]\lambda(t),$$

$$z_0(t) = z_0,$$

$$\frac{d}{dt}\lambda(t) = -\big[q_p + q_e\big]z(t),$$

$$\lambda(t_f) = \big[k_{pf} + k_{ef}\big]z(t_f).$$

The auxiliary variable $\lambda(t)$ will be calculated using the sweep method [22]. Let us find $\lambda(t)$ in the form

$$\lambda(t) = K(t)z(t), \quad \text{(A.5)}$$

where $K(t)$ is an unknown matrix. The total derivative of the expression (A.5) is given by

$$\frac{d}{dt}\lambda(t) = \left\{\frac{d}{dt}K(t)\right\}z(t) + K(t)\left\{\frac{d}{dt}z(t)\right\} =$$

$$= \left[\left\{\frac{d}{dt}K(t)\right\} + K(t)\left[-\frac{1}{r_p}I + \frac{1}{nr_e}\times\right.\right.$$

$$\left.\left.\times(\mathbf{1}_n \otimes \mathbf{1}_n^{\mathrm{T}} \otimes I_m)\right]K(t)\right]z(t)\Big]. \quad \text{(A.6)}$$

Equalizing the expressions (A.1) and (A.6), we obtain:

$$\frac{d}{dt}K(t) = -K(t)\left[-\frac{1}{r_p}I_m + \frac{1}{nr_e}(\mathbf{1}_n \otimes \mathbf{1}_n^{\mathrm{T}} \otimes I_m)\right]\times K(t) -$$

$$-\big[q_p + q_e\big]I_m, \quad K(t_f) = \big[k_{pf} + k_{ef}\big]I_m.$$

Due to (A.4) and (A.5), the equations take the form:

$$u_p^0(t) = -\frac{1}{r_p}K(t)z(t),$$

$$u_e^0(t) = -\frac{1}{nr_e}(\mathbf{1}_n^{\mathrm{T}} \otimes I_m)K(t)z(t).\blacklozenge$$

Proof of Theorem 3.3. Consider the integrand of the performance criterion

$$J_\Sigma(z(\cdot), u_p(\cdot), u_e(\cdot)) = \frac{1}{2}z^{\mathrm{T}}(t_f)Fz(t_f) +$$

$$+\frac{1}{2}\int_t^{t_f}\Big\{z^{\mathrm{T}}(t)Qz(t) + u_p^{\mathrm{T}}(t)Ru_p(t) - \left(\mathbf{1}_n \otimes u_e(t)\right)^{\mathrm{T}} \times$$

$$\times P\left(\mathbf{1}_n \otimes u_e(t)\right)^{\mathrm{T}} P\left(\mathbf{1}_n \otimes u_e(t)\right)\Big\}dt.$$

Substituting $d\big[z^{\mathrm{T}}(t)K(t)z(t)\big]/dt$ into this integrand and compensating the result outside the integral by $0.5\big[z^{\mathrm{T}}(t)K(t)z(t) - z^{\mathrm{T}}(t_f)K(t_f)z(t_f)\big]$, we have:

$$J_{\Sigma}(z(\cdot),u_p(\cdot),u_e(\cdot)) = \frac{1}{2}z^{\mathrm{T}}(t_f)Fz(t_f) +$$
$$+\frac{1}{2}\left[z^{\mathrm{T}}(t)K(t)z(t) - z^{\mathrm{T}}(t_f)K(t_f)z(t_f)\right] +$$
$$+\frac{1}{2}\int_t^{t_f}\left\{z^{\mathrm{T}}(t)Qz(t) + u_p^{\mathrm{T}}(t)Ru_p(t) -\right.$$
$$\left. -\left(\mathbf{1}_n \otimes u_e(t)\right)^{\mathrm{T}} P\left(\mathbf{1}_n \otimes u_e(t)\right)\right\}dt +$$
$$+\frac{1}{2}\int_t^{t_f}\left\{z^{\mathrm{T}}(t)\left\{\frac{dK(t)}{dt}\right\}z(t) +\right.$$
$$\left. +\left\{\frac{dz^{\mathrm{T}}(t)}{dt}\right\}K(t)z(t) + z^{\mathrm{T}}(t)K(t)\left\{\frac{dz(t)}{dt}\right\}\right\}dt. \quad (\text{A}.7)$$

Note that under the optimal controls

$$u_p^0(t) = -R^{-1}K(t)z(t), \;\; u_e^0(t) = -P^{-1}\left(\mathbf{1}_n^{\mathrm{T}} \otimes K(t)z(t)\right), \quad (\text{A}.8)$$

where

$$\frac{d}{dt}K(t) + K(t)\left[-R^{-1} + P^{-1}(\mathbf{1}_n \otimes \mathbf{1}_n^{\mathrm{T}})\right]\times$$
$$\times K(t) + Q = 0, \; K(t_f) = F, \quad (\text{A}.9)$$

the system's dynamics are described by

$$\frac{d}{dt}z(t) = \left[-R^{-1} + P^{-1}(\mathbf{1}_n^{\mathrm{T}} \otimes I_m)\right]K(t)z(t),$$
$$z(t_0) = z_0. \quad (A.10)$$

Recall that $J_{\Sigma}^0(t,z(t))$ denotes the minimax value of the criterion $J_{\Sigma}(z(t), u_p(t), u_e(t))$. Substituting (A.8) and (A.10) into (A.7) and taking (A.9) into account, we finally arrive in

$$J_{\Sigma}^0(t,z(t)) = \frac{1}{2}z^{\mathrm{T}}(t)K(t)z(t), \;\; t_0 \le t \le t_f. \;♦$$

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UNCONSTRAINED OPTIMIZATION OF A TIME-VARYING OBJECTIVE FUNCTION ON A DISCRETE TIME SCALE

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Abstract. This paper develops an approximate method to optimize a time-varying objective function on a discrete time scale. The method should provide an admissible (controllable) error value. The conditions to be satisfied by the time scale, the objective function, and the environment's parameters are established. The unconstrained optimization of a time-varying objective function that depends on the control vector components is considered on a discrete set of time instants. To find a solution, a discrete gradient constrained optimization method is proposed. Efficiency conditions for the gradient method are formulated. A lower bound on the solution error is obtained in terms of the time step, the rate of change of the objective function, and its first- and second-order derivatives with respect to the control vector components. The method is illustrated on a numerical example of an optimal controller design for a time-varying plant with a nonlinear objective function. According to the numerical experiments, the wide-range variations of the controller's parameters have no significant effect on the qualitative behavior of the resulting trajectory. The method can be used to calculate an optimal control function for a system with a discrete-valued objective function.

Keywords: time-varying system, optimal controller, unconstrained optimization, lower bound on error.

INTRODUCTION

Purposeful developing systems, such as the national and regional economy or large multiple product farms, use optimal management mechanisms to maximize a target indicator [1]. This indicator can be total output, added value, profit, etc. Under crisis conditions, in an unpredictable environment (economic sanctions, financial catastrophes, force majeure), the management methods based on knowledge of normal business processes do not provide the desired result. For example, the international division of labor, which usually plays a positive role, becomes useless under sanctions. In such a situation, successful economic management should primarily focus on its internal capabilities and closed technological cycles within the economic system, thereby being autonomous in some sense [2].

Mathematical models for managing developing systems under crisis conditions may have little or even no accuracy. In such cases, management has to be limited only to a set of target indicators. In addition, the available statistical reports usually provide economic indicators only for certain periods (month, quarter, or year). All these factors restrict the applicability of any methods involving a smooth objective function and should be considered when developing appropriate management and decision-making methods.

Under uncertain behavioral rules of an object, the most appropriate method is optimal control. It consists in determining and maintaining a mode of operation in which the optimal (minimum or maximum) value of some criterion characterizing the object's performance is achieved. The construction of management mechanisms for autonomous system models has much in common with the design of an optimal controller that automatically finds and maintains the optimal value of the controlled variable. It ensures some stability of the controlled object (often called plant). The optimal controller's applicability is restricted since we cannot manage the long-term consequences of its operation.

In addition, in the case of a limited amount of information about the object, its inertial properties may be neglected.

In the 1960s, optimal (extremal) control formed an independent branch in the theory of nonlinear automatic control systems [3], and optimal controllers became widespread. For example, they were used in optimal relay systems [4] and pulse self-adjusting (adaptive) and optimal automatic control systems [5]; when tuning resonance loops and automatic measuring devices; when finding the optimal parameters of tunable models; when controlling chemical reactors and heating units for flotation and crushing [6].

Depending on the available information about the plant, the control laws in optimal controllers involve various approaches, differing in their validity and convergence of the result to the optimum. For example, in the paper [7], a heuristic extreme regulation algorithm was proposed to simulate the metabolic process. At each iteration, this algorithm performs a random search for the best response. The convergence of the process was demonstrated using an example for a particular object. In the paper [8], the air supply u in the furnace was regulated using the optimal control of the inertial object's static characteristic f as follows:

$$\Delta u(i+1) = h \operatorname{sign} [\Delta f(i+1)\, \Delta u(i)],$$

where u and f are the scalar control parameter and the measurable response parameter (control criterion), respectively; i denotes the time instant. The convergence of the control process in the cited paper was also demonstrated experimentally. A particular case of an optimal controller with a nonlinear objective function was considered in [9] for a linear dynamic system described by an ordinary first-order differential equation. The control step was chosen constant, and its sign was inverse to that of the derivative with respect to the control variable. The convergence of the control process was proved under the exactly known system's dynamics and derivatives of the objective function with respect to the time and control variables. The algorithm proposed below does not require this knowledge.

Consider a time-varying autonomous system model. For this system, let a control u be designed by

$$F(y, u) \to \underset{u}{\text{opt}}$$

at each point t, where $u(\text{t})$ denotes the control vector; $F(y, u)$ is an objective function that satisfies smoothness and convexity in u; $y(t,\ p,\ x, u)$ denotes the system's state vector; p is the parameters vector; finally, $x(t)$ is the environment's state vector. If such an optimal control exists, it depends on the current state of the system and can be determined, e.g., using the gradient-based unconstrained optimization method from the first-order optimality condition

$$\frac{dF}{du} = 0\ .$$

If at the corresponding time instants only the values of the objective function are known, and the current state of the system is considered implicitly, we will find the control by the value of this function, denoting

$$f(t,\ u) = F(\ y(t,\ p,\ x(t), u), u)\,.$$

For the nonstationary problems of this type, the convergence of the gradient-based unconstrained optimization method was considered in the paper [10]. Assuming the exactly known gradient of the objective function, the convergence of the discrete-time iterative process

$$u(t_{k+1})\ =\ u(t_k)\ - \gamma_k \nabla_u f(t_k,\ u(t_k))$$

was established under the requirement

$$\exists a > 0 : \left\| u^*(t_{k+1}) - u^*(t_k) \right\| \le a\,,$$

where $u^*(t)$ is the optimum of the function $f(t, u)$ at the time instant t; a specifies the deviation of the limit value from the optimum $u^*(t_k)$ as $k \to \infty$; the step γ_k is determined by the properties of the matrix $\nabla_{uu} f$. The results of [10] were further developed in the paper [11] by weakening the convergence condition of the iterative process:

$$\left\| \nabla_u f(t_{k+1}, u) - \nabla_u f(t_k, u) \right\| \le a,\ \ a > 0\,.$$

When considering nonstationary unconstrained optimization problems in [10, 11], both the deviation of the solution from the exact value at the current time instant t and the limit deviation as $t \to \infty$ were controlled. As a disadvantage of purely gradient methods, note the relatively slow convergence to the exact solution, which can be explained by the following fact: when approaching the optimum of the objective function with smooth derivatives, the gradient norm $\left\| \nabla_u f(t_k, u) \right\|$ may tend to 0 faster than the growth of the step γ_k. Therefore, the approximate solution "lags" the exact counterpart at every step. The methods proposed in [8, 9] allow the advance of the exact solution, which does not reduce their errors but accelerates their convergence.

If the objective function values are measured at discrete instants, the derivative $\nabla_u f(t, u)$ can be estimated only approximately, for example, using the spline representation of the function $f(t, u)$ (ambiguously). The expected consequence of such assumptions is an increase in the solution error compared to the methods based on the accurate estimation of the objective function's derivatives. Moreover, the solution error will not vanish over time, i.e., it cannot be eliminated. Naturally, its value should increase with an increase in the discretization step of the time interval and the rate of change of the objective function. Accordingly, "the deviation of the solution from the exact value" becomes an incorrect concept due to the latter's ambiguity. It can be replaced by "the deviation of the solution from an exact value" or "the deviation of the solution from the set of exact values."

This paper develops an approximate method to optimize a time-varying objective function on a discrete time scale. The method should provide an admissible (controllable) error value. The conditions to be satisfied by the time scale, the objective function, and the environment's parameters are established. This method can be used to design an optimal controller for a system defined at discrete time instants.

1. PROBLEM STATEMENT

Let the objective function $f(t, u)$, where t is the scalar time and $u \in R^n$, be continuously differentiable with respect to both variables and convex in u. Also, let this function together with the vector $u(t)$ be given at discrete time instants $t_1 < t_2 < ... < t_i$.

Consider the unconstrained optimization of the objective function at the time instant t_{t+1}:

$$f(t_{i+1}, u) \to \underset{u}{\text{opt}}.$$

More precisely, the problem is to find the control vector value $u = u(t_{i+1})$ approximating the objective function value $f(t_{i+1}, u)$ to the optimum using the values $f(t_j, u(t_j))$, $j \le i$, and estimate the resulting error.

2. BASIC RESULTS

Let the function $f(t,u)$ have continuous first-order partial derivatives with respect to both variables. We introduce the following notations for the kth components of the vectors: $u^k_i = u^k(t_i)$, $k = 1,..., n$, $f_i = f(t_i, u_i)$, where $i = 1, 2,...$ The first-order partial derivatives on the two-point data have the approximations

$$\left.\frac{\partial f}{\partial u^k}\right|_i \cong \frac{f_{i+1} - f_i}{u^k_{i+1} - u^k_i} = \frac{\Delta f_i}{\Delta u^k_i},$$

written in the vector form as

$$\nabla_u f_i \cong \bar{\nabla}_u f_i.$$

Here the gradient and its approximations apply to the values of the variables t_i and u_i. Also, we denote by $\bar{\nabla}_f u_i$ the vector composed of $\frac{\Delta u^k_i}{\Delta f_i}$.

Proposition 1. Assume that:

1. The function $f(t,u)$ is continuously differentiable with respect to both variables.
2. The values of this function are given on the discrete set $\{t_i\}$ with the step Δt.
3. For each t_i, $\left.\frac{\partial f}{\partial t}\right|_{t_i} \ne 0$.
4. There exists a stationary point of this function in the variable u.

Then for some value h_i, the iterative method

$$u_{i+1} = u_i + h_i \bar{\nabla}_f u_i / \left(\left\|\bar{\nabla}_u f_i\right\| + \alpha_i\right),$$

$$0 \le \alpha_i \le h / \left(\frac{\partial f_i}{\partial t}\Delta t_i\right) - \left\|\bar{\nabla}_u f_i\right\|,$$

yields a sequence of values u_i deviating from the stationary point $\tilde{u}$ so that the differences $u^k_i - \hat{u}^k$ and $u^k_{i+1} - \hat{u}^k$, $k = 1,..., n$, are of opposite sign. Thus, the result u_i fluctuates around the stationary points $\nabla_u f|_{\hat{u}, t_i} = 0$.

Proposition 2. In addition to the conditions of Proposition 1, assume that:

1. The function (t, u) has second-order partial derivatives with respect to the variable u that form the matrix $\nabla_{uu} f$.
2. At each point (t_i, u_i), this matrix satisfies the strong convexity in u: $\left\|\nabla_{uu} f_i\right\| > 0$.

Then for $|h_i| \ge \left|\frac{\partial f}{\partial t}\left\|\bar{\nabla}_u f\right\|\Delta t\right|_i$ and $0 \le \alpha_i \le h / / \left(\frac{\partial f_i}{\partial t}\Delta t_i\right) - \left\|\bar{\nabla}_u f_i\right\|$, where the gradient approximation applies to the values of the variables t_i and u_i, the iterative method

$$u_{i+1} = u_i + h_i \bar{\nabla}_f u_i / \left(\left\|\bar{\nabla}_u f_i\right\| + \alpha_i\right)$$

yields a sequence of values u_i deviating from the stationary points alternately by each coordinate in the opposite directions by the value Δu_i. Moreover, the lower bound $\inf\left\|\Delta u_i\right\| = \left|\frac{h_i \Delta t \left\|\bar{\nabla}_f u_i\right\|}{\left\|\nabla_{uu} f_i\right\|}\right|^{\frac{1}{2}}$ of its norm is achieved for $|h_i| = \left|\frac{\partial f}{\partial t}\left\|\bar{\nabla}_u f\right\|\Delta t\right|_i$.

Proposition 3. In the maximization problem, the sign of the step h_i obeys the following rule: h_i has the same sign as $\left(\Delta f(t_i, u_i) - \Delta t_i \left.\frac{\partial f}{\partial t}\right|_{\hat{t} \in [t_i, t_{i+1}]}\right)$ if

$\Delta f(t_i, u_i) \geq 0$ and the opposite sign otherwise $(\Delta f(t_i, u_i) < 0)$.

The proofs of Propositions 1–3 are postponed to the Appendix.

Remark 1. Given an admissible solution error δ, the admissible class of all functions $f(t, u)$ for which $\|\Delta u_i\| \leq \delta$ must satisfy the inequality $\left| \dfrac{\left.\frac{\partial f}{\partial t}\right|_i \Delta t \|\nabla_f u_i\| \|\nabla_u f_i\|}{\|\nabla_{uu} f_i\|} \right|^{\frac{1}{2}} \leq \delta$ on the time interval under consideration (when needed, on the entire definitional domain). Therefore, greater values $\left| \left.\frac{\partial f}{\partial t}\right|_i \right|$ and Δt lead to greater errors.

Remark 2. According to Proposition 1, a fluctuating process approximates the optimum at the first iteration of the method. According to Remark 1, the process will not leave the tube $\|\Delta u_i\| \leq \delta$.

Remark 3. Proposition 3 is applicable if the derivative $\frac{\partial f}{\partial t}$ varies in the period Δt_i so that the sign of $\left(\Delta f(t_i, u_i) - \Delta t_i \left.\frac{\partial f}{\partial t}\right|_t\right)$ is fixed for $t \in [t_i, t_{i+1}]$. Under a fixed step Δt, the value $\Delta t \left.\frac{\partial f}{\partial t}\right|_{\hat{t}}$ can be estimated using the three-point approximation

$$\Delta t \left.\frac{\partial f}{\partial t}\right|_{\hat{t}} \cong$$

$$\cong \frac{(f_i - f_{i-1})(u_{i+1} - u_i) - (f_{i+1} - f_i)(u_i - u_{i-1})}{u_{i+1} - 2u_i - u_{i-1}}.$$

This estimate can be obtained from the system of equations

$$f_{i+1} - f_i \cong \frac{\partial f}{\partial u}(u_{i+1} - u_i) + \Delta t \left.\frac{\partial f}{\partial t}\right|_{\hat{t}},$$

$$f_i - f_{i-1} \cong \frac{\partial f}{\partial u}(u_i - u_{i-1}) + \Delta t \left.\frac{\partial f}{\partial t}\right|_{\hat{t}},$$

assuming that the derivative $\frac{\partial f}{\partial t}$ has a small variation in the period Δt.

Remark 4. Choosing the value α in the iterative method so that $\alpha_i \geq \alpha = \mathrm{const} \geq 0$, we obtain a greater range of the increment

$$\Delta u_{i+1}(\alpha) = h \overline{\nabla}_f u_i / \left(\|\overline{\nabla}_u f_i\| + \alpha \right) \geq \Delta u(\alpha_i),$$

for which the differences $u_i^k - \hat{u}_i^k$ and $u_{i+1}^k - \hat{u}_i^k$ have opposite signs. The greater the difference $\alpha_i - \alpha$ is, the greater the range of fluctuations $u_i - \hat{u}_i$ will be.

The admissible values of the parameters α_i and h_i can be chosen within a rather wide range without any accurate estimates of the values $\left.\frac{\partial f}{\partial t}\right|_i$ and $\|\nabla_{uu} f_i\|$. The closer the parameter α_i to 0 is, the greater the range of Δu_i will be. Decreasing the parameter h_i reduces the range of Δu_i; however, for very small h_i, the algorithm diverges: the total error increases between iterations.

Remark 5. Since the external factors affect the objective function through its derivative $\left.\frac{\partial f}{\partial t}\right|_{t_i}$, the range of control values directly depends on the value of this effect.

Remark 6. Since the values of the target function $f(t, u)$ are calculated (or measured) only at the nodes of the discrete time grid and for the corresponding values of the control vectors, it is possible to construct a spline approximation of this function of the required smoothness and apply exact methods of gradient descent for it [10, 11]. However, the accuracy of the obtained result will remain finite, since such a spline approximation is not unique. In addition, the computational complexity of this method will significantly exceed the complexity of the proposed approach.

3. NUMERICAL EXAMPLE

The approximate method for optimizing a discrete time-varying system is illustrated below by a numerical example of optimal controller design for a simplified model of production. This model is described by the following discrete-time finite difference relations. However, according to the problem statement, only the values of the objective function and control at the previous and current time instants are used for solution.

The control (and simultaneously the state parameter) is the production output $u(t)$. The objective function—profit––has the form

$$r(t) = f(t, u(t)) = p(t)u(t) - C_0 - cu^2(t) \to \max_u,$$

where $t = 0, 1, 2, \ldots$; $p(t)u(t)$ gives the income; C_0 are fixed costs; $C_0 + cu^2$ is an estimate of the total costs including production assets, remuneration of labor and direct expenditures; finally, $p(t)$ specifies the unit price of the products (the environment's parameter)

$$p(t) = d\, p(t-1), \quad p(0) = p_0,$$

where d is the growth coefficient, and p_0 is an initial price.

The marginal profit (the gradient of the objective function with respect to the control variable) is estimated as

$$e(t) = (r(t) - r(t-1)) / (u(t) - u(t-1)).$$

At the next step, the control is calculated using the proposed approximate optimization method:

$$u(t+1)=u(t)+h/((e(t)(|e(t)|+\alpha)).$$

Here α is the stabilization parameter, and h is the coupling coefficient.

The figure shows the simulation results for the production output model with the parameters $C_0 = 1$ and $c = 1$ and the optimal controller with the parameters $\alpha = 0.1$ and $h = 1$. The product price with the initial value $p_0 = 3$ varies with the constant rate $d = 1.03$. The exact optimal solution has the form

$$u(t)=\frac{p(t)}{2c}.$$

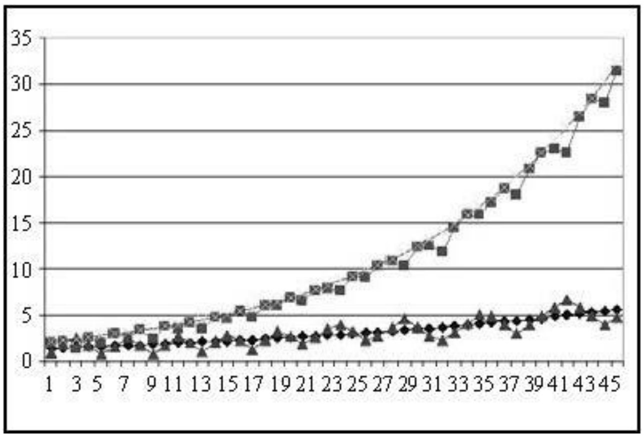

Fig. Simulation results for optimal controller:
optimal solution, calculated solution, optimal profit, and calculated profit.

The optimal controller quickly (in one step) approximates the production output the optimal one and, over time, tracks the optimal output within the method's error.

The range of fluctuations around the optimal output is conditioned by the discrete nature of the model, the nonstationary behavior of the product price, and an inaccurate choice of the controller's parameters. With an increase in the time derivative of the price, the solution error grows, which agrees with the approximation estimate presented above. At the initial steps, the value h_i has the estimate $|h_i|=\left|\frac{\partial f}{\partial t}\left\|\bar{\nabla}_u f\right\|\Delta t\right|_i \cong 1$ and the sign +1. An appropriate estimate of the coefficient α_i was selected from the stability considerations to satisfy $\left\|\bar{\nabla}_u f_i\right\| >> \alpha_i > 0$. They can be experimentally refined along the trajectory by maximizing the value achieved by the objective function $f(t, u(t))$ during several steps of the discrete algorithm. In the example, the values of α_i and h_i were constant along the entire trajectory.

CONCLUSIONS

The approximate optimization method proposed in this paper is not very critical to the choice of the parameters α and h. They can be determined in a particular way depending on the applied problem under consideration. In the numerical example, the value of the parameters corresponds to the rate of increase in the unit price of products. In addition, the parameter h can be estimated using finite-difference approximations for the derivatives of the function $f(t, u)$. In this case, the parameter α can be estimated as $\alpha \cong \left\|\bar{\nabla}_u f_0\right\|^2 \times$ $\times h / \max_t \left|\frac{\partial f}{\partial t}\right|$. According to numerical simulations, varying the controller's parameters in a rather wide range has an insignificant effect on the qualitative behavior of the calculated trajectory.

APPENDIX

Proof of Proposition 1. Using the linear part of the Taylor–Lagrange series for the function $f(t, u)$ at the point (t_{i+1}, u_{i+1}), where the remainder is given at the intermediate point $(t_i, \hat{u}^k), t_i \le \hat{t} \le t_{i+1}, \hat{u}^k \in [u^k_i, u^k_{i+1}]$, we obtain

$$f(t_{i+1}, u_{i+1}) - f(t_i, u_i) = .$$

$$= \sum_k \left.\frac{\partial f}{\partial u^k}\right|_{\hat{t},\hat{u}} (u^k_{i+1} - u^k_i) + \left.\frac{\partial f}{\partial t}\right|_{\hat{t},\hat{u}} (t_{i+1} - t_i).$$

According to the first-order optimality condition, let $\left.\frac{\partial f}{\partial u^k}\right|_{\hat{t},\hat{u}} = 0$. Then

$$\Delta u^k_i = u^k_{i+1} - u^k_i =$$

$$= (t_{i+1} - t_i)\left.\frac{\partial f}{\partial t}\right|_{\hat{t},\hat{u}} / \frac{f(t_{i+1}, u_{i+1}) - f(t_i, u_i)}{u^k_{i+1} - u^k_i},$$

which can be written in the vector form as

$$\Delta u_i = \Delta t_i \left.\frac{\partial f}{\partial t}\right|_{\hat{t},\hat{u}} \bar{\nabla}_f u_i. \quad (1)$$

Now we present a calculation method suitable for numerical implementation. Let α_i and h_i be determined from the relation

$$\frac{\partial f_i}{\partial t} = h_i / \left(\Delta t_i\left(\left\|\bar{\nabla}_u f_i\right\| + \alpha_i\right)\right). \quad (2)$$

Then, near the stationary point $\hat{u}_i \in [u_i, u_{i+1}]$, we have

$$\Delta u_i = h_i \bar{\nabla}_f u / \left(\left\|\bar{\nabla}_u f_i\right\| + \alpha_i\right).$$

Since $\hat{u}_i$ is an inner point of the interval $[u_i, u_{i+1}]$, the differences $u^k_i - \hat{u}^k_i$ and $u^k_{i+1} - \hat{u}^k_i$ have opposite signs. Thus, u_i will coordinate-wise fluctuate around the stationary points $\nabla_u f|^{\hat{u}}_i = 0$.

Proof of Proposition 2. We denote by $\overline{\Delta u}$ the method error at the current step due to the discrete time scale. Near the stationary point, the gradient can be estimated as

$$\bar{\nabla}_u f_i = \nabla_{uu} f_i \overline{\Delta u}\ .$$

If the value of the derivative $\frac{\partial f}{\partial t}$ is exactly known, then for the values h and α satisfying (2), we use condition (1) to obtain

$$\left\|\overline{\Delta u}\right\| = |h|\,\Delta t \left\|\bar{\nabla}_f u_i\right\| / \left(\left\|\nabla_{uu} f_i\right\|\left\|\overline{\Delta u}\right\| + \alpha\right).$$

Solving the quadratic equation for $\left\|\overline{\Delta u}\right\|$ yields

$$\left\|\overline{\Delta u}\right\| = -\frac{\alpha}{2\left\|\nabla_{uu} f_i\right\|} + \left(\frac{\alpha^2 + 4|h|\,\Delta t \left\|\bar{\nabla}_f u_i\right\|\left\|\nabla_{uu} f_i\right\|}{4\left\|\nabla_{uu} f_i\right\|^2}\right)^{\frac{1}{2}}.$$

Obviously, the function $\left\|\overline{\Delta u}\right\|(\alpha)$ is monotonically increasing.

From the relation (2) it follows that

$$h_i = \frac{\partial f_i}{\partial t}\left(\left\|\bar{\nabla}_u f_i\right\| + \alpha_i\right)\Delta t_i\,,$$

$$|h_i| \geq \left|\frac{\partial f}{\partial t}\left\|\bar{\nabla}_u f\right\|\Delta t\right|_i. \qquad (3)$$

For α = 0, the expressions (2) and (3) give

$$\left\|\overline{\Delta u}\right\|(\alpha = 0) = \left(\frac{|h|\,\Delta t\left\|\bar{\nabla}_f u_i\right\|}{\left\|\nabla_{uu} f_i\right\|}\right)^{\frac{1}{2}} =$$

$$= \left|\frac{\left.\frac{\partial f}{\partial t}\right|_i \Delta t \left\|\bar{\nabla}_f u_i\right\|\left\|\bar{\nabla}_u f_i\right\|}{\left\|\nabla_{uu} f_i\right\|}\right|^{\frac{1}{2}}.$$

Due to this equality, the estimate

$$\left\|\overline{\Delta u}\right\| = \left|\frac{\left.\frac{\partial f}{\partial t}\right|_i \Delta t \left\|\bar{\nabla}_f u_i\right\|\left\|\bar{\nabla}_u f_i\right\|}{\left\|\nabla_{uu} f_i\right\|}\right|^{\frac{1}{2}}$$

is a lower bound for the solution error.

Proof of Proposition 3. According to Proposition 1, the interval $\left[u_i,\, u_{i+1}\right]$ does not contain the stationary point if for some (or all) coordinates, the differences $u_{i+1} - u_i$ have the same signs.

In this case, assuming $\left.\frac{\partial f}{\partial u^k}\right|_{\hat{t}\,\in[t_i,\, t_{i+1}]} \neq 0$, we obtain

$$\Delta u^k_i = u^k_{i+1} - u^k_i =$$

$$= \frac{f(t_{i+1}, u_{i+1}) - f(t_i, u_i) - (t_{i+1} - t_i)\left.\frac{\partial f}{\partial t}\right|_{\hat{t}}}{\left.\frac{\partial f}{\partial u^k}\right|_{\hat{t}}},$$

which can be written in the vector form

$$\Delta u_i = \left(\Delta f(t_i,\, u_i) - \Delta t_i \left.\frac{\partial f}{\partial t}\right|_{\hat{t}}\right)\nabla_f u\Big|_{\hat{t}}.$$

In the maximization problem, for a fixed absolute value of the step h, the descent direction is chosen from the condition

$$\operatorname{sign}(h) = \operatorname{sign}\left(\Delta f(t_i,\, u_i) - \Delta t_i \left.\frac{\partial f}{\partial t}\right|_{\hat{t}}\right) \text{ if } \Delta f(t_i,\, u_i) > 0\,,$$

$$\operatorname{sign}(h) = -\operatorname{sign}\left(\Delta f(t_i,\, u_i) - \Delta t_i \left.\frac{\partial f}{\partial t}\right|_{\hat{t}}\right) \text{ if } \Delta f(t_i,\, u_i) < 0\,.$$

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MODELS OF EXPERIENCE

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Abstract. A generalized probabilistic model is proposed that uniformly describes the formation and development of individual, collective, and social experience at various human activity levels. Some of its particular cases are considered, covering many learning models known in mathematical psychology and models of developing and mastering technologies within the methodology of complex activity.

Keywords: experience, activity, knowledge, technology, culture, learning curve.

INTRODUCTION

Active systems. Let us separate two classes of systems that include a human. (Following the terminology of [1, 2], such systems will be called *active systems*.). These classes are:

– *Natural systems*, existing or emerging "independently," in the absence of an external source that forms or determines the goal of activity. Such systems have independent goal-setting, and their global goal is development (which requires preservation and possibly adaptation reproduction). In terms of systems engineering [3, 4], active systems with internal goal-setting are *systems of systems* (SoSs) belonging to collaborative or virtual classes.

– *Artificial systems* created by some subject to achieve his goals. In terms of systems engineering [3, 4], active systems with internal goal-setting are externally directed SoSs or externally acknowledged SoSs.

Depending on the presence of an explicit subject, one can distinguish between *subject systems* and *non-subject systems.* The former systems perform their activity, which is uniformly described by *the methodology of complex activity* (MCA) [5] regardless of their type. The latter systems perform no activity themselves; more precisely, their "activity" is the set of activities of their components.

Thus, we have three options (one of the four possible options is contradictory); see examples in (Table 1) below.

Table 1

Classification of systems

	Natural systems (internal goal-setting)	Artificial systems (external goal-setting)
Subject systems	Individual	Organization Enterprise Government Particular case: individual employee whose internal motives are coordinated with external goals
Non-subject systems	Social communities: Group Family Genus Tribe Society Ethnos People Economic communities: Market Set of independent interacting economic agents	–

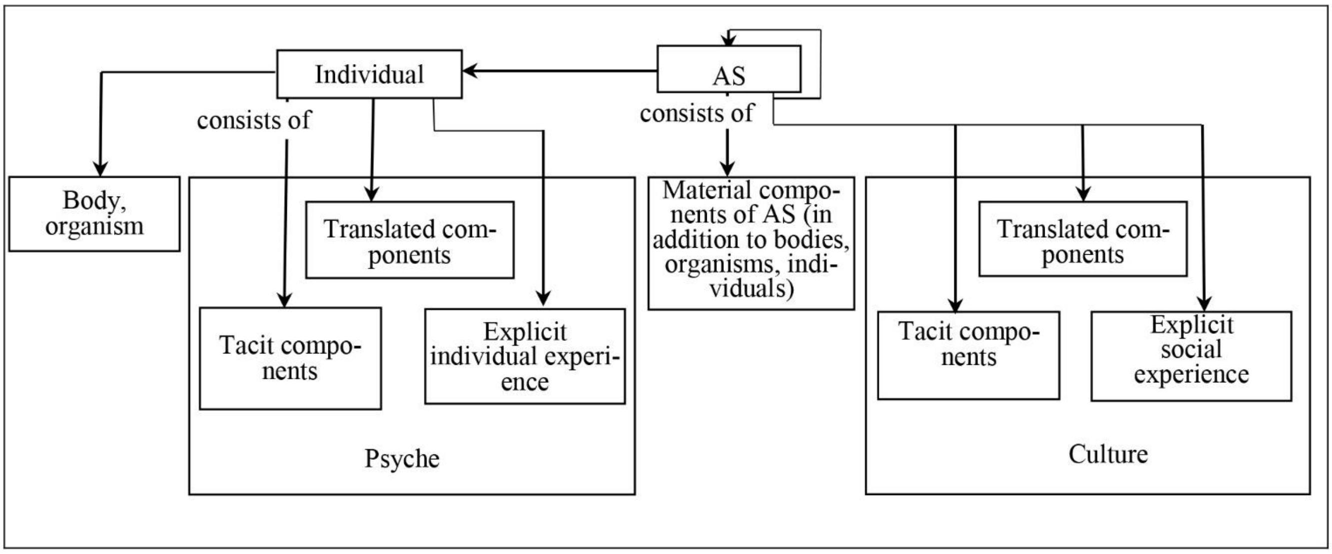

Fig. 1. Structure of active system.

As a digression, note that human activity can be considered within the ASs of different levels and scales (Table 2), and a promising task of MCA is their uniform description (probably, except for the two lower levels).

Table 2

Human activity

Level	Typical object	Dominant form of activity of elements
Cultural	Ethnos, people	Reproduction and development of activity
Political	Government, institution	Institutionalization of activity
Economic	Organization, enterprise	Collective practical activity
Social	Society	Communication activity
	Group, collective; family, genus, tribe	Collective practical activity
Psychic	Personality	Individual practical activity
	Individual	Internal activity
Biological	Organism	Life activity
Physical	Body	Movement

Moreover, the assignment of a particular system to a specific class depends on the aspect of its consideration. For example, a social group is itself a non-subject natural system. However, when studying the problems of managing such a group by other subjects (an individual, another group, or government), it must be considered an artificial subject system, together with the control subject.

For natural non-subject systems, the key factors are the mechanisms of their functioning (conditions, principles, norms, requirements, and criteria for assessing the activity of the system components, both separately and during their interaction [5, 6]). Recall that a *mechanism* is a system or device that determines the order of some activity [2]. The mechanisms of functioning form a multilevel system of nested feedback loops determining the dependence of conditions, principles, norms, etc. (including control actions) on previous and current performance results and uncertainty factors. As a rule, the mechanisms of functioning of non-subject systems are reflexive. Some examples include natural selection, competition, conflicts, dissemination of ideas, etc. These mechanisms provide (self) control of such systems.

In subsystems (ASs or individuals), let us separate the material component (for an individual, his body and material means of activity) and the immaterial component (for an individual, *psyche*; for a collective subject, *culture*). For details, see

Fig. 1). Experience is a significant part of the immaterial components.

Experience. *Experience* is understood [7–9] as:

1) a set of practically mastered knowledge, skills, abilities, and habits (individual experience);

2) the reflection of the objective world and social practice aimed at changing the world in the human mind (socio-historical experience, the individual experience of each individual).

The category of experience is closely related to other categories such as education, technology, and culture (Fig. 2).

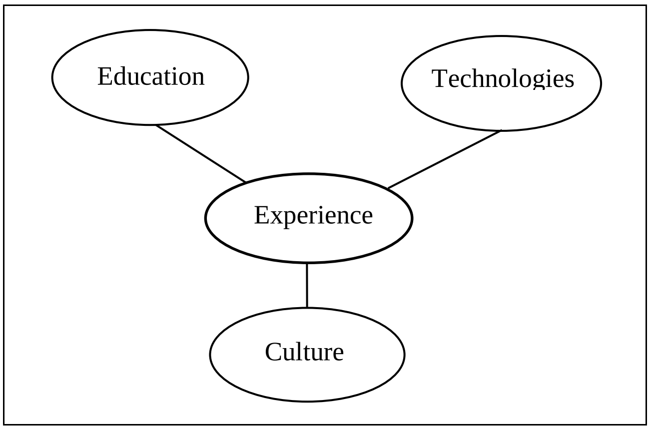

Fig. 2. Experience and related categories.

Indeed, *education* is the development of experience [10] and includes learning. (*Learning* is the process and result of acquiring individual experience [11].) A modern survey of mathematical models of learning can be found in [11, 12], and a survey of learning models in automatic control theory in [13].

Technologies are the operational reflection of a mass-practice proven and systematized practical experience [14]. (According to [5], technology is a system of conditions, criteria, forms, methods, and means for consistently achieving a set goal.)

Culture includes [7]:

– the objective results of human activity (machines, technical structures, results of cognition (books, works of art, legal and ethical norms, etc.), representing the first component of culture;

– the subjective human strengths and abilities realized in activities (sensations, perceptions, knowledge, skills, production and professional skills, the level of intellectual, aesthetic, and moral development, worldview, the methods and forms of mutual communication of people, etc.), representing the second component of culture [15].

The objective results of human activity (the first component of culture) are reflected in different forms of social consciousness such as language (understood in a broad sense–both natural native and foreign languages and artificial languages), everyday consciousness, political ideology, law, ethics, religion (or atheism as anti-religion), art, science, and philosophy [15].

The second component of culture is subjective human strengths and abilities. They are expressed in personal knowledge, including figurative, sensory knowledge, which is not transferred by words (concepts), as well as in skills, the development of certain individual abilities, the worldview of each person, etc. [7].

Here are some quotes and definitions that characterize the concept of culture:

– "a set of genetically non-inherited information in the field of human behavior" (Yu.M. Lotman);

– a set of sustainable forms of human activity (*organizational culture*);

– "Just as the embryo in the womb repeats in a fantastically accelerated time scale the entire evolution of life on Earth over a billion years, so a growing person in 20 years must assimilate the culture that mankind has created for 4 million years." [7, p. 32];

– a set of accepted standard norms of activity (ways of standardizing and regulating behavior) and the corresponding results. The main function of culture is the reproduction and construction (development) of activity.

Thus, culture can be viewed as a generalized experience proven by social practice [7, 12, 16, 17]; see Fig. 1.

Experience can be formed through independent acquisition by a subject (individual or collective) during his activity or through the development of someone else's experience during learning activity (Fig. 3).

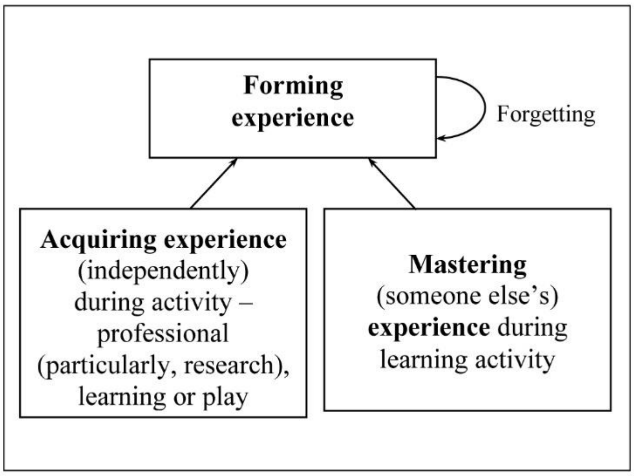

Fig. 3. Formation of experience.

Depending on the methods and means of fixing and translating the experience (or even more broadly—in the case of an individual—the components of the psyche, when relating ideas, beliefs, attitudes, personality worldview, etc. to the widely interpreted experience), we can distinguish among:

– *explicit experience*, which is often translated in the form of text (e.g., knowledge, or technology);

– *tacit experience* (tacit knowledge), which is often translated in non-verbal and non-textual forms (e.g., beliefs, or worldview);

– *nontranslated components*, which are, perhaps, translated "biologically" (e.g., biopsychic properties of an individual; the specific physiology of individuals, conditioned by climate, landscape, and lifestyle), but so slowly that they can be considered unchanged.

The types of experience are listed in Table 3.

Table 3

Types of experience

Level	Experience
Social system	Social
Group	Collective
Personality	Individual

In the process of his activity, a subject can participate (Fig. 4) in:

1) mastering social experience;
2) forming/acquiring individual experience;
3) mastering collective experience;
4) forming collective experience;
5) forming social experience.

Mastering social and/or collective experience can be conventionally regarded as "learning with a teacher," and forming individual experience as "learning without a teacher."

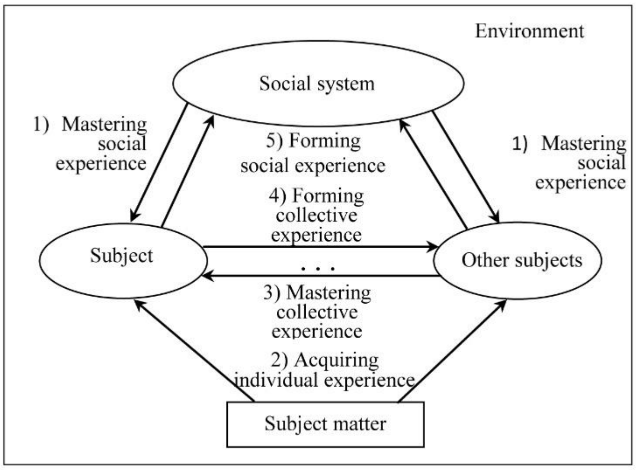

Fig. 4. Individual, collective, and social experience.

The goal of this study is to create a general experience formation model that would adequately and uniformly describe the processes of forming and mastering individual, collective, and social experience (Fig. 4), explicit or tacit, at various levels of activity (Table 2) in any classes of systems (Table 1).

The remainder of this paper is organized as follows. In Section 1, a general experience formation model is introduced. Section 2 provides a classification system for different models of experience. Sections 3 and 4 consider several particular models for forming/mastering individual and collective/social experience, respectively. The Conclusions section outlines some promising lines for further research.

1. GENERAL EXPERIENCE FORMATION MODEL

We extend the original learning model (see subsection 3.3.4 of the book [18]) by supplementing it with the following effects: environment variability, making experience outdated (or, equivalently, forgotten/lost), and a more complex formation of experience, with mastering experience by other subjects and the interaction of different subjects.[1]

Let an AS be composed of a given set $N = \{1, \ldots, n\}$ of active elements (AEs). (An AE is an element of an active system representing an individual or a lower-level AS.)

Assume that each AE observes one of K possible values of an *uncertainty factor* (UF) in each period. In the general case, the values observed by different AEs will differ. We introduce the concept of *a complex uncertainty factor*: its current value will be characterized by the aggregate of all states encountered by all AEs in period t.

A complex UF can be represented as a matrix $\boldsymbol{\omega}(t) = \|\omega_{ik}(t)\|$ with binary elements ($\omega_{ik}(t) \in \{0;\ 1\}$). Suppose that in a current period t, the UF for AE i has state $k(i)$. Then the elements $\omega_{ik(i)}(t)$ are 1, and the others are 0. Obviously, the matrix $\|\omega_{ik}(t)\|$ satisfies the condition $\sum_{k=1}^{K} \omega_{ik}(t) = 1,\ i = \overline{1, n}$.

We denote by $\boldsymbol{\Omega}$ the set of all such matrices. (Its cardinality is K^n). On the set $\boldsymbol{\Omega}$ we define a time-varying probability distribution $\{p_\omega(t)\}$ for the states of the complex UF of the environment, assuming that the current state $\omega(t)$ occurs independently of the previous ones. We number the elements of this set using the function $\iota(\omega) = \sum_{i=1}^{n} \sum_{k=1}^{K} k n^{i-1} \omega_{ik}$.

Therefore, $\sum_{\omega \in \Omega} p_\omega(t) \equiv \sum_{\omega \in \Omega} p_{\iota(\omega)}(t) \equiv 1$.

In a particular case, the states observed by each of the AEs are mutually independent. We denote by $p_{ik}(t)$ the probability of observing state k by AE i, where

[1] In the proposed model, we will not separate the effect of physiological forgetting of experience from the effect of rejecting the previously learned experience when new technologies appear. Separation of two effects, generally speaking, different by their nature and speed – the objective change in technologies and the resulting hard or soft rejection of experience (depending on the distribution of the AE parameters) and the subjective physiological forgetting of it together with a regular trend towards age-related changes in the parameters of the AE cognitive characteristics – will allow us to analyze several social effects (cultural interaction of generations, a decrease in collective experience (including culture) in revolutionary periods, etc.) by varying their speeds. The effects mentioned above can become the subject matter of further research.

$\sum_{k=1}^{K} p_{ik}(t) \equiv 1,\ i = \overline{1,n}$, and by $k(i)$ the state observed by AE i. Then $p_{\omega}(t) = \prod_{i=1}^{n} p_{ik(i)}(t)$.

Let us describe the AS state by a matrix $\mathbf{v}(t) = \| v_{ik}(t) \|$ as follows. Each element is a binary variable characterizing *the formation of experience* (within the mathematical model, this process will also be called mastering the technology) by the AE for various states of the UF. More precisely, each element $v_{ik}(t)$ takes value 1 if, after period t, AE i has formed/mastered the experience under state k of the UF.

Suppose that the AS evolves during period $(t + 1)$ under the following mechanism.

Let the complex UF have a state $\boldsymbol{\omega}(t+1)$, and let AE i encounter state k of the UF. Then:

- For any UF state l unmastered by AE i ($v_{il}(t) = 0$), *the experience is formed* ($v_{il}(t + 1) = 1$) with a probability $0 \le w_{ikl}(\{\boldsymbol{v}(\cdot) \mid t - \tau;\ t\}) \le 1$, which generally depends on time as well as on the current $\boldsymbol{v}(t)$ and τ previous AS states; the experience is not formed with the probability $1 - w_{ikl}(\{\boldsymbol{v}(\cdot) \mid t - \tau;\ t\})$, which implies $v_{il}(t + 1) = 0$. In the sequel, $\{\boldsymbol{v}(\cdot) \mid t_1;\ t_2\}$ will denote *the history*, i.e., an ordered set of values $\boldsymbol{v}(\cdot)$ on a time interval between periods t_1 and t_2 inclusive. (If $t_2 = t_1$, there is no history.)
- For any UF state l mastered by AE i ($v_{il}(t) = 1$), *the experience is forgotten* ($v_{il}(t + 1) = 0$) with a probability $0 \le u_{ikl}(\{\boldsymbol{v}(\cdot) \mid t - \tau;\ t\}) \le 1$, which generally depends on time as well as on the current $\boldsymbol{v}(t)$ and τ previous AS states; the experience is not forgotten with the probability $1 - u_{ikl}(\{\boldsymbol{v}(\cdot) \mid t - \tau;\ t\})$, which implies $v_{il}(t + 1) = 1$.

This mechanism is illustrated in Fig. 5.

The semantics of this model reflects the possibility of forming experience, particularly, mastering technology by an active element, transferring knowledge from one element to another, forgetting knowledge and/or making it outdated, among other things, due to the evolution of the environment and the repeated adaptation of the AS to changes in the environment, reflected by the realized UF values.

In this case, for each UF state, the process of forming-forgetting experience by each AE is supposed to be binary (possible states = <mastered | unmastered>) and random, which reflects its uncertainty. For different AEs and different UF states, the transitions between states occur independently of each other. By assumption, there can be no more than one event during one period: forming experience or forgetting it. At

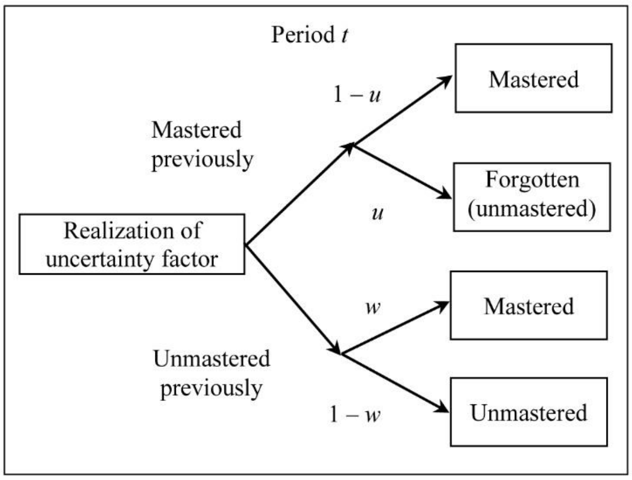

Fig. 5. Alternative events in period t.

the same time, as the probabilities of transition between states depend on the current and previous states of all AEs in the AS, the model describes rather complex laws of the AS behavior. For example, observing one state, an AE can generally form an experience corresponding to another UF state (by acquiring the experience from another AE).

Now we write dynamic equations for the probabilities of mastering experience and the expected experience maturity levels. Let $\mathbf{q}(t) = \| q_{ik}(t) \|$, where $q_{ik}(t) = \Pr(v_{ik}(t) = 1) = E[v_{ik}(t)]$ is the probability that state k of the UF is mastered by AE i after period t. Then by the rule of total probability yields

$$
\begin{aligned}
q_{ik}(t + 1) &= \Pr(v_{ik}(t + 1) = 1 \mid v_{ik}(t) = 0) \\
&\Pr(v_{ik}(t) = 0) + \Pr(v_{ik}(t + 1) = 1 \mid v_{ik}(t) = 1) \\
&\Pr(v_{ik}(t) = 1) = W_{ik}(\boldsymbol{q}(t))\,(1 - q_{ik}(t)) + \\
&+ (1 - U_{ik}(\boldsymbol{q}(t)))\, q_{ik}(t) = W_{ik}(\boldsymbol{q}(t)) + \\
&+ [1 - W_{ik}(\boldsymbol{q}(t)) - U_{ik}(\boldsymbol{q}(t))]\, q_{ik}(t), \qquad (1)
\end{aligned}
$$

where the functions $W_{ik}(\boldsymbol{v}(t))$ and $U_{ik}(\boldsymbol{v}(t))$ are the probabilities of mastering and forgetting, $w_{ikl}\{\cdot\}$ and $u_{ikl}\{\cdot\}$, respectively, averaged by the UF states considering their probabilities $p_m(t)$ and the probabilities of the AS states in the current and previous periods:

$$
\begin{aligned}
W_{ik}(\mathbf{q}(t)) &= \sum_{m=1}^{K^n} p_m(t) \times \\
&\times \sum_{\substack{\mathbf{z}(t)\ldots\mathbf{z}(t-\tau) \\ z_{ik}(t)=0}} w_{imk}\left(\{\mathbf{z}(t) \mid t - \tau;\ t\}\right) \times \qquad (2) \\
&\times \pi(\{\mathbf{z}(\cdot),\ \mathbf{q}(\cdot) \mid t - \tau;\ t\}, i^-, k^-),
\end{aligned}
$$

$$
\begin{aligned}
U_{ik}(\mathbf{q}(t)) &= \sum_{m=1}^{K^n} p_m(t) \times \\
&\times \sum_{\substack{\mathbf{z}(t)\ldots\mathbf{z}(t-\tau) \\ z_{ik}(t)=1}} u_{imk}\left(\{\mathbf{z}(t) \mid t - \tau;\ t\}\right) \times \qquad (3) \\
&\times \pi(\{\mathbf{z}(\cdot),\ \mathbf{q}(\cdot) \mid t - \tau;\ t\}, i^-, k^-),
\end{aligned}
$$

where $\pi(\{z(\cdot), q(\cdot)|t-\tau; t\}, i^{-}, k^{-})$ are the conditional probabilities that after periods $\{t-\tau; t\}$ the AE states have the values $z(\cdot)$ and the probabilities of mastering have the values $q(\cdot)$ given a known value of the experience maturity level under UF state k after period t. The conditional probabilities $\pi(\cdot)$ are calculated by formula (4) with the product taken over all triplets $< \alpha; \beta; \gamma >$ except for $< \alpha; \beta; \gamma > = < i; k; t >$:

$$\pi(\{\mathbf{z}(\cdot), \mathbf{q}(\cdot)\,|\,t-\tau;\ t\}, i^{-}, k^{-}) = \prod_{\substack{\alpha=1\ldots n;\ \beta=1\ldots K;\ \gamma=t-\tau\ldots t; \\ <\alpha;\beta;\gamma;s>\neq<i;k;t>}} (y_{\alpha\beta}(\gamma)\, z_{\alpha\beta}(\gamma) + (1-y_{\alpha\beta}(\gamma))\,(1-z_{\alpha\beta}(\gamma))). \quad (4)$$

In a particular case when the UF states observed by each AE are independent, the expressions (2) and (3) take the form

$$W_{ik}(\mathbf{y}(t)) = \sum_{\omega\in\Omega}\prod_{i=1}^{n} p_{ik(i)}(t) \times \sum_{\substack{\mathbf{z}(t)\ldots\mathbf{z}(t-\tau) \\ z_{ik}(t)=0}} w_{\iota(\omega)ik}\left(\{\mathbf{z}(t)\,|\,t-\tau; t\}\right) \times \pi(\{\mathbf{z}(\cdot), \mathbf{q}(\cdot)\,|\,t-\tau;\ t\}, i^{-}, k^{-})$$

and

$$U_{ik}(\mathbf{q}(t)) = \sum_{\omega\in\Omega}\prod_{i=1}^{n} p_{ik(i)}(t) \times \sum_{\substack{\mathbf{z}(t)\ldots\mathbf{z}(t-\tau) \\ z_{ik}(t)=1}} u_{\iota(\omega)ik}\left(\{\mathbf{z}(t)\,|\,t-\tau; t\}\right) \times \pi(\{\mathbf{z}(\cdot), \mathbf{q}(\cdot)\,|\,t-\tau;\ t\}, i^{-}, k^{-}).$$

Well, we have obtained the recurrence relations reflecting the dynamics of the expected experience maturity levels. To calculate their values in any period, it remains to specify the initial value matrix $\boldsymbol{q}(0)$. By default, suppose that in the initial (zero) period, the AE has no experience for any of the FN states.

Let *the individual experience criterion* $L_i(t)$ ("learning level") of AE i be the probability of realizing a UF value previously encountered, successfully mastered, and not forgotten by him (i.e., the expected share of the learned values):

$$L_i(t) = 1 - \sum_{k=1}^{K} p_{ik}(t)\,(1-q_{ik}(t)),\ i=\overline{1, n}. \quad (7)$$

By analogy, let *the collective experience criterion* $L_{\max}(t)$ be the probability of realizing a UF value previously encountered, successfully mastered, and not forgotten **by at least one AE**:

$$L_{\max}(t) = 1 - \prod_{i=1}^{n}(1-L_i(t)), \quad (8)$$

or the probability $L_{\min}(t)$ of realizing a UF value previously encountered, successfully mastered, and not forgotten **by each of the AEs**:

$$L_{\min}(t) = \prod_{i=1}^{n} L_i(t). \quad (9)$$

A sequence of experience criterion values will be called *an experience curve*, similar to the concept of a learning curve.

The collective experience criterion can be treated as an aggregate characteristic of the experience formed by the entire group.

Transition to continuous time. Let the AS and AE operate in continuous time: the processes of forming and forgetting experience are independent flows of elementary events, whose intensities (rates) $w_{ikl}(\{v(\cdot)\,|\,t-\tau; t\})$ and $u_{ikl}(\{v(\cdot)\,|\,t-\tau; t\})$ in a known way depend on the history of the AS states at the current and previous time instants $\{v(\cdot)\,|\,t-\tau; t\}$.

Assume that the UF changes its states somehow (in discrete or continuous time), independently of the AS, and the evolution of $\{p_{ik}(t)\}$ is known.

Then the system of difference equations (5), describing the AS dynamics with given initial conditions, can be replaced by the system of differential equations of the form

$$dq_{ik}(t) / dt = W_{ik}(\boldsymbol{q}(t)) - (W_{ik}(\boldsymbol{q}(t)) + U_{ik}(\boldsymbol{q}(t)))\, q_{ik}(t). \quad (10)$$

2. CLASSIFICATION OF EXPERIENCE MODELS

The expressions (1)–(10) describe the process and result of forming individual and collective experience in the most general case—under minimum assumptions. For an operational description and study, it is necessary to make some simplifications (additional assumptions about the structure and properties of the model). Therefore, we introduce a system of classifications based on the properties of the model components. (Note that classification bases 1–9 are mutually independent.)

1. Properties of complex UF states observed by AE in each period. For now, we will separate the general case (considered above) and a particular case in which all AEs observe the same realization of the UF state in each period. The UF properties will be described not by the n K-dimensional distribution $\{p_\omega(t)\}$, but by the K-dimensional one $\{p_k(t)\}$, where $\sum_{k=1}^{K} p_k(t) = 1$.

2. Dependence of complex UF states on time. Here, the general case (see above) is an arbitrary known dependence of the probability distribution of the complex UF states on time, and the particular case is a stationary (time-invariant) distribution.

3. Dependence of the probability of mastering on time. The general case (see above) is an arbitrary known dependence of the probability of mastering on time, and the particular case is no dependence.

4. Dependence of the probability of forgetting on time. Similar to item 3.

5. Dependence of the probability of mastering on process history. The general case (see above) is a known dependence of the probability of mastering on $\tau \le t$ previous states. Also, we will separate two subcases: $\tau = 0$ (the history-invariant probability of mastering) and $\tau = 1$ (the probability of mastering depends only on the previous state).

6. Dependence of the probability of forgetting on process history. Similar to item 5.

7. Dependence of the probability of mastering for AE *i* on the states of other AEs. The general case (see above) is a known dependence of the probability of mastering on the states of all AEs. An "intermediate" case is when the probability of mastering for AE *I* depends on the states of his "neighbors" – AEs from a known set $N_i(t)$. The particular case is when for each AE, the probability of mastering depends on his states only.

8. Dependence of the probability of forgetting for AE *i* on the states of other AEs. Similar to item 7.

9. Possible experience formation regardless of realized UF state. The general case is when, by transferring experience from other AEs, a specific AE can form his experience corresponding to a UF state that differs from the state observed by him. The particular case is when an AE forms an experience corresponding only to the UF states observed by him.

10. Number of AEs. The general case is a known number of AEs, $n > 1$. The particular case is $n = 1$.

Consider several models in the order of complication. (Models 1–6 correspond to individual experience, whereas models 7–12 to collective and social experience.)

3. MODELS OF INDIVIDUAL EXPERIENCE

Model 1 ([14, Section 2.2]), in which there is one AE, all parameters are time-invariant, the experience corresponding to the UF state observed by the AE is formed effectively (the probability of mastering is equal to 1), and there is no forgetting.

We denote by $p_k > 0$ the probability that in a next period, the AE will encounter UF state *k*. (Obviously, $\sum_{k=1}^{K} p_k = 1$.) The vector of these probabilities is $P = (p_1, \ldots, p_K)$.

In the case under consideration, $n = 1$ and $i = 1$, which implies $m = l$. Since the probability of mastering is 1, let $W_{mj}(\{v(\cdot) \mid t - \tau; t\}) = 1$ for $m = l$ and $W_{mj}(\{v(\cdot) \mid t - \tau; t\}) = 0$ for $m \ne l$ regardless of the history $\{v(\cdot) \mid t - \tau; t\}$, i.e., $W_{mj}(\{v(\cdot) \mid t - \tau; t\}) = \delta_{mj}$, where δ_m is the Kronecker delta. Due to no forgetting, we have $u_{mj}(\{v(\cdot) \mid t - \tau; t\}) \equiv 0$ and $U_j(q(t))$ $U_j(q(t)) \equiv 0$. From the expression (2) it follows that

$$U_j(q(t)) = \sum_{m=1}^{K} p_m(t) \times \times \sum_{\substack{z(t)\ldots z(t-\tau) \in X; \\ z_j(t)=0}} W_{mj}\left(\{z(t) \mid t-\tau;\, t\}\right) \times \times \pi\left(\{z(t) \mid t-\tau;\, t\},\ j,\ q(t)\right) = = \sum_{m=1}^{K} p_m(t) \sum_{\substack{z(t)\ldots z(t-\tau) \in X; \\ z_j(t)=0}} \delta_{mj} = p_j(t). \quad (11)$$

Really, the Kronecker delta is taken once in the sum $\sum_{\substack{z(t)\ldots z(t-\tau) \in X; \\ z_j(t)=0}} \delta_{mj}$ because $z_j(t) = 0$, and this condition reduces the sum to a single element for which $m = j$. As a result,

$$q_j(t + 1) = p_j(t) + [1 - p_j(t)]\, q_j(t).$$

Since the probabilities are stationary, $q_j(t + 1) = p_j + (1 - p_j)\, q_j(t)$, or $\Delta q_j(t + 1) = p_j\, (1 - q_j(t))$, or $1 - q_j(t + 1) = (1 - p_j)\, (1 - q_j(t + 1))$. According to (8), the experience criterion is

$$L(t) = 1 - \sum_{k=1}^{K} p_k\, (1 - q_k(t)) = 1 - \sum_{k=1}^{K} p_k\, (1 - p_k) \times \times (1 - q_k(t-1)) = 1 - \sum_{k=1}^{K} p_k\, (1 - p_k)^t. \quad (12)$$

Model 2. Consider a modification of Model 1 in which there is a unique UF state ($K = 1$), but the probability of mastering $w \in (0, 1]$ can be smaller than 1. Omitting the UF state subscript, by analogy with the expression (19) we obtain: $W(q(t)) = w$,

$$q_j(t + 1) = w + [1 - w]\, q_j(t),$$

or $1 - q_j(t + 1) = (1 - w)\, (1 - q_j(t + 1))$. From (11) it follows that the learning curve has the form

$$L(t) = 1 - (1 - w)^t. \quad (13)$$

(Also, see the expression (12) for $w = p_k = 1 / K$.)

Model 3. Consider a modification of Model 1 with the same stationary probability of mastering $w(q) \in (0, 1]$ for all UF states. By analogy with the expression (11) we obtain $W_j(q(t)) = w(q(t))\, p_j(t)$. Since the probabilities are stationary, from (5) it follows that $q_j(t + 1) = w(q(t))\, p_j + (1 - w(q(t))\, p_j)\, q_j(t)$, or $1 -$ $- q_j(t + 1) = (1 - w(q(t))\, p_j)\, (1 - q_j(t))$. Let the probability of mastering be a known function $g(\cdot)$ of the current experience criterion value, i.e.,

$w(q(t)) = g(L(q(t)))$. Then the equality $1 - q_j(t+1) = (1 - g(L(t))\, p_j)\,(1 - q_j(t))$ implies

$$1 - q_j(t) = \prod_{\tau=0}^{t-1} (1 - p_j\, g(L(\tau))\,.$$

Denoting $b_j(t) = 1 - q_j(t)$, we write $b_j(t+1) = (1 - g(L(t))\, p_j)\, b_j(t)$ and, consequently,

$$L(t) = 1 - \sum_{k=1}^{K} p_k b_k(t) = 1 - \sum_{k=1}^{K} p_k \prod_{\tau=0}^{t-1} (1 - p_k\ g(L(\tau))\,,$$

$$\Delta L(t) = g(L(t)) \sum_{k=1}^{K} p_k^2 \prod_{\tau=0}^{t-2} (1 - p_k\ g(L(\tau))\,.$$

In the uniform distribution case ($p_j = 1 / K,\ i = \overline{1,n}$), we have $b_j(t) = b(t)$ and

$$L(t) = 1 - \sum_{k=1}^{K} p_k b_k(t) = 1 - \sum_{k=1}^{K} \frac{1}{K} b_k(t) = 1 - b(t) = q(t).$$

Therefore,

$$\Delta L(t) = \sum_{k=1}^{K} p_k\ (b_k(t-1) - b_k(t)) =$$

$$= \sum_{k=1}^{K} \frac{1}{K} (b_k(t-1) - (1 - \frac{1}{K} g(L(t-1)))\, b_k(t-1)) =$$

$$= \sum_{k=1}^{K} \frac{1}{K} (1 - 1 + \frac{1}{K} g(L(t-1))\, b(t-1)) =$$

$$= \sum_{k=1}^{K} \frac{1}{K} g(L(t-1))\, b(t-1)) =$$

$$= \frac{1}{K}\ g(L(t-1))\, b(t-1)) =$$

$$= \frac{1}{K}\ g(L(t-1))\,(1 - L(t-1)),$$

which gives

$$\Delta L(t) = \frac{1}{K}\ g(L(t-1))\,(1 - L(t-1)). \qquad (14)$$

Depending on $g(\cdot)$, the solution of the difference equation (14) can be an exponential, power, or logistic curve; see the models of different learning curves and a survey in [14]:

Probability of mastering $g(\cdot)$	Difference equation	Learning curve
$g(L) = \gamma K$	$\Delta L(t) = \gamma\,(1 - L(t-1))$	Exponential
$g(L) = \mu K L$	$\Delta L(t) = \mu\, L(t-1)\,(1 - L(t-1))$	Logistic
$g(L) = \eta K (1 - L)^a$	$\Delta L(t) = \eta\,(1 - L(t-1))^{a+1}$	Power

Model 4 ([18], subsection 3.3.4]) is the intersection of particular cases for the nine classification bases above. It differs from Model 1 in the presence of stationary probabilities of mastering and forgetting, which are generally not equal to 1 and 0, respectively.

Suppose that during the first realization of UF state k, the corresponding experience is formed with a known probability $0 \le w_k \le 1$, where w_k is *the probability of mastering*, and is not with the probability $(1 - w_k)$. After forming component k of the experience, in each next period, it changes as follows:

– If the UF state realized differs from k, then the result of mastering state k remains the same.

– If UF state k is realized again, then component k of the experience is "lost" with *the probability of forgetting* $0 \le u_k \le 1$ and remains the same with the probability $(1 - u_k)$.

We construct the vectors of the probabilities of mastering and forgetting: $W = (w_1, \ldots, w_K)$ and $U = (u_1, \ldots, u_K)$. Generally speaking, these vectors do not satisfy the normalization condition.

By analogy with the expression (11), we obtain: $W_j(q(t)) = w_j\ p_j$, $U_j(q(t)) = u_j\ p_j$. Substituting this result into (5) yields

$$q_j(t+1) = p_j\, w_j + (1 - p_j\,(w_j + u_j))\, q_j(t). \qquad (15)$$

Let the initial conditions be $q_j(0) = \alpha_j \in [0, 1]$. Then, using the recursive formula (15), we find

$$q_i(t) = \frac{w_j}{w_j + u_j} (1 - (1 - p_j(w_j + u_j))^t) + \\ + (1 - p_j(w_j + u_j))^t \alpha_j. \qquad (16)$$

Substituting the sum (16) into (7), we finally arrive at

$$L(P,W,U,t) = \sum_{k=1}^{K} p_k \frac{w_k}{w_k + u_k} \left(1 - \left(1 - p_k\left(w_k + u_k\right)\right)^t\right) + \\ + \sum_{k=1}^{K} \alpha_k \left(1 - p_k\left(w_k + u_k\right)\right)^t = \sum_{k=1}^{K} p_k \frac{w_k}{w_k + u_k} + \\ + \sum_{k=1}^{K} \left(\alpha_k - p_k \frac{w_k}{w_k + u_k}\right)\left(1 - p_k\left(w_k + u_k\right)\right)^t. \qquad (17)$$

Suppose that the AE obtains a reward (payoff) h_k for successfully forming component k of his experience in a certain period. Then over T_0 periods, his total expected payoff from forming the experience during his work is given by

$$F(P, W, U,\ T) = \sum_{t=1}^{T_0} \sum_{k=1}^{K} p_k h_k \frac{w_k}{w_k + u_k} \times \\ \times \left(1 - \left(1 - p_k\left(w_k + u_k\right)\right)^t\right).$$

Calculating the sum of this geometric progression in time, we obtain

$$F(P,W,U,T_0)=\sum_{k=1}^{K}\frac{p_k h_k w_k}{w_k+u_k}\Big(T_0-\big(1-p_k\left(w_k+u_k\right)\big)\times$$

$$\times\frac{1-\big(1-p_k\left(w_k+u_k\right)\big)^{T_0}}{p_k\left(w_k+u_k\right)}\Bigg).$$

In the *homogeneous* case (for all states, the probabilities of mastering and forgetting are $w_k = w$ and $u_k = u$), from the expression (17) it follows that

$$L(P,\ w,\ u,\ t)=\frac{w}{w+u}\left(1-\sum_{k=1}^{K}p_k\big(1-p_k\left(w+u\right)\big)^t\right),$$

$$t = 0, 1, 2, \ldots \qquad (18)$$

In a particular homogeneous case ($w = 1$ and $u = 1$ under the uniform probability distribution), $L(t)=\frac{1}{2}-\frac{1}{2}\left(1-\frac{2}{K}\right)^t$. (The asymptote 0.5 means that the fact of forgetting is discovered during the repeated realization of the mastered UF state.)

Let us introduce the following assumption, known as *the learnability condition* [18]: the probabilities P, W, and U are such that

$$p_k\,(w_k + u_k) < 1,\ k=\overline{1,\,K}. \qquad (19)$$

As was demonstrated in [18], condition (19) can be violated at most for one UF state. Moreover, under assumptions I–VI:

– The initial value of the experience criterion is 0.

– The experience curve is not decreasing and asymptotically tends to $\sum_{k=1}^{K}p_k\frac{w_k}{w_k+u_k}$; in addition, its growth rate is monotonically decreasing.

Introducing a *threshold* $\rho\in[0,1/K]$, we denote by $P_{\rho,K} = \{P = (p_1, \ldots, p_K) \mid \sum_{k=1}^{K}p_k=1,\ p_k \geq \rho,\ k=\overline{1,K}\}$ the set of K-dimensional probability distributions whose values are all not smaller that ρ.

As was established in [14], an analog of the expression (17) achieves maximum over all possible probability distributions $P \in P_{\rho,K}$ at the uniform distribution. We present a similar result for the model under consideration.

Proposition 1. *If*

$$w_k \in (0, 1],\ u_k \in[0,1),\ k=\overline{1,\,K}, \qquad (20)$$

then $\forall\rho \in (0, 1/K]$ $\exists\, t(\rho, W, U) = \dfrac{2}{\rho\min\limits_{k=\overline{1,K}}\{w_k+u_k\}}-1$ *such that* $\forall\,\tau > t(\rho, W, U)$ *the function* (17) *is strictly concave in* $\{p_k\} \in P_{\rho,K}$.

Proof. Denoting

$$\alpha_k=\frac{w_k}{w_k+u_k}\in(0,1],\ \beta_k=w_k+u_k\in(0,\ 2),\ k=\overline{1,\,K},$$

we write the expression (25) as

$$x_t(P,W,U)=\sum_{k=1}^{K}\alpha_k\,p_k\left[1-\left(1-\beta_k\,p_k\right)^t\right]. \qquad (21)$$

Differentiating the expression (21) twice, we easily check that the condition

$$t>\frac{2}{\rho\min\limits_{k=\overline{1,\,K}}\{\beta_k\}}-1$$

guarantees the strict concavity of the function (21) in all variables $\{p_k\} \in P_{\rho,K}$. ♦

Corollary. *If*

$$u_k = \sqrt{w_k} - w_k,\ k=\overline{1,K}, \qquad (22)$$

then the uniform probability distribution $p_k = 1/K$, $k = \overline{1,K}$ *is a unique solution of the problem*

$$x_\tau(P,W,U)\ \to\max_{P\in P_{\rho,K}}. \qquad (23)$$

This fact follows from Proposition 1 and the symmetry of the function (17) in all variables $\{p_k\} \in P_{\rho,K}$ under condition (22). (Also, see the proof of Proposition 4 in [14]). Note that condition (22) implies the learnability condition (19).

Thus, in the presence of forgetting and the nonunitary probabilities of mastering, the uniform probability distribution is generally not optimal in the problem (23); a sufficient condition for its optimality is given by (22), where $w_k\in(0,1]$.

In the homogeneous case, for the uniform probability distribution to be optimal in the problem (23), it suffices to satisfy the relation $w + u = 1$, under which condition (19) always holds and a particular case of which is the basic model with $w = 1$ and $u = 0$.

Substituting the uniform probability distribution $p_k = 1/K$, $k = \overline{1,K}$, into the expression (18), we obtain

$$x_t(K,\ w,\ u)=\frac{w}{w+u}\left(1-\left(1-\frac{w+u}{K}\right)^t\right)=$$

$$=\frac{w}{w+u}[1-\exp\left(-\gamma(K,\,w,u)\ \ t\right)],$$

where $\gamma(K, w, u) = \ln(1 + 1/(K - (u + w)))$ is *the rate of forming experience*. Since $(w+u)\in(0,2)$, the learnability condition (19) will be satisfied if $K \geq 2$.

Model 5 (learning and productive activity). Assume that the AE has a foresight horizon T_0. In this horizon, the first $T\in\{0,\,1,\ldots,\,T_0\}$ periods are occupied by learning. In the initial period, the subject chooses an allocation $X = (X_1, \ldots, X_K)$ of his time (the same for all T future periods) among K possible activity types, where X_k is a share of his time for forming experience in activity type k and $X\in\Delta^K$ $=\{s\in\Re_+^K \mid \sum_{k=1}^{K}s_k=1\}$.

Suppose that there is no forgetting. Then from the expression (16) we obtain the following expectation that component k of the AE experience is successfully formed after period t:

$$q_k(X_k, t) = 1-(1-w_k X_k)^t,\ k=\overline{1,K}. \qquad (24)$$

The vector $q = (q_1, \ldots, q_K)$ in learning models will be called the AE's *qualification*.

Upon completing the learning process, the AE proceeds to productive activity. (By assumption, there is no learning during productive activity.) In each period of this activity, UF state k is realized with a probability p_k, which forces the AE to perform the complex activity of type k. If the experience corresponding to activity type k is formed by the given period, the AE performs this type of activity and obtains a reward h_k; otherwise, he obtains nothing.

Thus, in each period $t \in \{T+1, \ldots, T_0\}$ of his productive activity of type k, the AE obtains the expected "income" $h_k\, q_k(T)$, which is the expectation of obtaining the reward h_k in the case of successfully achieving the result of activity k. (If the corresponding experience is formed and not forgotten by the AE, the result is achieved.) The AE's objective function in period t has the value $f(X, t) = \sum_{k=1}^{K} h_k p_k q_k(X_k, T)$, $t \in \{T+1, \ldots, T_0\}$. (There is no learning during productive activity, and hence the probabilities of successfully achieving the positive results are determined by the learning results achieved by the end of learning.)

Consider *the AE's time allocation problem:* maximize the expected "income" per unit time of productive activity,

$$\sum_{k=1}^{K} h_k p_k q_k(X_k,\ T) \to \max_{X\in\Delta^K}. \qquad (25)$$

Substituting the difference (24) into (25), we write this problem as

$$\sum_{k=1}^{K} h_k p_k (1-w_k X_k)^T \to \min_{X\in\Delta^K}. \qquad (26)$$

$$h_k p_k w_k T(1-w_k X_k)^{T-1} = \lambda;$$

$$X_k = \frac{1}{w_k}\left(1-\left(\frac{\lambda}{h_k p_k w_k T}\right)^{\frac{1}{T-1}}\right);$$

$$\sum_{k=1}^{K} X_k = \sum_{k=1}^{K}\frac{1}{w_k} - \lambda^{\frac{1}{T-1}} \sum_{k=1}^{K}\frac{1}{w_k}\left(\frac{1}{h_k p_k w_k T}\right)^{\frac{1}{T-1}} = 1;$$

$$\lambda^{\frac{1}{T-1}} = \frac{\sum_{k=1}^{K}\frac{1}{w_k}-1}{\sum_{k=1}^{K}\frac{1}{w_k}\left(\frac{1}{h_k p_k w_k T}\right)^{\frac{1}{T-1}}}.$$

Solving the constrained optimization problem (26), we find the AE's optimal time allocation for learning:

$$X_k^* = \frac{1}{w_k}\left(1-\frac{\sum_{j=1}^{K}\frac{1}{w_j}-1}{(h_k p_k w_k)^{\frac{1}{T-1}}\sum_{j=1}^{K}\frac{1}{w_j}\left(h_j p_j w_j\right)^{\frac{1}{1-T}}}\right),$$

$$k=\overline{1,K}. \qquad (27)$$

In a particular case (the unitary probabilities of mastering and the same "incomes" from different activity types) we have

$$X_k^* = 1-\frac{K-1}{1+\left(p_k\right)^{\frac{1}{T-1}}\sum_{\substack{j=1\\ j\neq k}}^{K}\left(p_j\right)^{\frac{1}{1-T}}}. \qquad (28)$$

The solution (27) and (28) of the problem (26) with a fixed value T being available, we can formulate *the AE's optimal learning time problem* as follows. If in each period of learning the AE bears fixed costs $c \ge 0$, then the problem is to choose a period to terminate the learning process by maximizing the difference between the expected income and costs:

$$f(X^*, T+1)\ (T_0 - T) - c\,T \to \max_{T\in[0;\,T_0]}. \qquad (29)$$

Substituting the difference (28) into (29), we arrive in the scalar optimization problem

$$(T_0-T)\sum_{k=1}^{K} h_k p_k\left(1-\frac{\left(\sum_{j=1}^{K}\frac{1}{w_j}-1\right)^T}{(h_k p_k w_k)^{\frac{T}{T-1}}\left(\sum_{j=1}^{K}\frac{1}{w_j}\left(h_j p_j w_j\right)^{\frac{1}{1-T}}\right)^T}\right)$$

$$-\,c\,T \to \max_{T\in[0;\,T_0]}. \qquad (30)$$

The solution of the problem (30) will give the AE's expected payoff under sequential learning and productive activity. An alternative is *learning during work*: during all T_0 periods some UF states are realized, and the AE forms the corresponding experience of practical activity, achieving a positive result (and obtaining a "reward" for it) through mastering. Assume that under learning during work, the AE bears costs c in each period. In the absence of forgetting, due to the expression (25), his total expected payoff will be

$$F(P, W,\ T_0) - c\,T_0 = \sum_{k=1}^{K} p_k h_k \times$$

$$\times\left[T_0 - \left(1-p_k w_k\right)\left(p_k w_k\right)^{T_0-1}\right] - cT_0. \qquad (31)$$

The expressions (30) and (31) can be compared in each particular case (for specific values of the model parameters) to answer the following question: which strategy – sequential learning and productive activity or learning during work – is more beneficial for the AE in terms of the total expected payoff?

Example 1. Let the probabilities of mastering be 1, different activity types yield the same "income," and the probability distribution be uniform. Then from the expression (28) it follows that $X_k^* = 1/K$. The optimization problem (30) takes the form $(T_0 - T)h\left(1-(1-1/K)^T\right) - c\,T \to \max_{T\in[0;\,T_0]}$; see the concave curve in (28), illustrating the dependence of this objective function on T.

In this case, the AE's total expected payoff under learning during work (formula (31)) is $h[T_0 - (1 - 1/K)$ $\left(1/K\right)^{T_0-1}$ – cT_0; see the horizontal line in T.

Let $K = 10$, $T_0 = 100$, and $h = 4$. For $c = 2$, there exists an optimal learning time (18 periods) during which sequential learning and productive activity yield a higher total expected payoff (approximately 242.8) than learning during work (approximately 200.0). If the costs per unit time decrease (e.g., $c = 1$), the optimal choice is learning during work; see the optimal choice is learning during work; see Fig. 6*a* vs. Fig. 6*b*. ♦

A similar model can be constructed in the case of nonzero initial conditions for $\{q_j\}$, in which the optimal solution will depend on the AE's initial experience.

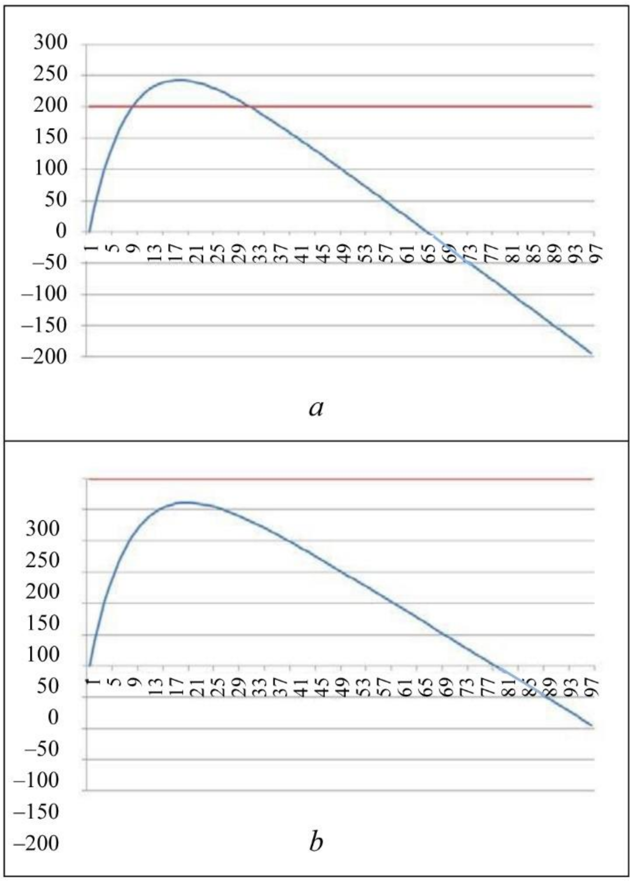

Fig. 6. AE's total expected payoffs in Example 1:
a) c = 2 and b) c = 1 (horizontal line corresponds to learning time T).

Model 6 (deterministic model with one subject). Consider a modification of Model 1 in which there is a unique UF state ($K = 1$) realized with the unitary probability, and the probability of mastering $w(q, t) \in [0,1]$ does not explicitly depend on the history. By analogy with (11), omitting the UF state subscript, we obtain $W(q(t), t) = w(q(t), t)$. The expression (5) gives the difference equation

$$q(t+1) = w(q(t), t) + (1 - w(q(t), t))\, q(t).$$

The corresponding differential equation (see (10)) has the form

$$\dot q(t) = w(q, t)\,(1 - q). \qquad (32)$$

The family of differential equations (32) with an initial condition $q(0) \in [0,1]$ and a Lipschitz function $w(\cdot,\cdot) \in [0,1]$ as the parameter possesses the following properties:

– The solution of equations (32) exists and is unique.

– The experience curve $q(t)$ is strictly monotonically increasing and $\forall t \ge 0\ \dot q(t) \le 1$ (its growth rate is bounded).

– The experience curve $q(t)$ is *slowly asymptotic*, i.e., $\lim_{t\to+\infty} q(t) = 1$ and $\lim_{t\to+\infty} \dot q(t) = 0$.

Allowing the effect of forgetting, we obtain the family of differential equations with an initial condition $q(0) \in [0,1]$ and two parameters – Lipschitz functions $w(\cdot,\cdot) \in [0,1]$ and $u(\cdot,\cdot) \in [0,1]$:

$$\dot q(t) = w(q, t)\,(1 - q) - u(q, t)\, q. \qquad (33)$$

Let us analyze the differential equations (33), characterizing the family of solutions. The following question is of particular interest: for which time-varying functions $q(t) \in [0,1]$ is it possible to find Lipschitz functions $w(\cdot,\cdot) \in [0,1]$ and $u(\cdot,\cdot) \in [0,1]$ so that the function $q : [0, +\infty) \to [0,1]$ will be the solution of (33)?

Proposition 2[2]. *A continuously differentiable function* $q : [0, +\infty) \to [0,1]$ *with a Lipschitz derivative* $\dot q$ *is the solution of equations* (33) *under some Lipschitz functions* $w(\cdot,\cdot) \in [0, 1]$ *and* $u(\cdot,\cdot) \in [0, 1]$ *if and only if*

$$\forall t \ge 0 \quad -q(t) \le \dot q(t) \le 1 - q(t). \qquad (34)$$

Proof. Conditions (34) are immediate from $q(t) \in [0,1]$ and the constraints imposed on $w(\cdot,\cdot)$ and $u(\cdot,\cdot)$. Conversely, let a function $g(t)$ satisfying the hypotheses of this proposition be the solution of equations (33). Choosing

$$w(t) := \dot q(t) + q(t),\ \ u(t) := 1 - \dot q(t) - q(t),\ \ t \ge 0, \qquad (35)$$

we have $w(t)(1-q(t)) - u(t)q(t) \equiv (\dot q(t) + q(t))(1 - q(t)) - (1 - \dot q(t) - q(t))q(t) \equiv \dot q(t)$. Moreover, from the relations (35)

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[2] This result was established by S.E. Zhukovskiy, Dr. Sci. (Phys.–Math.).

it follows that $w(t) \in [0,1]$ and $u(t) \in [0,1]$ for all $t \geq 0$. ♦

Let us find possible equilibria: the right-hand side of (33) vanishes for

$$q(t) = \frac{w(q(t),t)}{w(q(t),t) + u(q(t),t)}.$$

According to (35), in the case $q(0) = 0$, the unique experience curve with a stationary (time-invariant) probability of mastering $\gamma > 0$ is the exponential curve $q(t) = \gamma\,(1 - \exp(-t))$.

4. MODELS OF COLLECTIVE AND SOCIAL EXPERIENCE

Model 7 (mastering social experience; see arrow 1 in Fig. 4). The mastering of social experience by a subject can be described by modifying the general model from Section 1. In the absence of forgetting, assume that: "social experience" contains all the necessary information about optimal actions for any UF values and "guides" the subject's learning; the subject sequentially encounters the UF states (for convenience, in accordance with their numbering); the same state repeats until the probability of forming the corresponding experience component (see formula (13))

$$q_k^1(t) = 1 - (1 - w_k)^t,\ t = 0, 1, 2, \dots,$$

achieves a given threshold q^*. The time (the expected number of repetitions) required is

$$t_k^1(q^*) = \frac{\ln(1-q^*)}{\ln(1-w_k)}.$$

Hence, for all K possible UF values, the threshold q^* will be achieved in the time $t^1(q^*) = \sum_{k=1}^{K} \frac{\ln(1-q^*)}{\ln(1-w_k)}$. In the homogeneous case, $t^1(q^*) = \frac{\ln(1-q^*)^K}{\ln(1-w)}$.

Model 8 (forming individual experience; see arrow 2 in Fig. 4). Generally speaking, Models 1–6 all describe the formation of individual experience. We will consider a particular case: no forgetting and the uniform probability distribution ($p_k = 1 / K$) of various UF states. In this case, experience is formed by the rule

$$L_t^2(W) = 1 - \frac{1}{K}\sum_{k=1}^{K}\left(1 - \frac{w_k}{K}\right)^t,\ t = 0, 1, 2, \dots,$$

representing a particular case of formulas (16) and (17).

In the homogeneous case, $L_t^2 = 1 - \left(1 - \frac{w}{K}\right)^t$. Hence, the threshold q^* will be achieved in the time $t^2(q^*) = \frac{\ln(1-q^*)}{\ln(1-\frac{w}{K})}$.

Proposition 3. *The ratio* $\frac{t^2(q^*)}{t^1(q^*)} = \frac{\ln(1-w)}{\ln(1-\frac{w}{K})^K} \geq 1$, *characterizing the relative effectiveness of mastering social experience compared to forming individual experience, is independent of* q^* *and monotonic in* w *and* K.

Model 9 (mastering collective experience; see arrow 3 in Fig. 4). Joint activity of subjects within collectives implies the possibility of exchanging their experience acquired during the process of activity. (*A team* is a particular case of collectives [19].)

In the absence of forgetting, assume that: a team includes n AEs; for each AE, a certain UF state (same for all subjects) is realized with a given probability distribution P in each period; the subjects form their experience of activity for this state independently within the model (17); after that, the subjects completely exchange their information with each other (i.e., all team members will form their experience for a certain UF state if at least one team member does). The elements of the matrix $\mathbf{W} = \|w_{ik}\|$ can be interpreted as the effectiveness of "learning by one's own and someone else's experience" for different subjects under different UF states.

After t periods, team member i will not master UF state k with the probability $(1 - p_k\, w_{ik})^t$, and all team members will not master it with the probability $\prod_{i=1}^{n}(1 - p_k w_{ik})^t$. We obtain the following experience curve for the entire team and each team member (the probability that none of the team members will encounter a new UF state for the entire team):

$$L_t^3(P, \mathbf{W}) = 1 - \sum_{k=1}^{K} p_k \prod_{i=1}^{n}(1 - p_k w_{ik})^t,\ t = 0, 1, 2, \dots \quad (36)$$

In the case of homogeneous AEs and the uniform probability distribution, the expression (38) takes the form $L_t^3 = 1 - \left(1 - \frac{w}{K}\right)^{nt}$. The threshold q^* will be achieved in the time $t^3(y^*) = \ln(1-q^*) \big/ n\ln(1-\frac{w}{K})$. Therefore, $t^3(q^*) = \frac{1}{n} t^2(q^*)$.

Proposition 4. *The complete exchange of experience between the subjects reduces the time for forming their individual experience proportionally to the number of AEs participating in this exchange.*

This conclusion is valid under a constant probability of mastering w. The decreasing dependence of the probability of mastering $w(n)$ on the number of interacting subjects seems to be more realistic. A promising line is to consider models with the coefficients w_{ij} depending not on the UF states but on the pairs of interacting AEs (subject i acquiring experience from subject j).

Model 10 (forming collective experience; see arrow 4 in Fig. 4). Assume that for each of n AEs, a certain UF state (same for all subjects) is realized with a given probability distribution P in each period. The effect of forgetting will be described by the matrix $\mathbf{U} = ||u_{ik}||$. We denote by π_{ikt} the probability that team member i will master UF state k after t periods. According to the expression (16), $\pi_{ikt} = \frac{w_{ik}}{w_{ik}+u_{ik}}\left(1-\left(1-p_k\left(w_{ik}+u_{ik}\right)\right)^t\right)$. Hence, team member i will not master UF state k after t periods with the probability

$$1-\pi_{ikt} = 1-\frac{w_{ik}}{w_{ik}+u_{ik}}\left(1-\left(1-p_k\left(w_{ik}+u_{ik}\right)\right)^t\right) =$$

$$= \frac{u_{ik}}{w_{ik}+u_{ik}} + \frac{w_{ik}}{w_{ik}+u_{ik}}\left(1-p_k\left(w_{ik}+u_{ik}\right)\right)^t.$$

None of the team members will master this state with the probability

$$\prod_{i=1}^{n}(1-\pi_{ikt}) = \prod_{i=1}^{n}\left(\frac{u_{ik}}{w_{ik}+u_{ik}} + \frac{w_{ik}}{w_{ik}+u_{ik}}\left(1-p_k\left(w_{ik}+u_{ik}\right)\right)^t\right).$$

We obtain the following experience curve (the probability that at least one team member will form the experience for a new UF state realized in period $(t + 1)$; see formula (7)):

$L_{\max}(P, \mathbf{W}, U, t) = 1 -$

$$L_{\max}(P, \mathbf{W}, U, t) = 1-\sum_{k=1}^{K} p_k \prod_{i=1}^{n}\left(\frac{u_{ik}}{w_{ik}+u_{ik}} + \right.$$

$$\left. + \frac{w_{ik}}{w_{ik}+u_{ik}}\left(1-p_k\left(w_{ik}+u_{ik}\right)\right)^t\right), \ t = 0, 1, 2, \ldots \quad (37)$$

As noted above, the group/collective experience criterion can be either the probability $L_{\max}(t)$ (**at least one** of the AEs will encounter a previously known, successfully mastered, and not forgotten UF state; see the expression (37)), or the probability $L_{\min}(t)$ (**each** of the AEs will do so; see the expression (8)):

$$L_{\min}(P, \mathbf{W}, U, t) = \sum_{k=1}^{K} p_k \prod_{i=1}^{n}\left(\frac{w_{ik}}{w_{ik}+u_{ik}} \times \right.$$

$$\left. \times\left(1-\left(1-p_k\left(w_{ik}+u_{ik}\right)\right)^t\right)\right), \ t = 0, 1, 2, \ldots \quad (38)$$

Substantially, the formation of social/collective experience differs from the formation of individual experience in that, in order to "consolidate" the methods of effective activity under a certain UF state, many subjects must encounter this state many times. In the course of modeling, this feature can be reflected, e.g., by a low probability of mastering; see Model 12 below.

Consider the homogeneous case of identical AEs ($u_{ik} = 0$ and $w_{ik} = w$) and the uniform probability distribution. Assuming that the probability of mastering is less than 1 and there is no forgetting, we reduce the expressions (37) and (38) to the following form (also, see (36)):

$$L_{\max}(n, w, t) = 1-\left(1-\frac{w}{K}\right)^{nt}, \ t = 0, 1, 2, \ldots,$$

$$L_{\min}(n, w, t) = \left(1-\left(1-\frac{w}{K}\right)^t\right)^n, \ t = 0, 1, 2, \ldots$$

Based on the results for Model 8, we obtain the following experience curve of one AE with the unitary probability of mastering for all UF states in the absence of forgetting (also, see (13)):

$$L_t^2(w=1) = 1-\left(1-\frac{1}{K}\right)^t, \ t = 0, 1, 2, \ldots$$

Applying trivial transformations to the condition $L_{\max}(n, w, t) = L_t^2$, we establish the following fact.

Proposition 5. *In the case of no forgetting and* $w(n) = K - K\,(1-\frac{1}{K})^{\frac{1}{n}}$, *forming collective experience with the probability of mastering* $w(n)$ *is equivalent to forming individual experience by an AE with the unitary probability of mastering*.

An alternative version of Proposition 5 is as follows: in the case of no forgetting and

$$n(w) = \frac{\ln(1-\frac{1}{K})}{\ln(1-\frac{w}{K})},$$

forming individual experience by an AE with the unitary probability of mastering is equivalent to forming collective experience by $n(w)$ AEs with the same probability of mastering w.

Model 11 (deterministic model with several interacting subjects).

Assume that: there is a single UF state ($K = 1$) realized with the unitary probability; the probabilities of mastering and forgetting do not explicitly depend on the history (also, see Model 6). Then from the expression (1) we obtain the system of difference equations

$$q_i(t + 1) = W_i(\boldsymbol{q}(t), t) + [1 - W_i(\boldsymbol{q}(t), t) - U_i(\boldsymbol{q}(t), t)]\, q_i(t), \ i = \overline{1, n}, \quad (39)$$

and the corresponding system of differential equations

$$\dot{q}_i(t) = w_i(\boldsymbol{q}, t)\,(1 - q_i) - u_i(\boldsymbol{q}, t)\, q_i, \ i = \overline{1, n},$$

with an initial condition $\mathbf{y}(0) \in [0,1)^n$ and two parameters – Lipschitz vector functions $\mathbf{w}: [0;1]^n \times \Re_+^1 \to [0;1]^n$ and $\mathbf{u}: [0;1]^n \times \Re_+^1 \to [0;1]^n$.

The difference equation (39) has a specific structure with given constraints on the functions in the right-hand side. Therefore, we cannot write in its terms, e.g., *the linear models*

$$\Delta q_i(t+1) = \sum_{j \in N_i(t)} a_{ij}(q_j(t) - q_i(t)), \qquad (40)$$

where $\sum_{j \in N_i(t)} a_{ij} \le 1$ (see a survey in Section 3.2 of the book [10] and the work [200]) or *the threshold behavior models* [10]

$$q_i(t+1) = I\left(\frac{1}{|N_i(t)|} \sum_{j \in N_i(t)} q_j(t) \ge \delta_i(t)\right), \qquad (41)$$

where $N_i(t) \in 2^N$ denotes the set of "*neighbors*" of AE i in period t, and $\delta_i(t) \in [0, 1]$ is his "*threshold*."

Therefore, we will proceed by considering the influence of other AEs not on the state of a given AE (the expected level of his experience q) but his probability of mastering. Let the probability of mastering be represented as the sum of two functions,

$$W_i(\boldsymbol{q}(t), t) = d_i(q_i) + D_i(q_{-i}), \; i = \overline{1, n}, \qquad (42)$$

taking values from the range [0, 1] but in the sum not exceeding 1, where $q_{-i} = (q_1, \dots, q_{i-1}, q_{i+1}, \dots, q_n)$ is the opponents' experience profile for AE i.

Some practical examples of such dependencies are "the linear model" (with the superscript L)

$$W^L_i(\boldsymbol{q}(t)) = \alpha_i + \beta_i \sum_{j \in N_i(t)} a_{ij} q_j(t), \qquad (43)$$

which can be compared with (40), and "the threshold model" (with the superscript T)

$$W^T_i(\boldsymbol{q}(t)) = \alpha_i + \beta_i \, I\left(\frac{1}{|N_i(t)|} \sum_{j \in N_i(t)} q_j(t) \ge \delta_i(t)\right) \qquad (44)$$

with constants $\alpha_i + \beta_i \le 1, \; i = \overline{1, n}$, which can be compared with (41).

The first term in the right-hand sides of (42), (43), and (44) can be interpreted as reflecting an *explicit experience* (directly transferred to the subject and mastered by him), whereas the second term as a *tacit experience* (acquired and mastered by the subject indirectly, through interactions with other subjects).

In this class of models, *natural selection* (competition) can be considered, e.g., by letting $W_i(\boldsymbol{q}(t), t) \to 0$ as $q_i(t) << \frac{1}{|N_i(t)|} \sum_{j \in N_i(t)} q_j(t)$.

Example 2. Consider two active elements with the same stationary probability of forgetting u and the probabilities of mastering $W_i(\boldsymbol{q}(t), t) = \gamma_i \dfrac{q_i(t)}{q_i(t) + q_{3-i}(t)}$, $i = \overline{1, 2}$. In a practical interpretation, the AEs compete for a constant amount of a resource for each period, distributed between them proportionally to their experience. The probability of mastering in each period is proportional to the amount of the resource received in the past period. The coefficients of proportionality γ_i can be treated as the individual learning aptitudes of the AEs.

Assume that the AEs differ in the initial values of their experience: $q_1(0) = 0.1$ and $q_2(0) = 0.2$; see the solid and dashed lines in Fig. 7, corresponding to the first and second AEs, respectively. However, despite the worse "starting position," the first AE has a higher learning aptitude: $\gamma_1 = 0.2$ and $\gamma_2 = 0.1$.

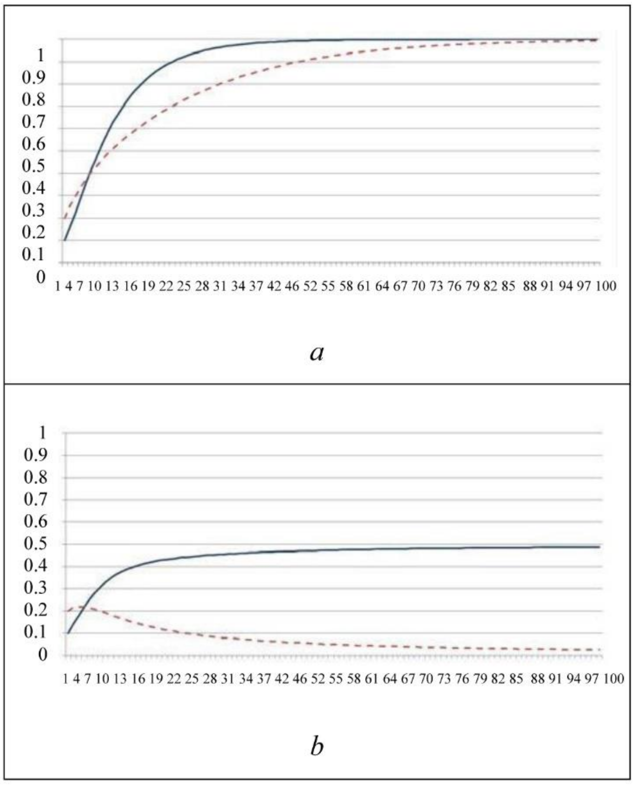

Fig. 7. Dynamics of AE's experience in Example 2:
a) u = 0 and b) u = 0.2 (horizontal line corresponds to time).

In the absence of forgetting, both AEs successfully learn and will equally share the resource on a sufficiently long horizon (see Fig. 7*a*). In the presence of forgetting (u = 0,2), the first AE wins the competition and will obtain the entire resource on a sufficiently long horizon (see Fig. 7*b*). ♦

Model 12 (forming social experience; see arrow 4 in Fig. 4). Within Model 10, collective experience is formed if at least one of the team members forms it. Model 12 rests on the assumption that social experience is formed only if all team members form it.

This approach makes the experience criterion multiplicative over subsets: in the homogeneous case of Model 10 (the identical AEs and the uniform probability distribution), if the society consists of two subsets: $N = N_1 \cup N_2,\ N_1 \cap N_2 = \varnothing,\ |N_1| = n_1,\ |N_2| = n_2$, then

$$L_{\min}(n, w, t) = L_{\min}(n_1, w, t) \quad L_{\min}(n_2, w, t).$$

The expressions $L_{\max}(n, w, t) = 1 - (1 - w/K)^{nt}$ and $L_{\min}(n, w, t) = \left(1 - \left(1 - \frac{w}{K}\right)^t\right)^n$, $t = 0, 1, 2, \ldots$, can be used for estimating the expected time required for achieving a given collective experience level q^*, depending on the number of AEs:

$$t(n, q^*) \sim \frac{\ln(1 - (q^*)^{1/n})}{\ln(1 - \frac{w}{K})}. \quad (45)$$

Clearly, the time (45) grows rather slowly with an increase in the number of AEs (approximately linearly in the logarithm of this number). Since collective experience grows linearly in the number of AE, we arrive at the following result.

Proposition 6. *Social experience is formed approximately by n* ln(*n*) *times slower than the collective one*.

Applying trivial transformations to $L_{\min}(n, w, t) = L_t^2$, we establish the following fact.

Proposition 7. *In the case of no forgetting and* $w(n,t) = K\left\{1 - \left[1 - \left(1 - \left(1 - \frac{1}{K}\right)^t\right)^n\right]^{\frac{1}{t}}\right\}$, *forming social experience with the unitary probability of mastering is equivalent to forming individual experience by one AE with the probability of mastering w*(*n*, *t*).

Due to Proposition 7, social experience can be viewed as the experience of one integral and virtual subject.

CONCLUSIONS

The results of Sections 3 and 4 show that the well-known learning models [11, 14, 18], including the learning curves (20)–(26), correspond to particular cases of the general experience model from Section 2. As was shown in [14, 18], the learning models, in turn, generalize the following models: testing of complex systems and checking their characteristics; increasing the efficiency of mass production during mastering (the model of T.P. Wright and his followers); software testing; dissemination of knowledge (ideas, theories, concepts) in society; knowledge management and extraction/acquisition; machine learning; iterative learning and testing of knowledge in pedagogy, psychology, and physiology of humans and animals. The surveys [14, 18] referred to numerous works on the experiment-based characterization of nontrivial processes that underlie learning models.

Let us outline some promising lines for further research:

- The formation of experience is an essential component of any activity. Therefore, it seems strategically important to develop a general mathematical model of complex activity (with operational decomposition and aggregation of particular models) within MCA, reflecting the active choice of subjects and considering the processes of forming their experience.

- The system of classification bases for experience models (see Section 2) yields various particular models of forming and mastering individual and collective experience. The development and study of such models are a well-founded "tactical" step. Here, some of the promising areas are as follows: exploring the joint formation of experience during work, optimizing the duration of learning before the transition to productive activity, analyzing the impact of forgetting and the history length, identifying the role of the time dependence of the uncertainty factor states, revealing the influence of the experience structure (the logical connections between its components), and optimizing and managing the experience formation process.

- The experience model proposed above is rich enough for describing many phenomena and processes:

– *personnel management* (recruitment, placement, *development*, promotion, and dismissal) and *human capital models*;

– *risk management* and *information security management*;

– *evolution* (including *adaptation*, *competition*, and *natural selection*) in biological systems (involving the conventional mathematical apparatus for this range of problems – inite automata[3] [21, 22], differential equations [23–25], evolutionary games [26, 27], etc.);

– different characteristic times and hierarchy for the translated (explicit and tacit) and nontranslated experience, which are considered within *psychological* and *sociological* approaches;

– selection, formation, mastering, consolidation, and transmission of social experience, considered as *cultural phenomena*.

[3] The areas of research mentioned here are very extensive. Without claiming to be exhaustive, we therefore refer to several classical monographs and/or modern surveys.

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TAX INCENTIVES FOR PROSOCIAL VOTING IN A STOCHASTIC ENVIRONMENT

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Abstract. Three income redistribution algorithms supporting the agents with prosocial voting are considered within the Voting in Stochastic Environment (ViSE) model of social dynamics. The first algorithm is income tax; the second one ensures that the income of each agent with the prosocial strategy is not smaller than the average income; the third one ensures that the average income of prosocial agents is not smaller than that of the entire society. The social utility of prosocial voting is analyzed. The three algorithms are compared with each other. The effectiveness of income tax depends on the environment. The second and third algorithms do not suffer from this disadvantage. However, under certain conditions, the second algorithm provides too many bonuses to prosocial agents. With any of these income redistribution algorithms, the egoists get more profit than in a society without any prosocial agents. Thus, whenever such taxation schemes motivate some participants to choose the prosocial voting strategy, this will increase the expected income of all agents.

Keywords: ViSE model, altruism, voting, social dynamics, tax, pit of losses, prosocial behavior.

INTRODUCTION

Basic elements of ViSE model

In the ViSE (Voting in Stochastic Environment) model [1], a *society* consisting of *n agents* is considered. Each agent is characterized by a social attitude determining his voting strategy and current wealth (*capital*), expressed as a real number. A strategy is an algorithm for using information about a proposal and society to support (or not) the proposal put to the vote. A *stochastic environment* generates a proposal to society—a vector of realizations of independent identically distributed random variables. Each *i*th component of this vector is a *proposed capital increment* for agent *i*. The proposal is put to the vote; each agent votes for or against it, following his voting algorithm. If the number of votes “for” exceeds 50%, the proposal is approved, and the corresponding proposal components are added to the agents’ capital. (In a more general case, the number of votes “for” must exceed αn, where α is a *relative voting threshold* and n is the number of agents). Otherwise, the capital of the participants remains the same. Proposals are put to the vote consecutively; voting on one proposal is called a *move* or *step* in a sequence of decisions. In a series of votes, the parameters of the distribution generating the proposals and the voting strategies of the agents are fixed. In this paper, the Gaussian distribution is considered a generator of proposals. The research aims to analyze the effectiveness of the voting strategies of agents and collective decision procedures by the criteria for increasing the individual capitals of agents and their sum.

In the papers [1, 2] and other publications on the ViSE model, many of its variants were considered by imposing additional conditions. Specific features of the model related to the subject of this study will be discussed below.

The ViSE model refers to the theory of voting, which, in turn, is part of social choice. Unlike several game-theoretical models, agents in the ViSE model are not treated as players maximizing their utility functions. They possess capital, but their behavior is not always reduced to capital maximization: generally speaking, their behavior has an arbitrary structure given by the researcher. The main element of this behavior is the agent's personal (and not necessarily constant) voting algorithm. Society is characterized by a mechanism for making collective decisions. The researcher analyzing, within the model assumptions, the effectiveness of the individual and collective decision mechanisms is a social designer trying to understand which of the identified patterns may be useful in real life.

Pit of losses

For the ViSE model, the following scenario is known: a society consisting of agents with an egoistic social attitude acts irrationally, approving proposals that are generally disadvantageous to it since they lead to a negative total capital increment [2]. An *egoistic strategy* is a strategy in which the participant supports a proposal if and only if it increases his capital. The effect of impoverishment and ruin of society in this scenario is usually called *the pit of losses*.

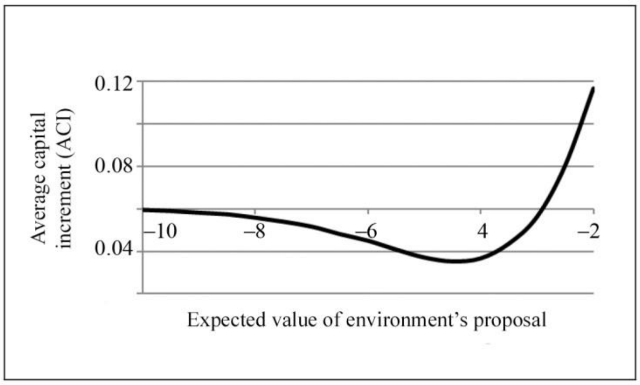

Fig. 1. Average capital increment in society of 25 agents with egoistic strategy.

This situation is illustrated in Fig. 1, where the vertical axis corresponds to the average capital increment (ACI) of the participant per one step in an unfavorable environment, and the horizontal axis corresponds to the expected value of the proposals. Throughout the paper, the components of an environment proposal are realizations of independent Gaussian random variables with σ = 12. In the situation under consideration, the "unbiased" proposals of the environment lead to the same result as the agenda manipulation in the Malishevskii paradox described, for example, in [3, pp. 92–95].

Indeed, the ruin of society due to implementing the decisions made by the majority of votes of its classically rational participants is, in some sense, a paradoxical effect. It has the following explanation: in the zone of moderately negative expected values, some proposals yield a small capital increase for most agents and a total decrease by absolute value for the rest of society. Such proposals are approved by a majority of votes, but the total welfare of society decreases during their implementation.

A way to secure against the pit of losses is to select the best voting threshold α [2]. In the case of an unfavorable environment, this optimum is usually above 50%. The corresponding dependence of the total capital on the voting threshold is comparable to the results of [4], where the influence of other social mechanisms (bargaining, bribes) on the effectiveness of decisions determining social dynamics was studied.

The influence of altruistic agents on social welfare

Another factor reducing the pit of losses is the presence in society of agents who, when voting, are guided not by personal interests but by those of the entire society. An agent's strategy supporting a proposal if and only if it increases the total welfare of society is called *altruistic*. Behavior that benefits society is also called *prosocial*. In the case shown in Fig. 1, replacing three egoists with altruistic participants appreciably increases the average capital increment of society; see Fig. 2.

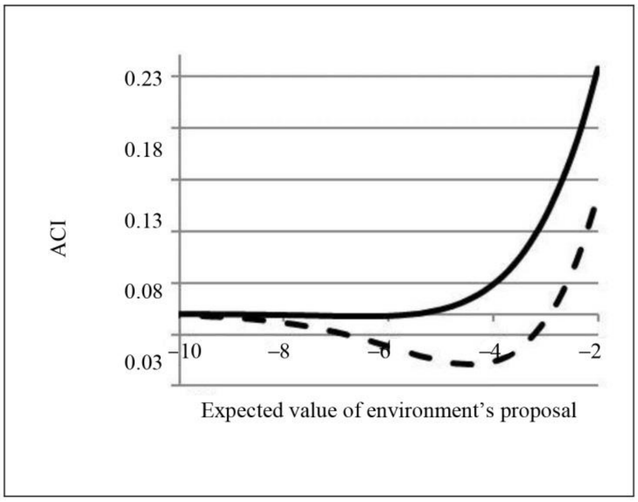

Fig. 2. Comparing effectiveness of societies composed of egoists only and egoists plus altruists: ——— 22 egoists and 3 altruists, – – – 25 egoists.

The results presented in this paper were obtained by simulations using *ViSE Experiment Module* [5]. Obtaining the same results analytically is a problem of at least high complexity: for the corresponding multiple integrals it is impossible to find a general expression in terms of standard functions.

The presence of altruistic agents always positively affects social welfare: the share of socially irrational decisions made is significantly reduced. However, the altruists themselves are outsiders in this case: according to Fig. 3, their capital is considerably smaller than the society average. This pattern is also observed for other values of the parameters.

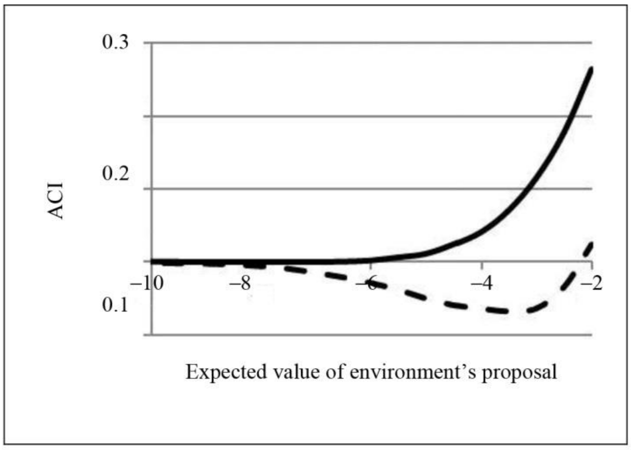

Fig. 3. Average capital increment in society of 22 egoists and 3 altrusists:
egoists, altruists.

This is because when voting, altruists neglect the change in their capital in the case of implementing the environment's proposal. Therefore, they support, among others, the proposals enriching society as a whole but reducing their capital. Thus, the presence of altruistic agents is beneficial to society, but their role is "sacrificial."

In real life, a society not encouraging the participants who secure it against ruin seems unfair. Moreover, in a version of the model where participants can change their voting principle to an individually more profitable one, providing prosocial agents with an income of at least the average will ensure that they do not change their strategy to selfish. If the position of prosocial agents becomes better than the society average, then the share of such agents in this version will grow, leading to an increase in social welfare.

Within the ViSE model, there is no need to represent any (particularly, prosocial) behavior by maximizing the agent's utility function: this would complicate the description of complex behavioral types common in real life. For example, a social attitude aimed at supporting the entire society is conditioned by philanthropy, the need to maintain reputation, etc., but such motivation can fade into the background under serious material losses and then return without external reasons.

The goal of this study is to analyze material support mechanisms for agents with prosocial strategies. The agents are allowed to change their voting strategies to individually more profitable ones, but specific mechanisms for such a change are not considered: this is not required to achieve the goal. The paper proposes and investigates several algorithms for supporting prosocial agents based on income redistribution (in other words, prosocial voting is motivated by taxes.) Also, the paper investigates the effectiveness of the prosocial strategy under various parameters of the environment.

Voting in a society of altruistic agents was studied in the earlier paper [6]. Like in the model considered below, the agent was assumed to maximize the welfare function during voting, the value of which monotonically increases with the growth of consumption of any agent (analogy of capital increase). The progressive taxation schemes quadratically dependent on production were put to the vote, and the presence of "self-approving" equilibrium was established.

In the paper [7], as a result of laboratory experiments, it was found that monetary incentives motivate prosocial behavior in the case of its private (non-public) nature. The work [8] examined reducing internal motivation for prosocial behavior with its monetary incentives on an example of "green" (environmental) taxes. The authors concluded the following: if a tax leads to positive changes in society, its introduction is justified even under decreasing the "moral" motivation.

Choosing an appropriate taxation scheme for agents by the majority voting was studied in [9]. The main result was the conclusion that progressive tax is beneficial to the "middle class." Also, choosing a linear income tax by voting was considered in [10].

1. EFFECTIVENESS OF ALTRUISTIC STRATEGY AS FUNCTION OF ENVIRONMENT'S FAVORABLENESS

As noted above, the presence of a small share of altruistic agents in society can significantly reduce or even eliminate the pit of losses. Let us identify the environment's parameters under which the presence of altruists increases the capital of society most of all. To do this, we compare the average capital increments for the societies consisting of 25 egoists and 22 egoists plus 3 altruists under $\sigma = 12$ and different expected values of the environment's proposals. The comparison results are demonstrated in Fig. 4, where "the benefit from the altruistic strategy" is the difference between the ACIs of the two societies mentioned.

As Fig. 4, agents with the altruistic strategy bring maximum benefits to society in a *neutral* environment that generates positive and negative proposals with equal probability. In an unfavorable environment most dangerous for society (the "bottom" of the pit of losses), the help of the three altruists is less in absolute terms. At the same time, it is enough to eliminate the pit of losses almost completely.

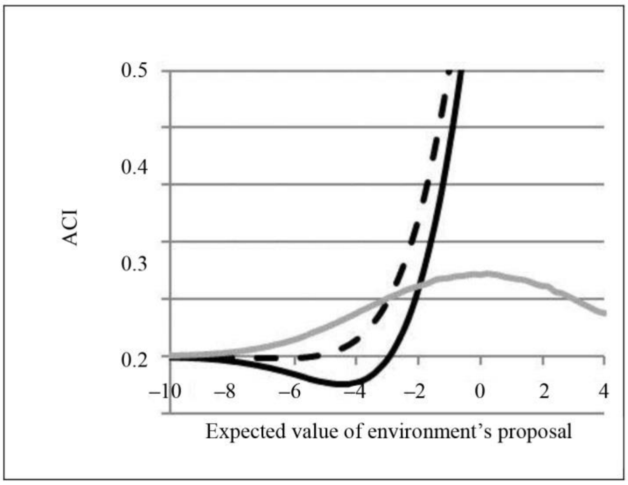

Fig. 4. Comparing average capital increments and their difference for societies of egoists only and egoists plus altruists:
25 egoists, 22 egoists and 3 altruists, benefit from altruistic strategy.

2. TAX INCENTIVES FOR ALTRUISTIC VOTING

As noted above, altruistic agents helping society make rational decisions in a moderately unfavorable environment need support to increase the relative welfare of altruists and, presumably, prevent their shift to egoism. Consider possible schemes for redistributing society's income in their favor. This redistribution can be treated as levying a tax. The simplest scheme is "flat" income tax. After each approved proposal, the egoists who have received a positive capital increment deduct v percent of their capital increment for the current step to the fund, and the fund is equally divided among the agents with the altruistic voting strategy. The effectiveness of a support method will be assessed by the increase in the altruist's ACI after introducing the tax and by the increase in the egoist's ACI compared to his increase in a society consisting of such agents only.

Figure 5 shows the average capital increments of different participants under 13% income tax. As before, society consists of 22 egoists and 3 altruists (12% of society). An income tax rate of 13% (further called *the first income redistribution algorithm* or *the first taxation scheme*) is applied. Clearly, due to the redistribution, the income of altruists *significantly* exceeds that of egoists. At the same time, the welfare of egoists remains higher than in the society without altruists. Thus, the agents voting altruistically do good to the entire society, becoming the main beneficiaries (the wealthy stratum): being altruistic[1] is very advantageous. If agents are allowed to change their strategy, then the egoists will be willing to vote altruistically to turn from taxpayers to tax fund recipients.

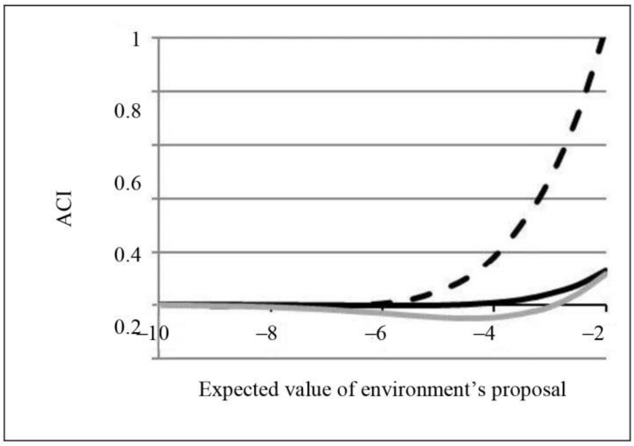

Fig. 5. Comparing average capital increments for different participants under 13% income tax:
altruists (society of 22 egoists and 3 altruists),
egoists (society of 22 egoists and 3 altruists),
egoists (society of 25 egoists).

In this regard, note that the difference between the welfare of egoists and altruists depends on the ratio of the number of agents with different strategies. The more altruists there are in the society, the less increase each of them will receive from the tax fund. As a result, the ACI curves of the two groups of participants will converge and finally match. The difference in the income of the different groups also depends on the environment's favorableness. For these reasons, it is natural to select the income tax rate depending on the parameters of society and the environment. A fixed rate can lead to insufficient or, conversely, excessive support for altruists. For example, in Fig. 5, there is an income gap that is difficult to justify.

Thus, an additional criterion for assessing support methods can be the dependence of the tax effect on the environment's parameters. The problem described above can be solved by more flexible taxation schemes. Here is one example, also called *the second income redistribution algorithm*, or *the second taxation scheme*.

After each approved proposal, calculate the difference ($\bar{c}$ – ACI) for the participant at the current step.

Calculate the sum of all positive excesses above the ACI over society: $S_{exc} = \sum_{i=1}^{n} I\left\{c_i > \bar{c}\right\} \cdot (c_i - \bar{c})$,

where c_i is the capital increment of agent i at the current step, and $I\{\cdot\}$ denotes the indicator function of an appropriate event. This function takes value 1 if the assertion within the curly brackets is true and 0 otherwise.

[1] In some cases, the term "altruists" is enclosed in quotation marks to emphasize that it refers to participants with the altruistic voting strategy. In view of the social support under consideration, the motivation for choosing this strategy can be mercantile, that is, selfish. The term "egoists" also refers only to the voting strategy.

Calculate the amount donated to the altruists for making their capital increments not smaller than the society average: $S_{don} = \sum_{i=1}^{n} altr(i) I\{c_i < \bar{c}\} \cdot \{\bar{c} - c_i\}$, where $Altr(i) = 1$ if agent i is altruist and $Altr(i) = 0$ otherwise.

Calculate *the income withdrawal rate* $u = \frac{S_{don}}{S_{exc}}$.

Charge the tax $(c_i - \bar{c})u$ from each agent i whose capital increment is greater than $\bar{c}$.

Redistribute the tax fund collected at this step among the altruists whose capital increments are smaller than $\bar{c}$, making them equal to $\bar{c}$.

Note that the coefficient u cannot exceed 1: the sum S_{exc} includes *all* excessive incomes, and S_{don} is the total income deficit (in relation to the average) only for the altruists who are proposed the capital increments smaller than the society average. Therefore, the income of the "lucky ones" with $c_i > \bar{c}$ cannot fall below the value $\bar{c}$.

Well, the second taxation scheme ensures that every altruist will obtain a capital increment of at least the society average from each proposal. Moreover, the tax is paid not only by egoists but also by altruists, who initially obtained a capital increment above the average. The expected capital increments of different participants under the second taxation scheme are shown in Fig. 6.

We emphasize that tax collection determines only the redistribution of capital within society: the decision-making process remains the same, and the taxes, therefore, do not affect the average capital of society. According to Fig. 6, the incomes of altruists, like in the case of the first taxation scheme, appreciably exceed those of egoists. The difference from the first taxation scheme is that in an unfavorable environment, the income of altruists turns out to be even higher. In fact, depending on the number of participants with the altruistic voting strategy, they can obtain either more or less income under the second taxation scheme compared to the first one (income tax).

Now consider *the third income redistribution algorithm* (*the third taxation scheme*), intended to reduce the gap between the incomes of altruists and egoists. This income is collected and redistributed as follows:

After each proposal approved, calculate the difference ($\bar{c}$ – ACI) of the participant at the current step and the ACIs $\bar{c}_{altr}$ of altruists. If $\bar{c}_{altr} \geq \bar{c}$, implement the current proposal without any changes; otherwise, pass to Step 2.

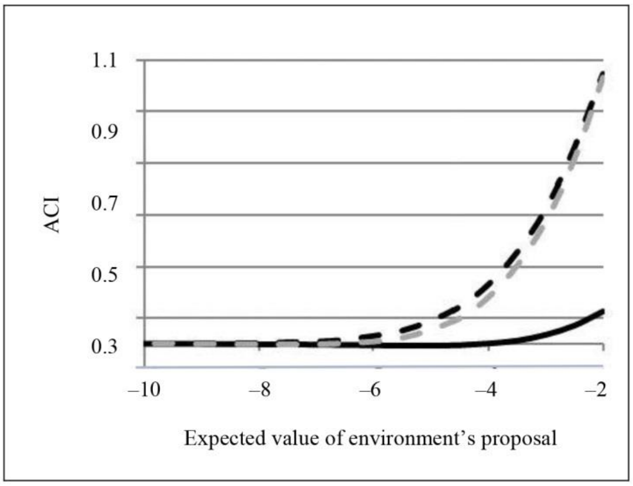

Fig. 6. Comparing average capital increments of different participants under different taxation schemes:
egoists (second taxation scheme), altruists (second taxation scheme), altruists (first taxation scheme).

Calculate the sum of all positive excesses above the average capital increment: $S_{exc} = \sum_{i=1}^{n} I\{c_i > \bar{c}\} \times$ $\times (c_i - \bar{c})$ by analogy with the second taxation scheme.

Calculate the amount donated to the altruists for making their capital increments not smaller than average over society: $S_{don} = \sum_{i=1}^{n} altr(i) I\{c_i < \bar{c}\} \cdot \{\bar{c} - c_i\}$, similar to the second taxation scheme.

Calculate *the income withdrawal rate* $\tilde{u} = \frac{(\bar{c} - \bar{c}_{altr}) n_{altr}}{S_{exc}}$, which ensures the altruists the average capital increment over society.

Calculate *the raise rate* $q = \frac{(\bar{c} - \bar{c}_{altr}) n_{altr}}{S_{don}}$.

Charge the tax $(c_i - \bar{c})\tilde{u}$ from each agent i whose capital increment is greater than $\bar{c}$.

For each altruist whose initial capital increment at this step is lower than the average one $\bar{c}$, pay the extra amount $(c_i - \bar{c})q$ from the tax fund.

This algorithm guarantees that at each step, the *average* capital increment of altruists is not smaller than $\bar{c}$ (the average capital increment in society). If this increment is initially smaller, then it is raised to the society average by payments from the tax fund; otherwise, it remains unchanged.

According to Fig. 7, the ACI of altruists under the third taxation scheme is appreciably smaller compared to the second one. The excess of the altruist's income over that of the egoist is also less.

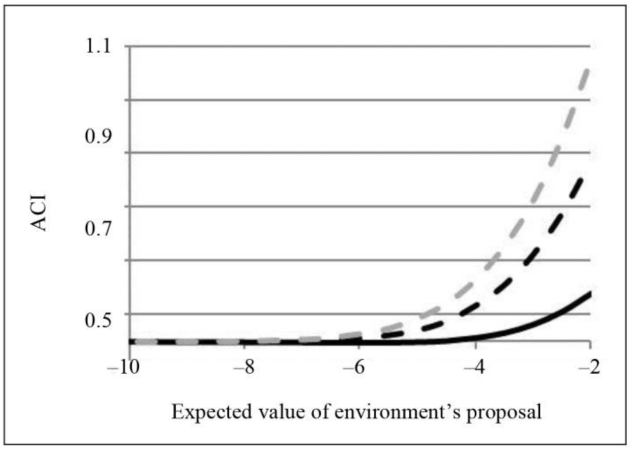

Fig. 7. Comparing average capital increments of different participants under different taxation schemes:
egoists (third taxation scheme), altruists (third taxation scheme), altruists (second taxation scheme).

We explain this pattern as follows. Suppose that the altruists obtained increments on average greater than the egoists per step. Then the third taxation scheme is not applied. At the same time, the second taxation scheme would provide a positive increase for those altruists whose initial income was below average. If a capital increment above $\bar{c}$ was only for the altruists, then the total income of the altruists under the second taxation scheme would not change; otherwise, it would increase due to egoists and become higher compared to the third taxation scheme.

Now consider the case in which the altruists obtained, on average, a smaller capital increase per step than the egoists. Under the third taxation scheme, after the redistribution of income, the ACIs of altruists and egoists will be equal to each other and the value $\bar{c}$. Under the second taxation scheme, every altruist who originally had a capital increment below $\bar{c}$ will receive an increment equal to $\bar{c}$. The increase in the altruist's capital, which initially exceeded the value $\bar{c}$, will remain above $\bar{c}$. Therefore, the average total capital increment of altruists under the second taxation scheme in each case will be not smaller compared to the third one. Due to the stochastic nature of proposals, proposals will be occasionally approved with probability 1, in which the second taxation scheme will provide altruists with a greater capital increase than the third one.

The above reasoning proves that the expected capital increment of altruists under the third taxation scheme (and non-zero variance σ^2) is always lower compared to the second one.

The results for all societies considered are summarized in Fig. 8.

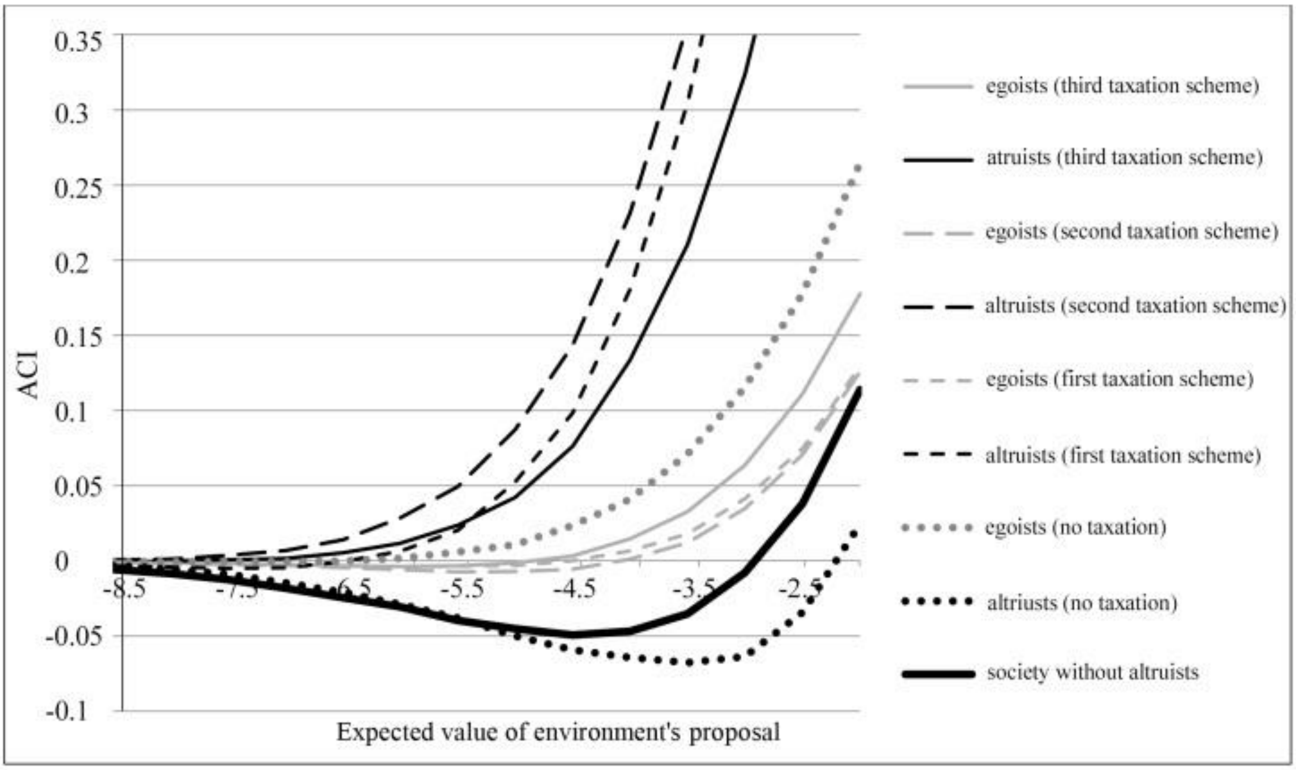

Fig. 8. Summarized data on average capital increments of different participants under different taxation schemes.

CONCLUSIONS

This paper has proposed and investigated some ways to support agents with prosocial behavior within the ViSE model. It has been established that altruists increase the capital of society, which helps to eliminate the pit-of-losses paradox. Without income redistribution, the welfare of prosocial agents is significantly smaller than that of egoistic ones. Hence, by a logical assumption, such agents would think about changing their strategy, thereby worsening society's state. Three income redistribution algorithms have been considered: income tax (the first taxation scheme), a tax with "pulling" at each step *each altruist's income* to the society average (the second taxation scheme), and a tax with "pulling" *the average income of all altruists* to the society average (the third taxation scheme). The application of each taxation scheme mentioned provides altruists with a greater average capital than egoists, which creates a material incentive for them to choose the altruistic voting beneficial for society. In this case, the benefit of society is that all participants, both egoists and altruists, obtain a greater average capital than in a society without altruists.

The problem of excessive bonuses to altruists may arise. Of the approaches considered, the third taxation scheme best secures against it, rewarding altruists on average in a smaller volume than the second taxation scheme. The consequences of introducing a flat-rate income tax (the first taxation scheme) strongly depend on the environment's favorableness and the share of altruistic agents, which indicates its inflexibility. At the same time, the administration of the second and third taxation schemes requires complete information on the income of participants and more complex calculations, which makes these taxes less transparent and somewhat complicates their practical application. In all income redistribution algorithms considered, egoists obtain a higher income than in a society without prosocial agents, which makes the appearance of altruists supported by tax attractive, particularly for egoists.

In the paper [11], an optimal taxation scheme was intended to ensure social welfare by maximizing the total utility function of society. In this paper, another criterion of tax optimality has been proposed and investigated: the degree of support for the agents whose strategy contributes to increasing social welfare. The patterns identified during this study can be used to develop real taxation algorithms.

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TWO-STAGE DUAL PHOTON SWITCHES IN AN EXTENDED SCHEME BASIS

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Abstract. This paper proposes a new method for constructing a two-stage dual photon switch with enhanced functional characteristics in the system basis of low-channel photon switches and photon multiplexers and demultiplexers. The method yields non-blocking switches with static self-routing. Also, the method ensures a significant speed-up for the switches with the same number of channels and a significant increase in the number of channels with the same speed compared to the non-blocking dual switches known previously. The non-blocking self-routing dual photon switches presented below have the highest possible speed and a maximum possible number of channels with almost the same complexity. As is shown in the paper, the dual switches have a switching complexity comparable with full switches and, at the same time, a lower channel complexity.

Keywords: physical level, photon switch, dual switch, multistage switch, conflict-free self-routing, non-blocking switch, static self-routing, quasi-complete digraph, switching properties, direct channels, scalability and speed.

INTRODUCTION

In the papers [1–3], a technique was proposed for constructing non-blocking photon switches with static self-routing for optical supercomputer systems. A system area network is non-blocking if for any packet permutation, conflict-free paths from sources to sinks can be built in it. A system area network is self-routing if conflict-free paths can be built locally over network nodes without their interaction based on routing information in packets only. Finally, self-routing is static if any source can independently choose conflict-free paths to its sink without interacting with other sources.

Until recently, the problem of constructing non-blocking system area networks has not been completely solved. At best, rearrangeable multistage Clos networks were proposed [4, 5]. In rearrangeable networks [4], a conflict-free implementation of any permutation of data packets is possible, but a conflict-free schedule for each permutation has to be compiled separately, and it is not self-routing.

Non-blocking Clos networks [6] are known. However, for such networks, no static or even dynamic routing procedures have been proposed so far. In addition, they have significantly greater complexity than rearrangeable Clos networks and are not applied in practice.

The p-ary r-dimensional generalized hypercube is a widely used system area network [7, 8]. However, for $r > 2$, it is neither non-blocking nor even rearrangeable. For making the generalized hypercube rearrangeable, the number of channels in it should be increased. For example, for $p = 2$, it suffices to double the number of channels of one dimension [9]. In this case, a double hypercube in which all channels are duplicated has conflict-free schedules for two permutations at once [10].

The three-dimensional hypercube with dynamic self-routing has been made non-blocking recently; see [11]. However, this required an almost threefold increase in the number of channels and the degree of constituent switches.

Nevertheless, the problem of constructing a non-blocking self-routing network has a solution in the special case of networks with the topology of a quasi-complete graph and digraph [12]. Unfortunately, the number N of users (processors) in such networks does not exceed the square of the degree p of composite switches: $N = p(p - 1)/\sigma + 1$ and $N = p^2$, respectively, where σ is the number of different channels between users. Moreover, they have a greater switching complexity and a smaller channel complexity compared to a complete graph. A quasi-complete graph is isomorphic to such a mathematical object as an incomplete balanced block design [13–17]; a quasi-complete digraph is isomorphic to a two-dimensional generalized hypercube (parallelogram) or a two-dimensional multiring. In particular, the two lowest dimensions of the four-dimensional hypercube Dragonfly (CRAY XC-30) [8] are implemented in the form of a 6×16 parallelogram and represent a non-blocking self-routing subnet.

Networks with the topology of quasi-complete graphs and digraphs can be extended by increasing the number of their users without changing the non-blocking and self-routing properties. This effect is achieved by the invariant extension method of system area networks [12, 18]. Unfortunately, such an extension further increases the switching complexity of extended networks, strongly restricting their scalability.

Note that most modern system area networks (Clos networks [5], generalized hypercubes [8], multidimensional tori [19], the hierarchy of complete graphs from IBM [20], and the Melanox thick tree [21, 22]) cannot implement an arbitrary permutation of packages in one session without conflict. It is often necessary to resend the packets blocked in buffers. At present, photon computers are being developed [23], whose photon networks contain no buffer memory for blocked packets in the channels.

In the papers [1–3], the concept of dual networks was introduced, and new dual networks were constructed for photon computers, implementing arbitrary permutations of packets without conflict at a slightly lower channel speed. The methodology for constructing these networks is based on four fundamental principles:

• Applying a four-channel switch of a new structure, which is *dual by the conflict resolution approach*. It combines the bus method (separation of conflicting signals to different cycles in one channel) and the switch method (separation of conflicting signals to different channels).

• Assuming the parallel transmission of signaling and control information for switches at different frequencies for each data bit. This assumption eliminates the problem of synchronizing signals from different channels.

• Cascading switches so that the *I*th channel of the *J*th switch on one stage is connected to the *J*th channel of the *I*th switch on the next stage. With such exchange links, the previous and next stages must include the same number of switches, each having the same number of channels. This method allows constructing multichannel switches with a small number of stages.

• Balancing the speed and complexity of a multistage switch using the invariant extension method of system area networks [4], which preserves the non-blocking property and speed of the switch with an increase in the number of its channels. This method involves an extended circuit base consisting of both $p \times p$ switches for p channels and pairs of $1 \times p$ multiplexers and $p \times 1$ demultiplexers, where $p \geq 2$.

In the papers [1–3], one of the block designs of a dual 4×4 switch is a two-stage circuit of four demultiplexers and four multiplexers with feedback links through the delay lines (Fig. 1). The switch stages are interconnected by exchange links.

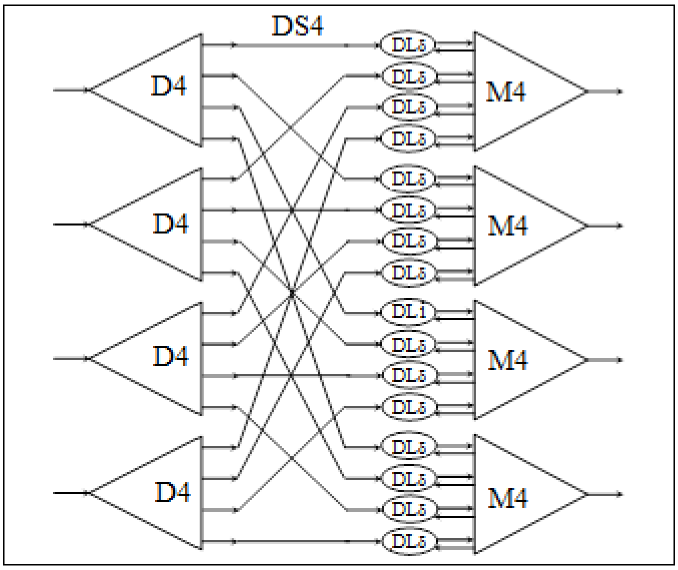

Fig. 1. Generalized block design of dual switch SS4:
D4 – four-input demultiplexer, M4 – four-output multiplexer, DLδ – delay line of length δ signals.

If the complexity of multiplexers M4 and demultiplexers D4 is measured by the number of switching points (equal to 4), then the switching complexity of the switch is $S_1 = 32$.

The combination of two control frequencies uniquely determines the demultiplexer's mode in which the information signal can be directed to one of the four outputs. Possible combinations of control signals are presented in Table 1 below.

Table 1

Control frequencies for photon 4×4 switch

Output no.	Control frequencies
1	$\lambda_1\lambda_1$
2	$\lambda_1\lambda_2$
3	$\lambda_2\lambda_1$
4	$\lambda_2\lambda_2$

The signals from the demultiplexer's outputs are supplied to the multiplexer's inputs. One of them is passed to the output, and the rest return to their delay lines DLδ. The switch implements dynamic signal delay using the feedback links through DLδ.

The dual switch SS4 provides non-blocking with static self-routing under an appropriate length δ of the delay line. The value δ depends on the number of the stage in which the switch SS4 is used.

For the first stage, δ = 1. Let four signals of duration T_0 be simultaneously supplied to the inputs of the switch SS4, all received in one cycle. With dynamic signal delay at the switch outputs, there are four possible options of signal placement: one at each output, two signals in a row at two outputs, one and three signals in a row at two outputs, and four signals in a row at one output. They are shown in Fig. 2. As a result, the switch SS4 will be non-blocking for any input traffic when the period $\underline{T}_1$ of information signals is four cycles.

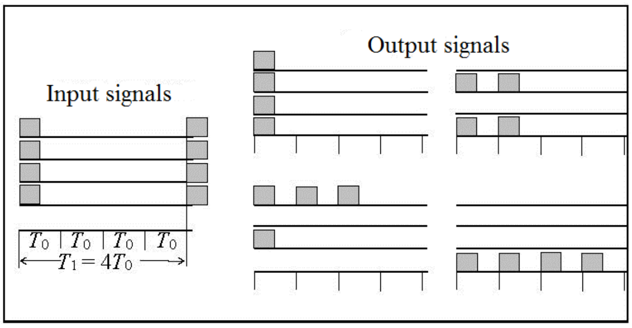

Fig. 2. Different distributions of input signals among lines and cycles.

Therefore, the non-blocking self-routing switch SS4 has the following performance characteristics: the signal period $\underline{T}_1 = 4 = \underline{N}_1$ cycles, the number of channels $\underline{N}_1 = 4$, and the switching complexity $\underline{S}_1 = 32 = \underline{N}_1^{5/2}$.

In the papers [1–3], the two-stage 16×16 switch $S_2$16 with exchange links was considered, consisting of four switches SS4 on each stage. The first stage includes DL1, and the second stage includes DL0 (no delay lines). For an arbitrary permutation of packets, $S_2$16 turned out to be a non-blocking self-routing switch with the following performance characteristics: the number of channels $\underline{N}_2 = 16$, the signal period $\underline{T}_2 = 4 = \underline{N}_2^{1/2}$ cycles, and the switching complexity $\underline{S}_2 = 2{\cdot}4{\cdot}32 = 256 = \underline{N}_2^2$.

In the papers [1–3], the four-stage 256×256 switch $S_4$256 with exchange links was considered, consisting of 16 switches $S_2$16 on each stage. It consists of four stages of switches SS4. The first stage includes DL1, the second stage includes DL4, the third stage includes DL15, and the fourth stage includes DL0 (no delay lines). For an arbitrary permutation of packets, $S_4$256 turned out to be a non-blocking self-routing switch with the following performance characteristics: the signal period $\underline{T}_4 = 49 \approx 3\underline{N}_4^{1/2}$ cycles, the number of channels $\underline{N}_4 = 256$, and the switching complexity $\underline{S}_4 = 2{\cdot}16{\cdot}256 = 8192 = \underline{N}_4^{1.625}$. Note the large digit period (low speed) and small complexity of this switch.

In the papers [1–3], the "speed–complexity" ratio was balanced using an invariant extension of small-period switches. In particular, the switch $S_2$16 was extended through external multiplexers M4 and demultiplexers D4. As a result, the non-blocking self-routing switch $S_3$64 was constructed, consisting of 16 switches $S_2$16 and 64 demultiplexers D4 and multiplexers M4. This switch has the following performance characteristics: the number of channels $\underline{N}_3 = 64$, the signal period $\underline{T}_3 = 4 = N_3^{1/3}$ cycles, and the switching complexity $\underline{S}_3 = 16{\cdot}256{+}4{\cdot}128 = 4\,608 = \underline{N}_3^{2.028}$.

In this paper, a non-blocking switch with the quasi-complete digraph topology will be used not only to extend two-stage switches but immediately to construct them. The resulting switches have a higher speed (a smaller signal period) and more channels than those proposed in [1–3].

In Section 1, we present the structure and characteristics of a switch with the quasi-complete graph topology for any number p of ports. In Section 2, we discuss the main idea of increasing the speed and the number of channels for $p = 2$ on an example of constructing a non-blocking three-stage switch. In Section 3, this idea is fully realized by constructing a non-blocking two-stage switch for $p = 2$. In Section 4, we construct a similar two-stage switch for any p. In Section 5, we describe an extension of a two-stage switch into switches with more channels and a constant signal period.

1. NON-BLOCKING SELF-ROUTING SWITCH WITH QUASI-COMPLETE DIGRAPH TOPOLOGY

Consider $N_1 = p^2$ dual $p{\times}p$ switches (SSp). For $p = 4$, the block design of each of them is shown in Fig. 1. In the general case, they represent a two-stage block design with exchange links: the first stage includes p demultiplexers of the type $1\times p$ (Dp), and the second stage includes p multiplexers of the type $p\times 1$ (Mp) with feedback links through the delay lines DLδ for each input. Each switch SSp has the switching complexity $S_1 = 2p^2$.

From $N_1 = p^2$ switches SSp, N_1 multiplexers Mp without delay lines, and N_1 demultiplexers Dp, it is possible to construct a non-blocking self-routing switch $N_1\times N_1$ with the quasi-complete digraph topology—the switch SFN_1. The interconnections in this switch are specified by an incidence table; for $p = 4$, see the example in Table 2. The circuit of the switch SF16 is provided in Fig. 3.

Table 2

Interconnections in 16×16 switch SF16 with quasi-complete digraph topology

Simplex channels from users				4×4 switches SS4	Simplex channels to users			
1	2	3	4	1	1	5	9	13
2	3	4	1	2	2	6	10	14
3	4	1	2	3	3	7	11	15
4	1	2	3	4	4	8	12	16
5	6	7	8	5	5	9	13	1
6	7	8	5	6	6	10	14	2
7	8	5	6	7	7	11	15	3
8	5	6	7	8	8	12	16	4
9	10	11	12	9	9	13	1	5
10	11	12	9	10	10	14	2	6
11	12	9	10	11	11	15	3	7
12	9	10	11	12	12	16	4	8
13	14	15	16	13	13	1	5	9
14	15	16	13	14	14	2	6	10
15	16	13	14	15	15	3	7	11
16	13	14	15	16	16	4	8	12

The switch SFN_1 is non-blocking on packet permutations and for any distribution of N_1 packets among outputs when no more than p packets are sent to each output.

The switch SFN_1 has the following performance characteristics: the number of channels $N_1 = p^2$, the signal period $\tau = p$, and the switching complexity $\Sigma = S_1N_1 + 2pN_1 = 2p^2N_1 + 2pN_1 = 2p^3(p + 1)$. For any switches, we will also introduce the layout complexity by the number of simplex channels in them. For SFN_1, this complexity is given by $\Lambda = 2pN_1 = 2p^3$. All performance characteristics of the switch are combined in Table 3. We consider the switch SFN_1 to be one-stage, according to the number of stages of the switches SSp. Below, the number of stages will be interpreted by the number of stages of the switches SSp.

We emphasize that the switch SFN_1 is non-blocking on packet permutations and for any distribution of N_1 input packets among outputs when at most p packets are sent to each output.

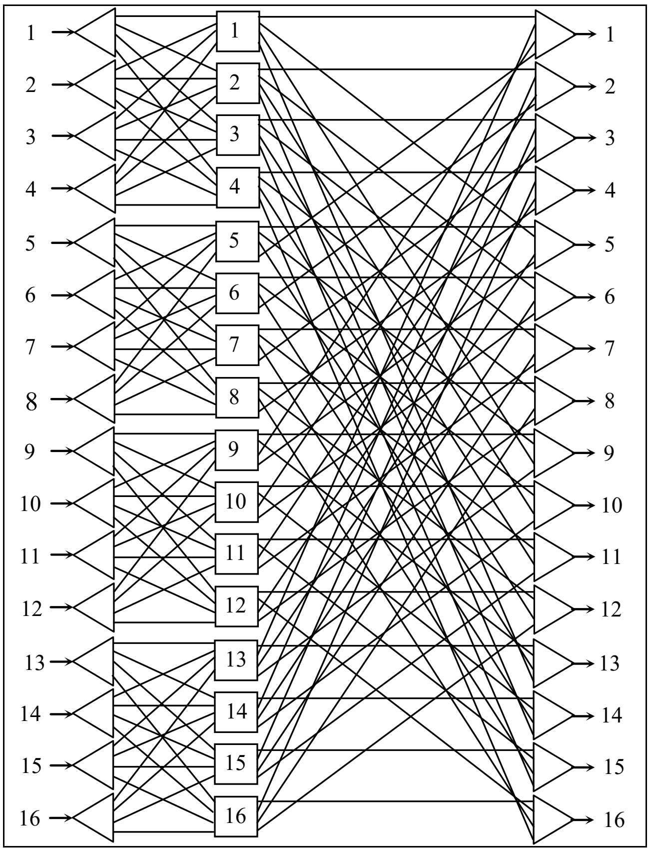

Fig. 3. Switch SF16 with quasi-complete digraph topology. Boxes correspond to switches SS4, and triangles to multiplexers M4 and demultiplexers D4.

Table 3

Performance characteristics of switch SFN_1

p	2	3	4	5	6	7	8
N_1	4	9	16	25	36	49	64
Σ	48	216	640	1500	3024	5488	9216
Σ	$N_1^{2.79}$	$N_1^{2.45}$	$N_1^{2.33}$	$N_1^{2.27}$	$N_1^{2.24}$	$N_1^{2.21}$	$N_1^{2.19}$
Λ	16	54	128	250	432	686	1024
Λ	N_1^{2}	$N_1^{1.82}$	$N_1^{1.75}$	$N_1^{1.72}$	$N_1^{1.69}$	$N_1^{1.68}$	$N_1^{1.67}$

The table is extended to $p = 8$ since a dual photon 8×8 switch, SS8, has already been developed; see [24].

In Sections 2 and 3, the dual switches SS2 and SF4, shown in Fig. 4, will be used. Note that the one-stage switch SF4 has the same signal period and number of channels as the two-stage switch constructed from SS4 in the papers [1–3].

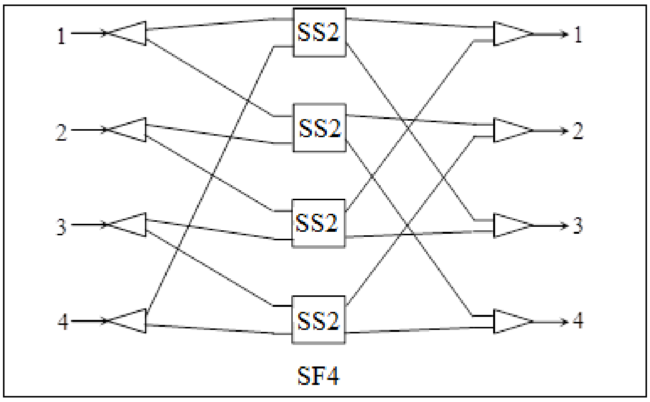

Fig. 4. Non-blocking self-routing switch SF4.

2. NON-BLOCKING SELF-ROUTING THREE-STAGE SWITCH

From the dual switch SF4, we construct a self-routing two-stage network with exchange links—the network C_216 shown in Fig. 5. It consists of two stages, and each stage includes four switches SF4. Unfortunately, this network is not a 16-channel non-blocking switch: it can have signal conflicts on the stage of the multiplexers M2, highlighted in gray. Signals in the first and second cycles may conflict. To resolve the conflicts, it suffices to provide these multiplexers with DLδ, $\delta = 2$ (Fig. 6). Then, conditionally speaking, they will form the second stage of dual switches without demultiplexers. The result is a three-stage non-blocking self-routing switch S_316. It has the following performance characteristics: the number of channels $N_3 = 16 = 2^4$, the signal period $T_3 = 4$, the switching complexity $S_3 = 2N_1\Sigma\big|_{p=2} = 384 = N_3^{2.15}$, and the layout complexity $L_3 = 2N_1\Lambda + N_1\big|_{p=2} = 144 = N_3^{1.79}$.

Let us compare the performance characteristics of this three-stage switch and the two-stage switch described in the introduction (the one composed of SS4). They have 16 channels each, the signal period ratio is $\gamma = T_3/\underline{T}_2 = 2/4 = 0.5$, and the switching complexity ratio is $\sigma = S_3/\underline{S}_2 = 384/256 = 1.5$. The product $\gamma\sigma = 0.75$ shows how many times the reduction in the signal period is less than the increase in the switching complexity.

Using the switches SF16 in a similar way, we can construct the two-stage self-routing network N_3256 with exchange links. It consists of two stages of 16 switches SF16 on each stage. This network may have conflicts on the multiplexers M4 of the first stage. For making N_3256 a three-stage non-blocking self-routing switch, it suffices to provide these multiplexers with DLδ, $\delta = 4$ (Fig. 7). The result is a three-stage non-blocking self-routing switch S_3256. It has the following performance characteristics: the number of channels $N_3 = 256 = 4^4$, the signal period T3 = 11, the switching complexity $S_3 = 2N_1\Sigma\big|_{p=4} = 20\,480 = N_3^{1.79}$, and the layout complexity $L_3 = 2N_1\Lambda + N_1\big|_{p=4} = 4352 = N_3^{1.51}$. The values δ and T3 are justified in Lemma 1 below.

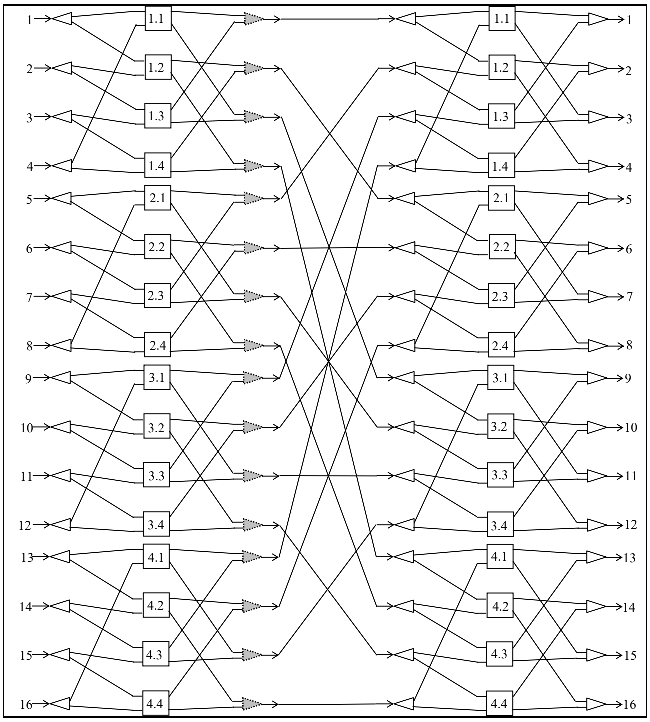

Fig. 5. Two-stage network N_216.

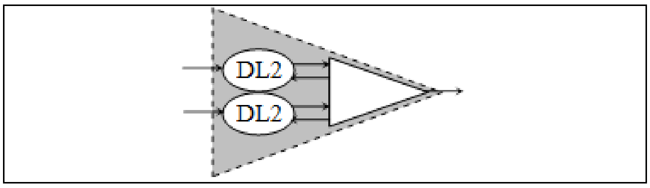

Fig. 6. Multiplexer M2 with delay lines for first stage.

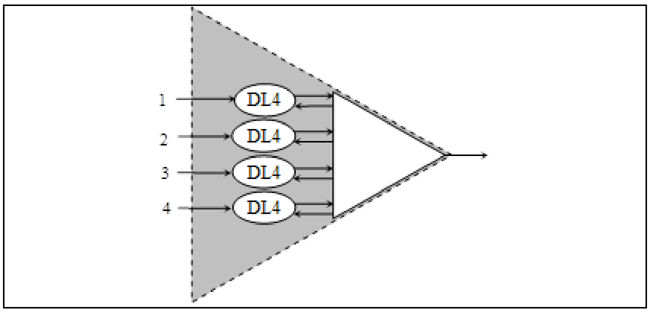

Fig. 7. Multiplexer M4 with delay lines for first stage.

Let us compare the performance characteristics of this three-stage switch and the four-stage switch described in the introduction in the case $p = 4$. They have 256 channels each, the signal period ratio is $\gamma = T_3/\underline{T}_2 = 10/49 \approx 0.204$, and the switching complexity ratio is $\sigma = S_3/\underline{S}_2 = 20\,480/8192 = 2.5$. The product $\gamma\sigma = 0.51$ shows how many times the reduction in the signal period is less than the increase in the switching complexity.

For an arbitrary $p > 2$, the three-stage switch S_3p^4 is constructed by analogy from the switches SFp^2. It has the following property.

L e m m a 1. *The switch* S_3p^4 *is non-blocking if the multiplexer* Mp *of the first stage has* DLδ *with* $\delta = p$ *and the signal period*

$$T_3 = p^2. \quad (1)$$

P r o o f. At their inputs, Mp receive p signals with their possible distribution on the interval from 1 to p cycles. In a conflict situation, they are distributed maximally on the interval from 1 to p cycles. For ensuring that in any conflicts, the primary signals are not superimposed on the signals delayed in the DLδ, a sufficient condition is $\delta = p$.

The maximum number of conflicting signals occurs when all signals are distributed among p cycles. In this case, conflict resolution will require retransmitting the conflicting signals through the DLδ p times. As a result, they will be distributed on the interval from 1 to p^2 cycles. And the conclusion follows. ♦

Formula (1) was confirmed by numerical simulations on a switch model with the synchronous generation of arbitrary permutations of packets.

For an arbitrary $p \geq 2$, the switches S_3p^4 have the performance characteristics shown in Table 4.

Table 4

Performance characteristics of switches S_3N_3

p	δ	$N_3 = p^4$	$T_3 = p^2$	S_3	L_3
2	2	16	4	$384 = N_3^{2.15}$	$144 = N_3^{1.79}$
3	3	81	9	$3888 = N_3^{1.88}$	$1053 = N_3^{1.58}$
4	4	256	16	$20\,480 = N_3^{1.79}$	$4352 = N_3^{1.51}$
5	5	625	25	$75\,000 = N_3^{1.74}$	$13\,125 = N_3^{1.47}$
6	6	1296	36	$217\,728 = N_3^{1.71}$	$32\,400 = N_3^{1.45}$
7	7	2401	49	$537\,824 = N_3^{1.70}$	$69\,629 = N_3^{1.43}$
8	8	4096	64	$1\,179\,648 = N_3^{1.68}$	$135\,168 = N_3^{1.42}$

3. NON-BLOCKING SELF-ROUTING BINARY TWO-STAGE SWITCH

The three-stage switch S_316 can be turned into the non-blocking two-stage switch S_316 by *the internal parallelization method*: we should cut the M2 stage with DLp from the first stage (Fig. 5) and separate the conflicting signals across two copies of the second stage (Fig. 8). The outputs of these copies are combined by an additional M2 stage. The switch S_216 switch has the following performance characteristics: the number of channels $N_2 = 16$, the signal period $T_2 = 2$, the switching complexity $S_2 = N_1(\Sigma - pN_1) + pN_2\big|_{p=2} = 576 = N_2^{2.29}$, and the layout complexity $L_2 = (p + 1)\, N_1\Lambda + pN_2\big|_{p=2} = 224 = N_3^{1.95}$.

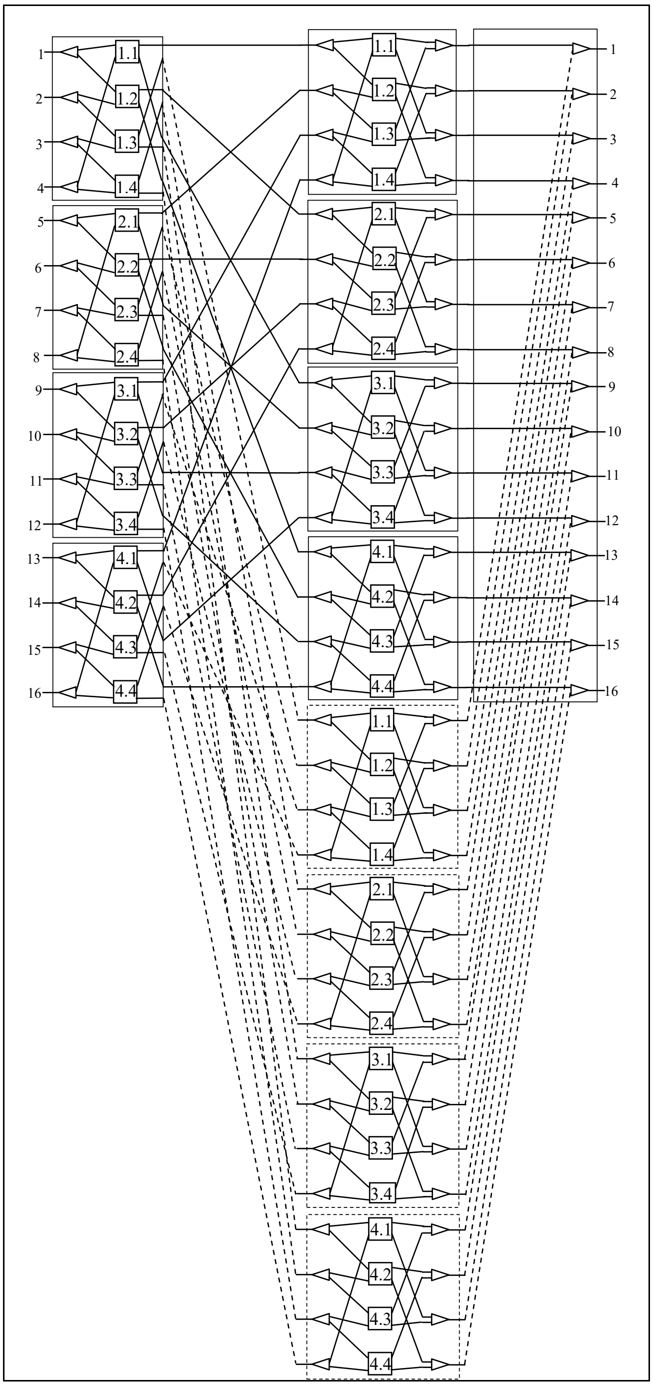

Fig. 8. Non-blocking self-routing binary two-stage switch S_216. Channels to second stage copies and from them are indicated by dotted lines.

Note that the switch S_216 has slightly greater complexity than S_316. In addition, the two-stage switch S_216 and the two-stage switch composed of the switches SS4 [1–3] have the same number of channels, but the former switch has half the signal period of the latter.

4. NON-BLOCKING SELF-ROUTING *P*-ARY TWO-STAGE SWITCH

Using *the internal parallelization method*, from the switches SFN_1 ($N_1 = p^2$, $p > 2$), we can construct the non-blocking self-routing p-ary two-stage switch $S_2N_1^2$ or S_2N_2 (Fig. 9) with the number of channels $N_2 = p^4$. In Fig. 9, the trapezoidal block $B_{2,1}$ indicates the switch SFN_1 without the output multiplexers Mp, which has N_1 inputs and N_1 groups of outputs with p outputs each. The square block $B_{2,2}$ indicates the complete switch SFN_1 with N_1 inputs and N_1 outputs. The triangular block $B_{2,3}$ indicates the multiplexer Mp.

A set of N_1 blocks $B_{2,1}$ is indicated by B_2^*. In this set, the outputs of blocks $B_{2,1}$ are numbered by I, J, and k, where I ($1 \le I \le N_1$) gives the number of $B_{2,1}$ in B_2^*, J ($1 \le J \le N_1$) gives the group number of p outputs of $B_{2,1}$, and k ($1 \le k \le p$) gives the output number in the group.

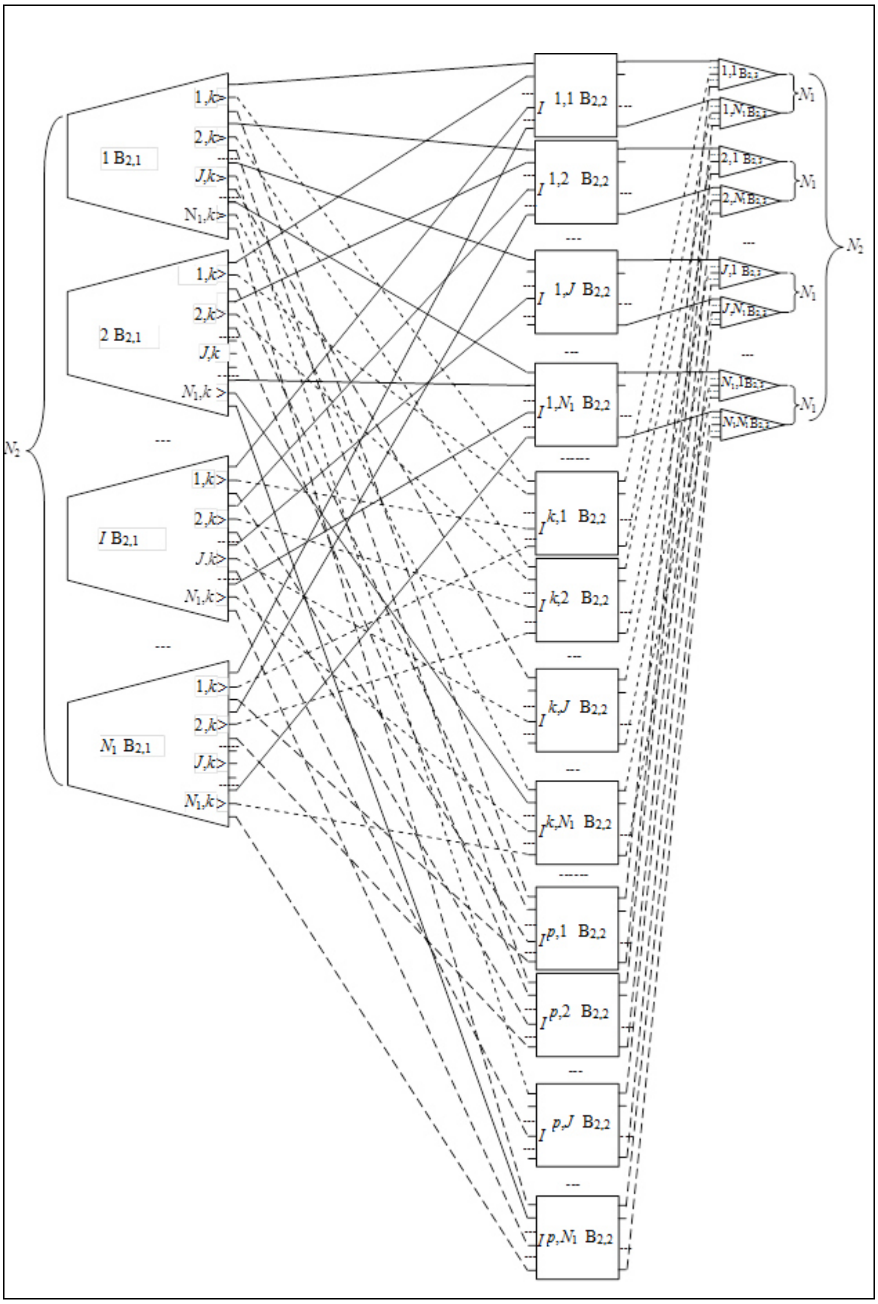

Fig. 9. Non-blocking self-routing *p*-ary two-stage switch S_2N_2. Channels to *k*th and *p*th copies of second stage and from them are indicated by short and long dotted lines, respectively.

A set of N_1 blocks $B_{2,2}$ is indicated by B_2. In this set, the inputs of blocks $B_{2,2}$ are numbered by I and J, where I ($1 \le I \le N_1$) gives the number of $B_{2,2}$ in B_2, and J ($1 \le J \le N_1$) gives the input number of $B_{2,2}$. There are p copies of the set B_2.

Between the outputs of blocks $B_{2,1}$ in the set B_2^* and the inputs of blocks $B_{2,2}$ in each of the p copies of the set B_2, there are exchange links, in which the outputs no. I, J, and k of the set B_2^* are connected to the inputs no. J and I of the kth copy of the set B_2.

The set B_2^* and each set B_2 in the kth group have exchange links: the outputs no. I, J, and k of blocks $B_{2,1}$ are connected to inputs no. J and I of blocks $B_{2,2}$ in the kth copy of sets B_2. The outputs of k copies of sets B_2 are combined by N_2 blocks into the outputs of the switch S_2N_2

Each block $B_{2,1}$ has the switching complexity $S_{2,1} = \Sigma - pN_1 = 2p^2N_1 + pN_1$ and the layout complexity $L_{2,1} = \Lambda = 2pN_1$ along with the output lines on a copy of $B_{2,2}$. Each copy of $B_{2,2}$ has the switching complexity $S_{2,2} = \Sigma = 2p^2N_1 + 2pN_1$ and the layout complexity $L_{2,2} = \Lambda = 2pN_1$. Each block $B_{2,3}$ has the switching complexity $S_{2,3} = p$ and the layout complexity $L_{2,3} = p$ along with the input lines from copies of $B_{2,2}$. As a result, the switching and layout complexities of the switch S_2N_2 are $S_2 = N_1S_{2,1} + pN_1S_{2,2} + N_2S_{2,3} = 2N_2(p^3 + 2p^2 + p) = 2(N_2^{7/4} + 2N_2^{3/2} + N_2^{5/4})$ and $L_2 = N_1L_{2,1} + pN_1L_{2,2} + N_2L_{2,3} = N_2(2p^2 + 2p + p) = 2N_2^{3/2} + 3N_2^{5/4}$, respectively. Table 5 shows the performance characteristics of the switches S_2N_2 compared to the switches S_3N_3 (Table 4).

According to Table 5, for the same p, the two-stage switches S_2N_2 have a considerably greater number of channels ($N_2 = \underline{N}_2^2$) than the two-stage switches composed of the switches SSp [1–3]. They both have the same signal period ($T_2 = \underline{T}_2 = p$) but different switching complexities ($S_2 \approx N_2^2$ vs. $\underline{S}_2 = 2\underline{N}_2^{3/2}$, respectively).

5. EXTENSIONS OF NON-BLOCKING SELF-ROUTING *P*-ARY SWITCHES

This section will apply the invariant extension method of system area networks [18], which is intended to enlarge the number of users by increasing the network complexity without changing transmission delays. To extend the switches S_2N_2 in this method, separate multiplexers Md and demultiplexers Dd are used, where d is a divider of N_2, i.e., $d = p^r$ (r = 1, 2, …). In the papers [1–3], this method was adopted in the case $r = 1$ for extending the switches constructed in the purely switching scheme base.

The invariant extension method consists in the following. Let us take d^2 switches S_2N_2. Each of them is divided into $n = N_2/d$ zones with d ports in each. (A port is an "input–output" pair.) Parts of the switches S_2N_2 in any zone are also $d \times d$ switches. All together, they are the backbone element of the switch SF_1d^2 with the quasi-complete digraph topology. (For example, see Fig. 3 for $d = p = 4$ and Fig. 4 for $d = p = 2$.) To form each such switch, it suffices to connect d^2 inputs to it through multiplexers Md, and d^2 outputs through demultiplexers Dd according to the corresponding interconnection table (for example, see Table 2 for $d = p = 4$). These connections yield the extended switch SE_2R_2, where $R_2 = nd^2 = dN_2$. In this switch, from any input the signal comes to the unique (!) copy of the switch S_2N_2, then passing to any given output without additional delays. Therefore, SE_2R_2 is a non-blocking self-routing switch, like S_2N_2.

An example of extending the switch SE_216 to the switch SE_232 with $d = p = 2$ is illustrated in Table 6 and Fig. 10.

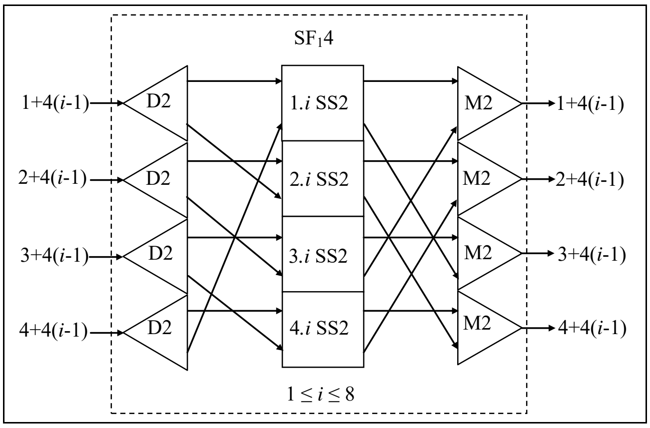

Fig. 10. Interconnection diagram of inputs and outputs of switch SE_232 in *i*th zone of switch SE_216.

The switch SE_2R_2 has the switching complexity $S_2^* = d^2S_2 + 2R_2d = d^2 2(N_2^{7/4} + 2N_2^{3/2} + N_2^{5/4} + N_2)$ and the layout complexity $L_2^* = d^2L_2 + 2R_2d = d^2(2N_2^{3/2} + 3N_2^{5/4} + N_2)$. Table 7 presents the performance characteristics of SE_2R_2 with some values of p and d. Note that the extended switch preserves its specific complexity (per channel) when increasing the number of its channels and maintaining its speed (signal period).

Table 5

Performance characteristics of switches S_2N_2

p	$N_2 = p^4$	$T_2 = N_2^{1/4}$	S_2	L_2	$\gamma = S_2/S_3$	$\sigma = T_3/T_2$	γ/σ
2	16	2	$576 = N_2^{2.29}$	$224 = N_2^{1.95}$	1.50	2	0.75
3	81	3	$7776 = N_2^{2.04}$	$2187 = N_2^{1.75}$	2.00	3	0.67
4	256	4	$51\,200 = N_2^{1.96}$	$11\,264 = N_2^{1.68}$	2.50	4	0.63
5	625	5	$225\,000 = N_2^{1.91}$	$40\,625 = N_2^{1.65}$	3.00	5	0.60
6	1296	6	$762\,048 = N_2^{1.89}$	$116\,640 = N_2^{1.63}$	3.50	6	0.58
7	2401	7	$2\,152\,296 = N_2^{1.87}$	$285\,719 = N_2^{1.61}$	4.00	7	0.57
8	4096	8	$5\,308\,416 = N_2^{1.86}$	$622\,592 = N_2^{1.60}$	4.50	8	0.56

Table 6

Arranging switches SS2 in extended switch SE_232 with $d = p = 2$

Copy of S_216	Ports of switch S_216															
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
	Composition of switches SE_232 from switches SS2															
1	1 SS2		2 SS2		3 SS2		4 SS2		5 SS2		6 SS2		7 SS2		8 SS2	
2	1 SS2		2 SS2		3 SS2		4 SS2		5 SS2		6 SS2		7 SS2		8 SS2	
3	1 SS2		2 SS2		3 SS2		4 SS2		5 SS2		6 SS2		7 SS2		8 SS2	
4	1 SS2		2 SS2		3 SS2		4 SS2		5 SS2		6 SS2		7 SS2		8 SS2	

Table 7

Performance characteristics of switch SE_2R_2 with some values of *p* and *d*

p		2	3	4	5	6	7	8
T_2		2	3	4	5	6	7	8
d = p	R_2	32	243	1024	3125	7776	16 807	32 768
	S_2^*	$R_2^{2.25}$	$R_2^{2.03}$	$R_2^{1.97}$	$R_2^{1.93}$	$R_2^{1.91}$	$R_2^{1.90}$	$R_2^{1.89}$
	L_2^*	$R_2^{1.89}$	$R_2^{21.74}$	$R_2^{1.69}$	$R_2^{1.66}$	$R_2^{1.65}$	$R_2^{1.64}$	$R_2^{1.63}$
$d = p^2$	R_2	64	729	4096	15 625	46 656	117 649	262 144
	S_2^*	$R_2^{2.21}$	$R_2^{2.03}$	$R_2^{1.97}$	$R_2^{1.94}$	$R_2^{1.93}$	$R_2^{1.92}$	$R_2^{1.91}$
	L_2^*	$R_2^{1.91}$	$R_2^{1.78}$	$R_2^{1.74}$	$R_2^{1.72}$	$R_2^{1.71}$	$R_2^{1.7}$	$R_2^{1.69}$
$d = p^3$	R_2	128	2187	16 384	78 125	279 936	823 543	2 097 152
	S_2^*	$R_2^{2.18}$	$R_2^{2.02}$	$R_2^{1.98}$	$R_2^{1.95}$	$R_2^{1.94}$	$R_2^{1.93}$	$R_2^{1.92}$
	L_2^*	$R_2^{1.92}$	$R_2^{1.81}$	$R_2^{1.78}$	$R_2^{1.76}$	$R_2^{1.75}$	$R_2^{1.74}$	$R_2^{1.74}$
$d = p^4$	R_2	256	6561	65 536	390 625	1 679 616	5 764 801	16 777 216
	S_2^*	$R_2^{2.16}$	$R_2^{2.02}$	$R_2^{1.98}$	$R_2^{1.96}$	$R_2^{1.95}$	$R_2^{1.94}$	$R_2^{1.93}$
	L_2^*	$R_2^{1.93}$	$R_2^{1.84}$	$R_2^{1.8}$	$R_2^{1.79}$	$R_2^{1.78}$	$R_2^{1.77}$	$R_2^{1.77}$

CONCLUSIONS

This paper has proposed a technique for constructing non-blocking self-routing photon switches of wide scalability based on new dual photon switches with a small number *p* of channels. In dual switches, signal conflicts are resolved by distributing them either among different channels or among different cycles. The latter method requires increasing the signal period by *p* times for a *p*-channel dual $p \times p$ switch.

Dual photon switches of wide scalability have been constructed with the minimum possible signal period, which is *p* times greater than the signal duration. Scalability is achieved using switches with the quasi-complete digraph topology and the invariant extension of any networks based on them. This method allows increasing the number of network channels while maintaining such properties as the signal period, non-blocking, and self-routing through its parallelization with increasing complexity. This method involves an extended element base consisting of $p \times p$ switches, $1 \times p$ demultiplexers, and $p \times 1$ multiplexers.

Three- and two-stage non-blocking self-routing *N*-channel $N \times N$ switches with $N = p^4$ have been constructed. In the first of these, the period–complexity balance is shifted towards a lower complexity under a large period. In the two-stage switch, the balance is shifted towards the minimum period under high complexity. Its switching complexity turned out to be comparable to the complexity of a full switch, and its layout complexity turned out to be significantly less than that of a full switch. These switches have only two stages of $p \times p$ switches and two stages of $1 \times p$ demultiplexers and $p \times 1$ multiplexers.

The above switches have been extended to *R*-channel $R \times R$ switches with $R = N^r$ ($r = 1, 2, 3, 4$) using another stage of $1 \times d$ demultiplexers and $d \times 1$ multiplexers with $d = p^r$. These *R*-channel switches are non-blocking self-routing switches with a minimum signal period. Their switching complexity is comparable to that of a full switch, and the channel complexity is significantly less.

In comparison with the papers [1–3], the main results of this work are a significant reduction in the signal period with the same number of channels and a significant increase in the number of channels with the same single period. In other words, the key novelty consists in developing a method for constructing non-blocking self-routing dual photon switches with the minimum possible signal period and the feasibility of achieving the maximum number of channels under almost the same complexity.

This paper, together with [1–3], has presented a technique for designing fundamentally new dual system networks with the following properties:

– They are non-blocking networks with static self-routing of packets, i.e., networks with conflict-free self-routing on arbitrary packet permutations.

– They have the widest scalability with the maximum speed reached on them and the complexity comparable to that of a full switch.

– Their maximum speed is only 2–4 times less than the physically achievable speed.

– They allow balancing the speed–complexity ratio with decreasing their complexity to that of the non-blocking Clos network (which has no conflict-free self-routing procedures) but at a significantly lower speed of the non-blocking network.

In the future, we expect to make dual networks channel fault-tolerant by replacing the quasi-complete digraph topology with the quasi-complete graph one. In addition, when cascading dual networks, we expect to use internal parallelization (Sections 3 and 4) instead of external parallelization (invariant extension). This approach will enable the scaling of dual networks with smaller costs.

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