CALCULATING THE SPECTRAL ENTROPY OF A STATIONARY RANDOM PROCESS

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Abstract. The problem of calculating the spectral entropy of a stationary random process is solved. The spectral entropy (σ-entropy) of a signal is understood as a scalar value characterizing the noise color; it describes the class of signals affecting a system depending on the band under study. By assumption, the random process is defined by a shaping filter, with the Gaussian white noise with a unit covariance matrix supplied at its input, or by an autocorrelation function. The spectral entropy of the stationary random process is analytically derived using a known mathematical model of the shaping filter in the form of a log-determinant function that depends on the transfer matrix and the observability Gramian of the filter. An algorithm for calculating the σ-entropy of stationary random processes with a known autocorrelation function is proposed. The method reduces to reconstructing the mathematical model of the shaping filter using its spectral density factorization. A numerical example is provided: spectral entropy is calculated for a disturbance describing the velocity of wind gusts that affect an aircraft.

Keywords: spectral entropy, stationary random process, spectral density, autocorrelation function, shaping filter.

INTRODUCTION

Representing a main characteristic of an automatic control system, the dynamic accuracy of signal transmission or transformation is defined by the difference between the desired and actual values of a signal over time (or a certain functional of this difference).

Any automatic control system must transmit or transform, in a required way, not a particular control signal but an entire set of such signals. Moreover, the nature of changes of each signal cannot be fully predicted in advance. Hence, it is necessary to study the statistical characteristics of the whole set of signals, which represent random functions of time. When investigating dynamic accuracy, one should consider system characteristics, e.g., the sample-to-sample scatter of parameters tolerance bounds or their random change within certain limits during operation, including, of course, random changes in the system structure [1, 2].

On the other hand, when designing closed-loop control systems, there is the need to suppress random exogenous disturbances affecting the system, i.e., to specify the desired dynamic accuracy of the closed-loop system. The Gaussian white noise is the most common way to define a random influence applied to the system input. However, this noise is known to be a physically unrealizable random process. The so-called colored random processes are closest to the real processes affecting a system [1, 3–5]. In the design and analysis of control systems, such processes can be implemented by solving the shaping filter problem. As a rule, a shaping filter is represented as a linear time-invariant system that receives the Gaussian white noise at the input and generates a signal with required statistical characteristics at the output. This class of random signals allows modeling the dynamics of a closed-loop system under noises close to real-life random processes.

In σ-entropy analysis and control problems, the sets of all possible random processes affecting a system are given by a scalar nonnegative value called the spectral entropy (σ-entropy) of a signal [6, 7]. However, when investigating systems, the following question arises inevitably: how should one determine the value of the spectral entropy of a random signal required for further study? This question can be settled analytically if the statistical characteristics of a random disturbance are defined in the form of a shaping filter with the Gaussian white noise as the input.

Therefore, we set a topical problem: it is required to determine the value of the spectral entropy of a random signal generated from the Gaussian white noise using a linear stationary filter. The remainder of this paper is organized as follows. Section 1 is devoted to the problem statement and the main theoretical results related to the σ-entropy theory. In Section 2, we analytically derive the spectral entropy of a stationary random process using a known mathematical model of the shaping filter or a known autocorrelation function. An illustrative numerical example is provided in Section 3.

1. BACKGROUND AND PROBLEM STATEMENT

This paper considers a classical shaping filter in the following form:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bv(t) \\ w(t) = Cx(t), \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ and $w(t) \in \mathbb{R}^m$ are the state and output of the filter; $v(t) \in \mathbb{R}^m$ is Gaussian white noise with zero mean and a unit intensity matrix; A, B, and C are constant real matrices of compatible dimensions. By assumption, system (1) is minimum-phase.

Recall that the spectral entropy of a stationary random process $w(t)$ is given by [6]

$$\mathfrak{S}(w) = -\frac{1}{2\pi}\int_{-\infty}^{+\infty} \varphi(\omega) \ln\det \frac{4\pi m \omega_0 S_w(\omega)}{\int_{-\infty}^{+\infty} \mathrm{tr} S_w(\lambda) d\lambda} d\omega, \tag{2}$$

where $S_w(\omega)$ denotes the spectral density of the signal $w(t)$; m is the dimension of the random process; ω and λ are integration variables; finally, $\varphi(\omega)$ is a scaling function of the form

$$\varphi(\omega) = \frac{\omega_0}{\omega_0^2 + \omega^2}. \tag{3}$$

The parameter ω_0 has the dimension of frequency and is chosen by the system designer based on the significant range of frequencies when investigating the random process of interest. If the influence of a random process on a linear control system is considered, then the value of this parameter should be several times greater than the system bandwidth.

The spectral entropy calculation problem can be formulated as follows.

Let a Gaussian white noise with a unit covariance matrix be supplied to the shaping filter (1). Given the scaling function (3), it is required to derive formulas for calculating the spectral entropy $\mathfrak{S}(w)$ (2).

To solve this problem, we will utilize the following well-known results [8, 9].

Lemma 1 (the Cauchy integral formula). *Let D be a domain on the complex plane with a piecewise-smooth or rectifiable boundary $\Gamma = \partial D$, $f(z)$ be a holomorphic function in $\overline{D}$, and z_0 be a point inside the domain D. Then*

$$f(z_0) = \frac{1}{2\pi i}\int_{\Gamma} \frac{f(z)}{z - z_0} dz. \tag{4}$$

Lemma 2 (on the absolute value of the logarithm of the determinant). *For a transfer matrix $G \in RH^{\infty}$,*

$$\ln\det\left(G^*(i\omega)G(i\omega)\right) = 2\ln\left|\det G(i\omega)\right|,$$

where $G^*(i\omega)$ denotes the Hermitian conjugate of the matrix G.

2. THE MAIN RESULT

To obtain the main result, we find the spectral density of the signal $w(t)$ at the output of the shaping filter (1). It is determined by the expression

$$S_w(\omega) = G^*(i\omega)G(i\omega)S_v(\omega),$$

where $S_v(\omega)$ denotes the spectral density of the Gaussian white noise $v(t)$ and $G(i\omega) = C\left((i\omega)I - A\right)^{-1}B$ is the transfer function of the shaping filter.

By the problem statement, the Gaussian white noise $v(t)$ has a unit covariance matrix; therefore,

$$S_v(\omega) = I_m.$$

In this case,

$$S_w(\omega) = G^*(i\omega)G(i\omega).$$

The problem of calculating the spectral entropy of the shaping filter (1) reduces to calculating the integral (2).

Before transforming the expression (2), we recall the identity $\det(\alpha \cdot T) = \alpha^m \det T$ for a matrix $T \in \mathbb{R}^{m\times m}$ and a real scalar value α. Note also that $\ln\alpha^m = m\ln\alpha$ for $\alpha > 0$.

Then the expression (2) can be transformed as follows:

$$-\frac{1}{2\pi}\int_{-\infty}^{+\infty}\varphi(\omega)\ln\det\frac{4\pi m\omega_0 S_w(\omega)}{\int_{-\infty}^{+\infty}\mathrm{tr}S_w(\lambda)d\lambda}d\omega$$

$$=-\frac{1}{2\pi}\int_{-\infty}^{+\infty}\varphi(\omega)\ln\det S_w(\omega)d\omega \qquad (5)$$

$$+\frac{m}{2\pi}\int_{-\infty}^{+\infty}\varphi(\omega)\ln\frac{\int_{-\infty}^{+\infty}\mathrm{tr}S_w(\lambda)d\lambda}{4\pi m\omega_0}d\omega.$$

Consider each term of (5) separately. For the first term, we obtain

$$\frac{m}{2\pi}\int_{-\infty}^{+\infty}\varphi(\omega)\ln\frac{\int_{-\infty}^{+\infty}\mathrm{tr}S_w(\lambda)d\lambda}{4\pi m\omega_0}d\omega$$

$$=\frac{m}{2\pi}\int_{-\infty}^{+\infty}\frac{\omega_0}{\omega_0^2+\omega^2}\ln\frac{\int_{-\infty}^{+\infty}\mathrm{tr}\left(G^*(i\lambda)G(i\lambda)\right)d\lambda}{4\pi m\omega_0}d\omega$$

$$=\frac{m}{2\pi}\int_{-\infty}^{+\infty}\frac{\omega_0}{\omega_0^2+\omega^2}\ln\frac{2\pi\|G\|_2^2}{4\pi m\omega_0}d\omega=\frac{m}{2}\ln\left(\frac{\|G\|_2^2}{2m\omega_0}\right),$$

where $\|G\|_2^2$ is the square of the H_2 norm of the transfer function $G(s)$ (s denotes the Laplace transform variable). It has the formula

$$\|G\|_2^2=\mathrm{tr}\left(B^{\mathrm{T}}PB\right),$$

where the observability Gramian P is a solution to the Lyapunov equation

$$A^{\mathrm{T}}P+PA+C^{\mathrm{T}}C=0.$$

For the second term,

$$-\frac{1}{2\pi}\int_{-\infty}^{+\infty}\varphi(\omega)\ln\det S_w(\omega)d\omega$$

$$=-\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\omega_0}{\omega_0^2+\omega^2}\ln\det\left(G^*(i\omega)G(i\omega)\right)d\omega$$

$$=-\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{\omega_0}{\omega_0^2+\omega^2}\ln\left|\det G(i\omega)\right|d\omega.$$

To calculate the last integral, we integrate $f(z)=\frac{\omega_0}{\omega_0^2+z^2}\ln\left|\det G(z)\right|$ along the closed contour Γ consisting of a semicircle of radius R centered at the origin and the diameter of this semicircle lying on the real axis. The desired integral will be found by letting $R\to\infty$. As a result,

$$-\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{\omega_0}{\omega_0^2+\omega^2}\ln\left|\det G(i\omega)\right|d\omega$$

$$=\lim_{R\to\infty}\left[-\frac{1}{\pi}\int_{\Gamma}\frac{\omega_0}{\omega_0^2+z^2}\ln\left|\det G(z)\right|dz\right].$$

The function $f(z)$ has poles at the points $\pm i\omega_0$ and is analytic inside the entire domain bounded by the curve Γ.

Hence,

$$f(z)=\frac{\omega_0}{\omega_0^2+z^2}\ln\left|\det G(z)\right|$$

$$=\frac{\omega_0\ln\left|\det G(z)\right|}{(z-i\omega_0)(z+i\omega_0)}=\frac{f_1(z)}{z-i\omega_0},$$

where $f_1(z)=\frac{\omega_0\ln\left|\det G(z)\right|}{z+i\omega_0}$.

Then the integral can be written as

$$-\frac{1}{\pi}\int_{\Gamma}\frac{\omega_0}{\omega_0^2+z^2}\ln\left|\det G(z)\right|dz=-\frac{1}{\pi}\int_{\Gamma}\frac{f_1(z)}{z-i\omega_0}dz.$$

Since the function $f_1(z)$ is analytic inside the closed contour Γ, the Cauchy integral formula (4) can be applied:

$$-\frac{1}{\pi}\int_{\Gamma}\frac{\omega_0}{\omega_0^2+z^2}\ln\left|\det G(z)\right|dz$$

$$=-\frac{1}{\pi}2\pi i\, f_1(\omega_0)=-2i\frac{\omega_0\ln\left|\det G(\omega_0)\right|}{2i\omega_0}$$

$$=-\ln\left|\det G(\omega_0)\right|.$$

Due to the minimum-phase property of system (1), we finally obtain

$$-\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{\omega_0}{\omega_0^2+\omega^2}\ln\left|\det G(i\omega)\right|d\omega=-\ln\det G(\omega_0),$$

where $G(\omega_0)=C\left(\omega_0 I-A\right)^{-1}B$.

Let us formulate the following result.

Theorem 1. *The spectral density of a stationary random sequence $w(t)$ generated by the shaping filter* (1) *from the Gaussian white noise $v(t)$ with zero mean and a unit covariance matrix is given by*

$$\mathfrak{S}(w)=-\ln\det\frac{\sqrt{2m\omega_0}G(\omega_0)}{\sqrt{\mathrm{tr}\left(B^{\mathrm{T}}PB\right)}}, \qquad (6)$$

where $G(\omega_0)=C\left(\omega_0 I-A\right)^{-1}B$ and P is the solution to the Lyapunov equation

$$A^{\mathrm{T}}P+PA+C^{\mathrm{T}}C=0.$$

P r o o f. Based on the above considerations, we have

$$\mathfrak{S}(w) = -\frac{1}{2\pi}\int_{-\infty}^{+\infty}\varphi(\omega)\ln\det\frac{4\pi m\omega_0 S_w(\omega)}{\int_{-\infty}^{+\infty}\mathrm{tr}S_w(\lambda)d\lambda}d\omega$$

$$= -\ln\det G(\omega_0) + \frac{m}{2}\ln\left(\frac{\mathrm{tr}\left(B^{\mathrm{T}}PB\right)}{2m\omega_0}\right)$$

$$= -\left[\ln\det G(\omega_0) - \ln\left(\frac{\mathrm{tr}\left(B^{\mathrm{T}}PB\right)}{2m\omega_0}\right)^{\frac{m}{2}}\right].$$

The last expression directly implies expression (6), and the proof is complete. ♦

The above theorem provides a method for determining the spectral entropy of a stationary random signal with a known mathematical model of the shaping filter. However, this result can be applied also under known (or experimentally determined) statistical characteristics of the random disturbance affecting the system, e.g., its autocorrelation function. For this purpose, it is necessary to identify the shaping filter. Here is one method for solving such a problem in the one-dimensional case [1].

Let $R_w(\tau)$ be a known analytical expression for the autocorrelation function of a stationary signal. Then its spectral density can be found from the relation

$$S_w(\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} R_w(\tau)\mathrm{e}^{-i\omega\tau}d\tau.$$

As is known, when a stationary random signal passes through a linear stable time-invariant system (in our case, the shaping filter), the spectral density $S_w(\omega)$ of the steady-state random process at the system output is given by

$$S_w(\omega) = \left|W_{sf}(i\omega)\right|^2 S_v(\omega),$$

where $S_v(\omega)$ is the spectral density of the input; $W_{sf}(i\omega)$ is the transfer function of the system. Since the input signal is a Gaussian white noise, then $S_v(\omega) = 1$.

Thus,

$$S_w(\omega) = \left|W_{sf}(i\omega)\right|^2,$$

i.e., the square of the amplitude-frequency response of the shaping filter must coincide (within a constant factor) with the spectral density of the signal to be formed.

To proceed, we emphasize that the spectral density of a stationary random process is a real even nonnegative function of ω under real values of ω.

Since the spectral density under consideration is fractional rational,

$$S_w(\omega) = \frac{Q(\omega)}{R(\omega)},$$

where $Q(\omega)$ and $R(\omega)$ are polynomials with real coefficients containing only even degrees of ω. (This fact follows from the evenness of the spectral density.)

For real values of ω, the spectral density $S_w(\omega)$ can be represented as

$$S_w(\omega) = \frac{B(i\omega)B(-i\omega)}{A(i\omega)A(-i\omega)}$$

$$= W_{sf}(i\omega)W_{sf}(-i\omega) = \left|W_{sf}(i\omega)\right|^2.$$

Obviously, the function $W_{sf}(i\omega)$ possesses all the properties of the transfer function of a stable linear time-invariant minimum-phase system.

Consequently,

$$W_{sf}(i\omega) = \frac{B(i\omega)}{A(i\omega)}.$$

Thus, by decomposing the spectral density of the generated signal into the complex conjugate factors, we easily determine the transfer function of the shaping filter. After that, it is necessary to get the state-space representation of the system, e.g., using, the canonical Frobenius form, and then apply Theorem 1.

3. A NUMERICAL EXAMPLE

Consider the problem of calculating the spectral entropy of exogenous disturbances for a flying missile guided to a target [1]. A missile in the atmosphere is subjected to various types of air flows, e.g., constant winds, upward and downward winds, wind gusts, swirls, etc. Wind gusts increase congestion. In a sufficiently limited domain of space and time, wind can be considered a stationary spatiotemporal process. The disturbing moments due to lift force variations are functions of the magnitude and direction of the velocity of wind gusts. However, only the velocity of wind gusts determines the value of missile deflection. The autocorrelation function of the velocity of wind gusts affecting an aircraft approximately equals

$$R_w(\tau) = \gamma\mathrm{e}^{-\alpha\tau},$$

where γ and α are constant values [1, 3]. The spectral density corresponding to this correlation function has the form

$$S_w(\omega) = \frac{2\gamma\alpha}{\alpha^2 + \omega^2}.$$

Factorizing the last expression yields

$$S_w(\omega) = \frac{\sqrt{2\gamma\alpha}}{\alpha + i\omega} \cdot \frac{\sqrt{2\gamma\alpha}}{\alpha - i\omega},$$

and the transfer function of the shaping filter modeling the velocity of wind gusts is given by

$$W_{sf}(i\omega) = \frac{B_{sf}(i\omega)}{A_{sf}(i\omega)} = \frac{\sqrt{2\gamma\alpha}}{\alpha + i\omega}.$$

The corresponding state-state representation is

$$\begin{cases} \dot{x}(t) = -\alpha x(t) + v(t) \\ w(t) = \sqrt{2\gamma\alpha}\, x(t). \end{cases}$$

The random process has the dimension $m = 1$. We apply Theorem 1 to calculate its spectral entropy. In this case,

$$G(\omega_0) = C(\omega_0 I - A)^{-1} B = \frac{\sqrt{2\gamma\alpha}}{\omega_0 + \alpha}, \quad P = \gamma.$$

Therefore, the spectral entropy of this process is analytically derived in the form

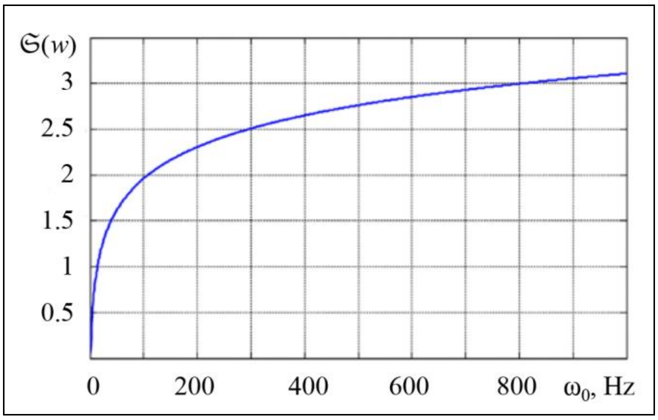
$$\mathfrak{S}(w) = -\ln\det \frac{\sqrt{2m\omega_0}\, G(\omega_0)}{\sqrt{\mathrm{tr}\left(B^{\mathrm{T}} P B\right)}}$$

$$= -\ln \frac{\sqrt{2m\omega_0}\sqrt{2\gamma\alpha}}{\sqrt{\gamma}\,(\omega_0 + \alpha)} = -\ln \frac{2\sqrt{\alpha\omega_0}}{\omega_0 + \alpha}.$$

Note that as $\omega_0 \to \infty$, the expression under the logarithm vanishes (hence, the spectral entropy tends to infinity). The choice of the frequency ω_0 determines the bandwidth of interest.

The figure below shows the graph of the spectral entropy depending on the parameter ω_0 under $\alpha = 0.5$.

Thus, the main result of this paper can be applied in practice within the σ-entropy approach to analyze and design linear control systems with exogenous stationary noise. ♦



The spectral entropy depending on the frequency ω_0.

CONCLUSIONS

In this paper, we have proposed a method for calculating the spectral entropy of a stationary random process using a known mathematical model of the shaping filter or a known autocorrelation function of the process. For the known mathematical model of the shaping filter, by assumption, the Gaussian white noise with a unit covariance matrix is supplied to its input. If the random process is specified by an autocorrelation function, the paper has proposed an algorithm for constructing a mathematical model of the shaping filter using the Fourier transform to obtain the spectral density of the random process and perform its subsequent factorization.

The main result of this paper can be applied to analyze and design linear time-invariant control systems with random disturbances, using the σ-entropy approach proposed in [6].

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