DOI: http://doi.org/10.25728/cs.2021.1.5

MODELS OF EXPERIENCE

M.V. Belov[1] and D.A. Novikov[2]

[1] Skolkovo Institute of Science and Technology, Moscow, Russia

[2] V.A. Trapeznikov Institute of Control Sciences, Russian Academy of Sciences, Moscow, Russia

[1]✉ mbelov59@mail.ru, [2]✉ novikov@ipu.ru

Abstract. A generalized probabilistic model is proposed that uniformly describes the formation and development of individual, collective, and social experience at various human activity levels. Some of its particular cases are considered, covering many learning models known in mathematical psychology and models of developing and mastering technologies within the methodology of complex activity.

Keywords: experience, activity, knowledge, technology, culture, learning curve.

INTRODUCTION

Active systems. Let us separate two classes of systems that include a human. (Following the terminology of [1, 2], such systems will be called *active systems*.). These classes are:

– *Natural systems*, existing or emerging “independently,” in the absence of an external source that forms or determines the goal of activity. Such systems have independent goal-setting, and their global goal is development (which requires preservation and possibly adaptation reproduction). In terms of systems engineering [3, 4], active systems with internal goal-setting are *systems of systems* (SoSs) belonging to collaborative or virtual classes.

– *Artificial systems* created by some subject to achieve his goals. In terms of systems engineering [3, 4], active systems with internal goal-setting are externally directed SoSs or externally acknowledged SoSs.

Depending on the presence of an explicit subject, one can distinguish between *subject systems* and *non-subject systems.* The former systems perform their activity, which is uniformly described by *the methodology of complex activity* (MCA) [5] regardless of their type. The latter systems perform no activity themselves; more precisely, their “activity” is the set of activities of their components.

Thus, we have three options (one of the four possible options is contradictory); see examples in (Table 1) below.

Table 1

Classification of systems

	Natural systems (internal goal-setting)	Artificial systems (external goal-setting)
Subject systems	Individual	Organization Enterprise Government Particular case: individual employee whose internal motives are coordinated with external goals
Non-subject systems	Social communities: Group Family Genus Tribe Society Ethnos People Economic communities: Market Set of independent interacting economic agents	–

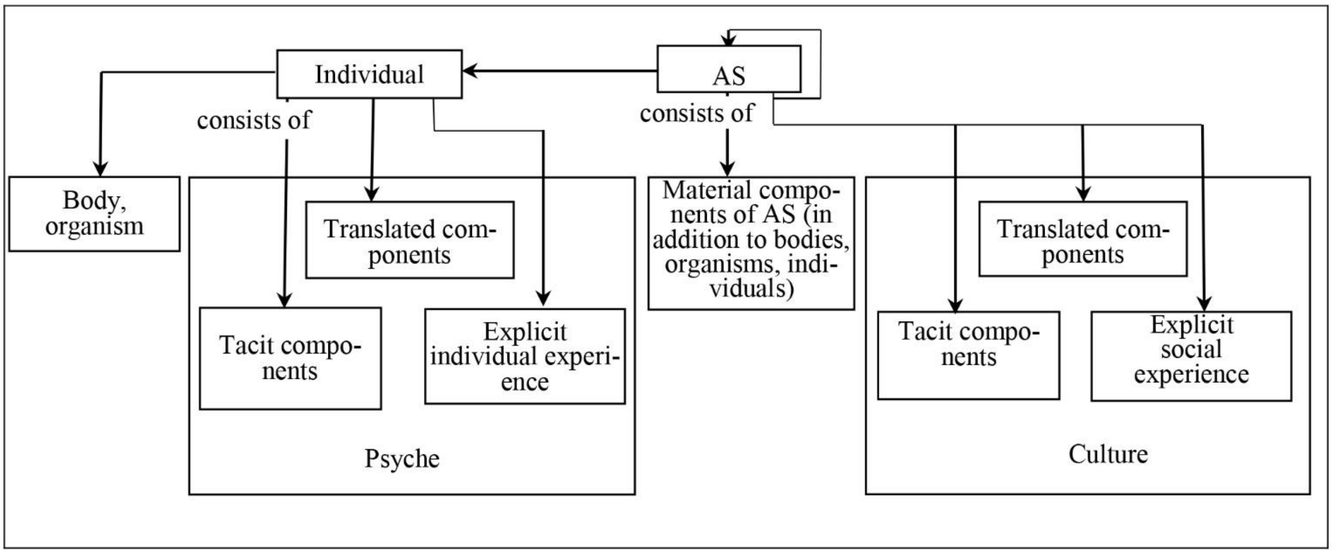

Fig. 1. Structure of active system.

As a digression, note that human activity can be considered within the ASs of different levels and scales (Table 2), and a promising task of MCA is their uniform description (probably, except for the two lower levels).

Table 2

Human activity

Level	Typical object	Dominant form of activity of elements
Cultural	Ethnos, people	Reproduction and development of activity
Political	Government, institution	Institutionalization of activity
Economic	Organization, enterprise	Collective practical activity
Social	Society	Communication activity
	Group, collective; family, genus, tribe	Collective practical activity
Psychic	Personality	Individual practical activity
	Individual	Internal activity
Biological	Organism	Life activity
Physical	Body	Movement

Moreover, the assignment of a particular system to a specific class depends on the aspect of its consideration. For example, a social group is itself a non-subject natural system. However, when studying the problems of managing such a group by other subjects (an individual, another group, or government), it must be considered an artificial subject system, together with the control subject.

For natural non-subject systems, the key factors are the mechanisms of their functioning (conditions, principles, norms, requirements, and criteria for assessing the activity of the system components, both separately and during their interaction [5, 6]). Recall that a *mechanism* is a system or device that determines the order of some activity [2]. The mechanisms of functioning form a multilevel system of nested feedback loops determining the dependence of conditions, principles, norms, etc. (including control actions) on previous and current performance results and uncertainty factors. As a rule, the mechanisms of functioning of non-subject systems are reflexive. Some examples include natural selection, competition, conflicts, dissemination of ideas, etc. These mechanisms provide (self) control of such systems.

In subsystems (ASs or individuals), let us separate the material component (for an individual, his body and material means of activity) and the immaterial component (for an individual, *psyche*; for a collective subject, *culture*). For details, see

Fig. 1). Experience is a significant part of the immaterial components.

Experience. *Experience* is understood [7–9] as:

1) a set of practically mastered knowledge, skills, abilities, and habits (individual experience);

2) the reflection of the objective world and social practice aimed at changing the world in the human mind (socio-historical experience, the individual experience of each individual).

The category of experience is closely related to other categories such as education, technology, and culture (Fig. 2).

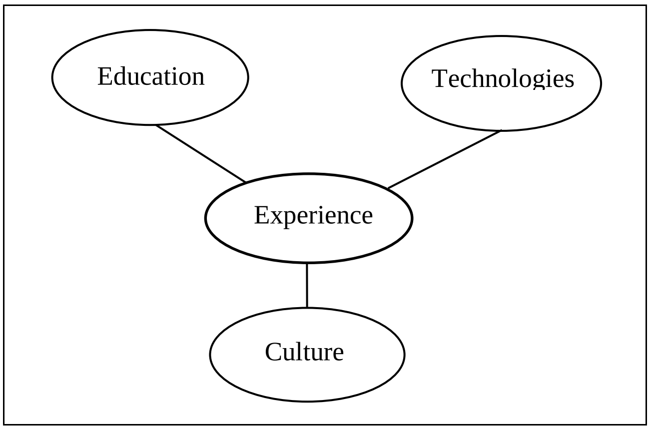

Fig. 2. Experience and related categories.

Indeed, *education* is the development of experience [10] and includes learning. (*Learning* is the process and result of acquiring individual experience [11].) A modern survey of mathematical models of learning can be found in [11, 12], and a survey of learning models in automatic control theory in [13].

Technologies are the operational reflection of a mass-practice proven and systematized practical experience [14]. (According to [5], technology is a system of conditions, criteria, forms, methods, and means for consistently achieving a set goal.)

Culture includes [7]:

– the objective results of human activity (machines, technical structures, results of cognition (books, works of art, legal and ethical norms, etc.), representing the first component of culture;

– the subjective human strengths and abilities realized in activities (sensations, perceptions, knowledge, skills, production and professional skills, the level of intellectual, aesthetic, and moral development, worldview, the methods and forms of mutual communication of people, etc.), representing the second component of culture [15].

The objective results of human activity (the first component of culture) are reflected in different forms of social consciousness such as language (understood in a broad sense–both natural native and foreign languages and artificial languages), everyday consciousness, political ideology, law, ethics, religion (or atheism as anti-religion), art, science, and philosophy [15].

The second component of culture is subjective human strengths and abilities. They are expressed in personal knowledge, including figurative, sensory knowledge, which is not transferred by words (concepts), as well as in skills, the development of certain individual abilities, the worldview of each person, etc. [7].

Here are some quotes and definitions that characterize the concept of culture:

– "a set of genetically non-inherited information in the field of human behavior" (Yu.M. Lotman);

– a set of sustainable forms of human activity (*organizational culture*);

– "Just as the embryo in the womb repeats in a fantastically accelerated time scale the entire evolution of life on Earth over a billion years, so a growing person in 20 years must assimilate the culture that mankind has created for 4 million years." [7, p. 32];

– a set of accepted standard norms of activity (ways of standardizing and regulating behavior) and the corresponding results. The main function of culture is the reproduction and construction (development) of activity.

Thus, culture can be viewed as a generalized experience proven by social practice [7, 12, 16, 17]; see Fig. 1.

Experience can be formed through independent acquisition by a subject (individual or collective) during his activity or through the development of someone else's experience during learning activity (Fig. 3).

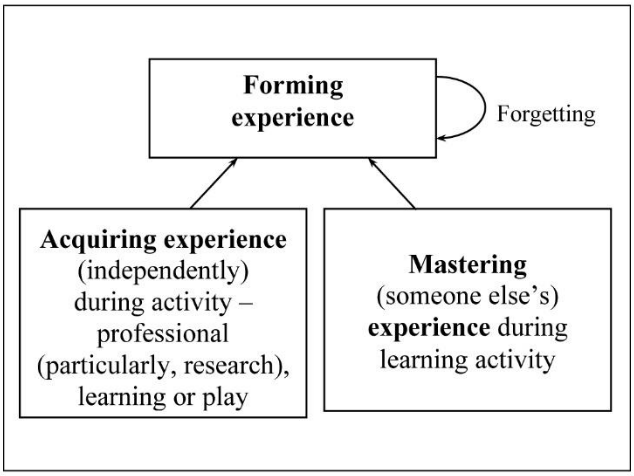

Fig. 3. Formation of experience.

Depending on the methods and means of fixing and translating the experience (or even more broadly—in the case of an individual—the components of the psyche, when relating ideas, beliefs, attitudes, personality worldview, etc. to the widely interpreted experience), we can distinguish among:

– *explicit experience*, which is often translated in the form of text (e.g., knowledge, or technology);

– *tacit experience* (tacit knowledge), which is often translated in non-verbal and non-textual forms (e.g., beliefs, or worldview);

– *nontranslated components*, which are, perhaps, translated "biologically" (e.g., biopsychic properties of an individual; the specific physiology of individuals, conditioned by climate, landscape, and lifestyle), but so slowly that they can be considered unchanged.

The types of experience are listed in Table 3.

Table 3

Types of experience

Level	Experience
Social system	Social
Group	Collective
Personality	Individual

In the process of his activity, a subject can participate (Fig. 4) in:

1) mastering social experience;
2) forming/acquiring individual experience;
3) mastering collective experience;
4) forming collective experience;
5) forming social experience.

Mastering social and/or collective experience can be conventionally regarded as “learning with a teacher,” and forming individual experience as “learning without a teacher.”

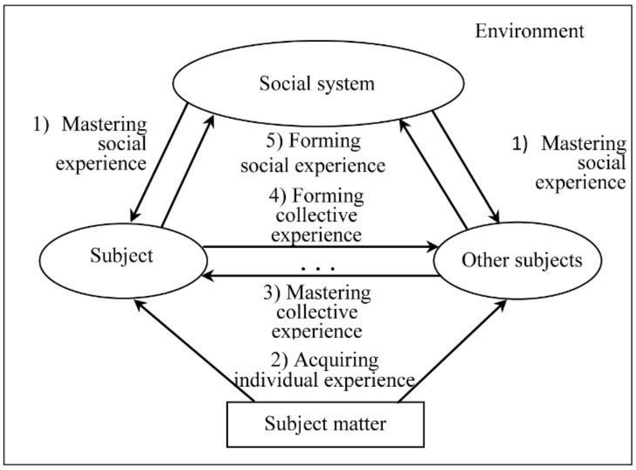

Fig. 4. Individual, collective, and social experience.

The goal of this study is to create a general experience formation model that would adequately and uniformly describe the processes of forming and mastering individual, collective, and social experience (Fig. 4), explicit or tacit, at various levels of activity (Table 2) in any classes of systems (Table 1).

The remainder of this paper is organized as follows. In Section 1, a general experience formation model is introduced. Section 2 provides a classification system for different models of experience. Sections 3 and 4 consider several particular models for forming/mastering individual and collective/social experience, respectively. The Conclusions section outlines some promising lines for further research.

1. GENERAL EXPERIENCE FORMATION MODEL

We extend the original learning model (see subsection 3.3.4 of the book [18]) by supplementing it with the following effects: environment variability, making experience outdated (or, equivalently, forgotten/lost), and a more complex formation of experience, with mastering experience by other subjects and the interaction of different subjects.[1]

Let an AS be composed of a given set $N = \{1, \ldots, n\}$ of active elements (AEs). (An AE is an element of an active system representing an individual or a lower-level AS.)

Assume that each AE observes one of K possible values of an *uncertainty factor* (UF) in each period. In the general case, the values observed by different AEs will differ. We introduce the concept of *a complex uncertainty factor*: its current value will be characterized by the aggregate of all states encountered by all AEs in period t.

A complex UF can be represented as a matrix $\boldsymbol{\omega}(t) = ||\,\omega_{ik}(t)\,||$ with binary elements ($\omega_{ik}(t) \in \{0;\ 1\}$). Suppose that in a current period t, the UF for AE i has state $k(i)$. Then the elements $\omega_{ik(i)}(t)$ are 1, and the others are 0. Obviously, the matrix $||\,\omega_{ik}(t)\,||$ satisfies the condition $\sum_{k=1}^{K} \omega_{ik}(t) = 1,\ i = \overline{1, n}$.

We denote by $\boldsymbol{\Omega}$ the set of all such matrices. (Its cardinality is K^n). On the set $\boldsymbol{\Omega}$ we define a time-varying probability distribution $\{p_\omega(t)\}$ for the states of the complex UF of the environment, assuming that the current state $\omega(t)$ occurs independently of the previous ones. We number the elements of this set using the function $\iota(\omega) = \sum_{i=1}^{n}\sum_{k=1}^{K} k n^{i-1}\omega_{ik}$.

Therefore, $\sum_{\omega\in\Omega} p_\omega(t) \equiv \sum_{\omega\in\Omega} p_{\iota(\omega)}(t) \equiv 1$.

In a particular case, the states observed by each of the AEs are mutually independent. We denote by $p_{ik}(t)$ the probability of observing state k by AE i, where

[1] In the proposed model, we will not separate the effect of physiological forgetting of experience from the effect of rejecting the previously learned experience when new technologies appear. Separation of two effects, generally speaking, different by their nature and speed – the objective change in technologies and the resulting hard or soft rejection of experience (depending on the distribution of the AE parameters) and the subjective physiological forgetting of it together with a regular trend towards age-related changes in the parameters of the AE cognitive characteristics – will allow us to analyze several social effects (cultural interaction of generations, a decrease in collective experience (including culture) in revolutionary periods, etc.) by varying their speeds. The effects mentioned above can become the subject matter of further research.

$\sum_{k=1}^{K} p_{ik}(t) \equiv 1,\ i = \overline{1,n}$, and by $k(i)$ the state observed by AE i. Then $p_\omega(t) = \prod_{i=1}^{n} p_{ik(i)}(t)$.

Let us describe the AS state by a matrix $\mathbf{v}(t) = \| v_{ik}(t) \|$ as follows. Each element is a binary variable characterizing *the formation of experience* (within the mathematical model, this process will also be called mastering the technology) by the AE for various states of the UF. More precisely, each element $v_{ik}(t)$ takes value 1 if, after period t, AE i has formed/mastered the experience under state k of the UF.

Suppose that the AS evolves during period $(t + 1)$ under the following mechanism.

Let the complex UF have a state $\boldsymbol{\omega}(t+1)$, and let AE i encounter state k of the UF. Then:

- For any UF state l unmastered by AE i ($v_{il}(t) = 0$), *the experience is formed* ($v_{il}(t + 1) = 1$) with a probability $0 \le w_{ikl}(\{\boldsymbol{v}(\cdot) \mid t - \tau;\, t\}) \le 1$, which generally depends on time as well as on the current $\boldsymbol{v}(t)$ and τ previous AS states; the experience is not formed with the probability $1 - w_{ikl}(\{\boldsymbol{v}(\cdot) \mid t - \tau;\, t\})$, which implies $v_{il}(t + 1) = 0$. In the sequel, $\{\boldsymbol{v}(\cdot) \mid t_1;\, t_2\}$ will denote *the history*, i.e., an ordered set of values $\boldsymbol{v}(\cdot)$ on a time interval between periods t_1 and t_2 inclusive. (If $t_2 = t_1$, there is no history.)
- For any UF state l mastered by AE i ($v_{il}(t) = 1$), *the experience is forgotten* ($v_{il}(t + 1) = 0$) with a probability $0 \le u_{ikl}(\{\boldsymbol{v}(\cdot) \mid t - \tau;\, t\}) \le 1$, which generally depends on time as well as on the current $\boldsymbol{v}(t)$ and τ previous AS states; the experience is not forgotten with the probability $1 - u_{ikl}(\{\boldsymbol{v}(\cdot) \mid t - \tau;\, t\})$, which implies $v_{il}(t + 1) = 1$.

This mechanism is illustrated in Fig. 5.

The semantics of this model reflects the possibility of forming experience, particularly, mastering technology by an active element, transferring knowledge from one element to another, forgetting knowledge and/or making it outdated, among other things, due to the evolution of the environment and the repeated adaptation of the AS to changes in the environment, reflected by the realized UF values.

In this case, for each UF state, the process of forming-forgetting experience by each AE is supposed to be binary (possible states = <mastered | unmastered>) and random, which reflects its uncertainty. For different AEs and different UF states, the transitions between states occur independently of each other. By assumption, there can be no more than one event during one period: forming experience or forgetting it. At the same time, as the probabilities of transition between states depend on the current and previous states of all AEs in the AS, the model describes rather complex laws of the AS behavior. For example, observing one state, an AE can generally form an experience corresponding to another UF state (by acquiring the experience from another AE).

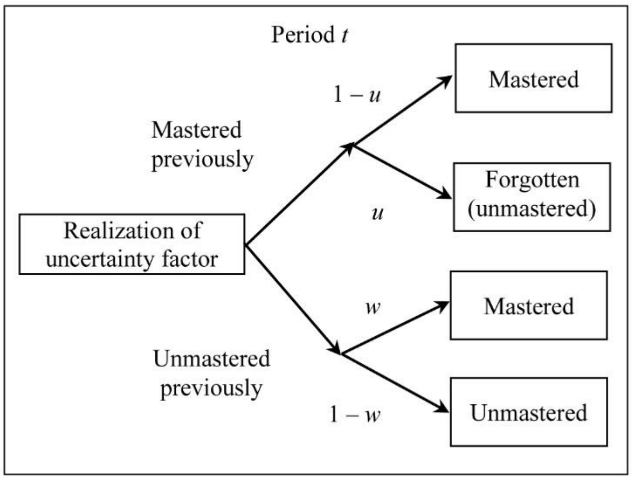

Fig. 5. Alternative events in period t.

Now we write dynamic equations for the probabilities of mastering experience and the expected experience maturity levels. Let $\mathbf{q}(t) = \| q_{ik}(t) \|$, where $q_{ik}(t) = \Pr(v_{ik}(t) = 1) = E[v_{ik}(t)]$ is the probability that state k of the UF is mastered by AE i after period t. Then by the rule of total probability yields

$$
\begin{aligned}
q_{ik}(t + 1) &= \Pr(v_{ik}(t + 1) = 1 \mid v_{ik}(t) = 0) \\
&\Pr(v_{ik}(t) = 0) + \Pr(v_{ik}(t + 1) = 1 \mid v_{ik}(t) = 1) \\
&\Pr(v_{ik}(t) = 1) = W_{ik}(\boldsymbol{q}(t))\,(1 - q_{ik}(t)) + \\
&+ (1 - U_{ik}(\boldsymbol{q}(t)))\, q_{ik}(t) = W_{ik}(\boldsymbol{q}(t)) + \\
&+ [1 - W_{ik}(\boldsymbol{q}(t)) - U_{ik}(\boldsymbol{q}(t))]\, q_{ik}(t), \qquad (1)
\end{aligned}
$$

where the functions $W_{ik}(\boldsymbol{v}(t))$ and $U_{ik}(\boldsymbol{v}(t))$ are the probabilities of mastering and forgetting, $w_{ikl}\{\cdot\}$ and $u_{ikl}\{\cdot\}$, respectively, averaged by the UF states considering their probabilities $p_m(t)$ and the probabilities of the AS states in the current and previous periods:

$$
\begin{aligned}
W_{ik}(\mathbf{q}(t)) &= \sum_{m=1}^{K^n} p_m(t) \times \\
&\times \sum_{\substack{\mathbf{z}(t)\ldots\mathbf{z}(t-\tau) \\ z_{ik}(t)=0}} w_{imk}\left(\{\mathbf{z}(t) \mid t-\tau;\, t\}\right) \times \qquad (2) \\
&\times \pi(\{\mathbf{z}(\cdot),\, \mathbf{q}(\cdot) \mid t-\tau;\, t\},\, i^-,\, k^-),
\end{aligned}
$$

$$
\begin{aligned}
U_{ik}(\mathbf{q}(t)) &= \sum_{m=1}^{K^n} p_m(t) \times \\
&\times \sum_{\substack{\mathbf{z}(t)\ldots\mathbf{z}(t-\tau) \\ z_{ik}(t)=1}} u_{imk}\left(\{\mathbf{z}(t) \mid t-\tau;\, t\}\right) \times \qquad (3) \\
&\times \pi(\{\mathbf{z}(\cdot),\, \mathbf{q}(\cdot) \mid t-\tau;\, t\},\, i^-,\, k^-),
\end{aligned}
$$

where $\pi(\{z(\cdot), q(\cdot)|t-\tau; t\}, i^-, k^-)$ are the conditional probabilities that after periods $\{t-\tau; t\}$ the AE states have the values $z(\cdot)$ and the probabilities of mastering have the values $q(\cdot)$ given a known value of the experience maturity level under UF state k after period t. The conditional probabilities $\pi(\cdot)$ are calculated by formula (4) with the product taken over all triplets $<\alpha; \beta; \gamma>$ except for $<\alpha; \beta; \gamma> = <i; k; t>$:

$$\pi(\{\mathbf{z}(\cdot), \mathbf{q}(\cdot)\,|\,t-\tau;\ t\}, i^-, k^-) = \\ = \prod_{\substack{\alpha=1\ldots n;\ \beta=1\ldots K;\ \gamma=t-\tau\ldots t;\\ <\alpha;\beta;\gamma;s>\neq<i;k;t>}} (y_{\alpha\beta}(\gamma)\, z_{\alpha\beta}(\gamma) + \\ + (1-y_{\alpha\beta}(\gamma))\,(1-z_{\alpha\beta}(\gamma))). \tag{4}$$

In a particular case when the UF states observed by each AE are independent, the expressions (2) and (3) take the form

$$W_{ik}(\mathbf{y}(t)) = \sum_{\omega\in\Omega}\prod_{i=1}^{n} p_{ik(i)}(t) \times \\ \times \sum_{\substack{\mathbf{z}(t)\ldots\mathbf{z}(t-\tau)\\ z_{ik}(t)=0}} w_{\iota(\omega)ik}\left(\{\mathbf{z}(t)\,|\,t-\tau;t\}\right)\times \\ \times\pi(\{\mathbf{z}(\cdot), \mathbf{q}(\cdot)\,|\,t-\tau;\ t\}, i^-, k^-)$$

and

$$U_{ik}(\mathbf{q}(t)) = \sum_{\omega\in\Omega}\prod_{i=1}^{n} p_{ik(i)}(t) \times \\ \times \sum_{\substack{\mathbf{z}(t)\ldots\mathbf{z}(t-\tau)\\ z_{ik}(t)=1}} u_{\iota(\omega)ik}\left(\{\mathbf{z}(t)\,|\,t-\tau;t\}\right)\times \\ \times\pi(\{\mathbf{z}(\cdot), \mathbf{q}(\cdot)\,|\,t-\tau;\ t\}, i^-, k^-).$$

Well, we have obtained the recurrence relations reflecting the dynamics of the expected experience maturity levels. To calculate their values in any period, it remains to specify the initial value matrix $\boldsymbol{q}(0)$. By default, suppose that in the initial (zero) period, the AE has no experience for any of the FN states.

Let *the individual experience criterion* $L_i(t)$ ("learning level") of AE i be the probability of realizing a UF value previously encountered, successfully mastered, and not forgotten by him (i.e., the expected share of the learned values):

$$L_i(t) = 1 - \sum_{k=1}^{K} p_{ik}(t)\,(1-q_{ik}(t)),\ i=\overline{1,n}. \tag{7}$$

By analogy, let *the collective experience criterion* $L_{\max}(t)$ be the probability of realizing a UF value previously encountered, successfully mastered, and not forgotten **by at least one AE**:

$$L_{\max}(t) = 1 - \prod_{i=1}^{n}(1-L_i(t)), \tag{8}$$

or the probability $L_{\min}(t)$ of realizing a UF value previously encountered, successfully mastered, and not forgotten **by each of the AEs**:

$$L_{\min}(t) = \prod_{i=1}^{n} L_i(t). \tag{9}$$

A sequence of experience criterion values will be called *an experience curve*, similar to the concept of a learning curve.

The collective experience criterion can be treated as an aggregate characteristic of the experience formed by the entire group.

Transition to continuous time. Let the AS and AE operate in continuous time: the processes of forming and forgetting experience are independent flows of elementary events, whose intensities (rates) $w_{ikl}(\{v(\cdot)\,|\,t-\tau;\ t\})$ and $u_{ikl}(\{v(\cdot)\,|\,t-\tau;\ t\})$ in a known way depend on the history of the AS states at the current and previous time instants $\{v(\cdot)\,|\,t-\tau;\ t\}$.

Assume that the UF changes its states somehow (in discrete or continuous time), independently of the AS, and the evolution of $\{p_{ik}(t)\}$ is known.

Then the system of difference equations (5), describing the AS dynamics with given initial conditions, can be replaced by the system of differential equations of the form

$$dq_{ik}(t) / dt = W_{ik}(\boldsymbol{q}(t)) - \\ - (W_{ik}(\boldsymbol{q}(t)) + U_{ik}(\boldsymbol{q}(t)))\, q_{ik}(t). \tag{10}$$

2. CLASSIFICATION OF EXPERIENCE MODELS

The expressions (1)–(10) describe the process and result of forming individual and collective experience in the most general case—under minimum assumptions. For an operational description and study, it is necessary to make some simplifications (additional assumptions about the structure and properties of the model). Therefore, we introduce a system of classifications based on the properties of the model components. (Note that classification bases 1–9 are mutually independent.)

1. Properties of complex UF states observed by AE in each period. For now, we will separate the general case (considered above) and a particular case in which all AEs observe the same realization of the UF state in each period. The UF properties will be described not by the n K-dimensional distribution $\{p_\omega(t)\}$, but by the K-dimensional one $\{p_k(t)\}$, where $\sum_{k=1}^{K} p_k(t) = 1$.

2. Dependence of complex UF states on time. Here, the general case (see above) is an arbitrary known dependence of the probability distribution of the complex UF states on time, and the particular case is a stationary (time-invariant) distribution.

3. Dependence of the probability of mastering on time. The general case (see above) is an arbitrary known dependence of the probability of mastering on time, and the particular case is no dependence.

4. Dependence of the probability of forgetting on time. Similar to item 3.

5. Dependence of the probability of mastering on process history. The general case (see above) is a known dependence of the probability of mastering on $\tau \le t$ previous states. Also, we will separate two subcases: $\tau = 0$ (the history-invariant probability of mastering) and $\tau = 1$ (the probability of mastering depends only on the previous state).

6. Dependence of the probability of forgetting on process history. Similar to item 5.

7. Dependence of the probability of mastering for AE *i* on the states of other AEs. The general case (see above) is a known dependence of the probability of mastering on the states of all AEs. An "intermediate" case is when the probability of mastering for AE *I* depends on the states of his "neighbors" – AEs from a known set $N_i(t)$. The particular case is when for each AE, the probability of mastering depends on his states only.

8. Dependence of the probability of forgetting for AE *i* on the states of other AEs. Similar to item 7.

9. Possible experience formation regardless of realized UF state. The general case is when, by transferring experience from other AEs, a specific AE can form his experience corresponding to a UF state that differs from the state observed by him. The particular case is when an AE forms an experience corresponding only to the UF states observed by him.

10. Number of AEs. The general case is a known number of AEs, $n > 1$. The particular case is $n = 1$.

Consider several models in the order of complication. (Models 1–6 correspond to individual experience, whereas models 7–12 to collective and social experience.)

3. MODELS OF INDIVIDUAL EXPERIENCE

Model 1 ([14, Section 2.2]), in which there is one AE, all parameters are time-invariant, the experience corresponding to the UF state observed by the AE is formed effectively (the probability of mastering is equal to 1), and there is no forgetting.

We denote by $p_k > 0$ the probability that in a next period, the AE will encounter UF state *k*. (Obviously, $\sum_{k=1}^{K} p_k = 1$.) The vector of these probabilities is $P = (p_1, \ldots, p_K)$.

In the case under consideration, $n = 1$ and $i = 1$, which implies $m = l$. Since the probability of mastering is 1, let $W_{mj}(\{v(\cdot) \mid t-\tau;\, t\}) = 1$ for $m = l$ and $W_{mj}(\{v(\cdot) \mid t-\tau;\, t\}) = 0$ for $m \ne l$ regardless of the history $\{v(\cdot) \mid t-\tau;\, t\}$, i.e., $W_{mj}(\{v(\cdot) \mid t-\tau;\, t\}) = \delta_{mj}$, where δ_m is the Kronecker delta. Due to no forgetting, we have $u_{mj}(\{v(\cdot) \mid t-\tau;\, t\}) \equiv 0$ and $U_j(q(t))$ $U_j(q(t)) \equiv 0$. From the expression (2) it follows that

$$\begin{aligned} U_j(q(t)) &= \sum_{m=1}^{K} p_m(t) \times \\ &\times \sum_{\substack{z(t)\ldots z(t-\tau)\in X;\\ z_j(t)=0}} W_{mj}\left(\{z(t) \mid t-\tau;\, t\}\right) \times \\ &\times \pi\left(\{z(t) \mid t-\tau;\, t\},\ j,\ q(t)\right) = \\ &= \sum_{m=1}^{K} p_m(t) \sum_{\substack{z(t)\ldots z(t-\tau)\in X;\\ z_j(t)=0}} \delta_{mj} = p_j(t). \end{aligned} \tag{11}$$

Really, the Kronecker delta is taken once in the sum $\sum_{\substack{z(t)\ldots z(t-\tau)\in X;\\ z_j(t)=0}} \delta_{mj}$ because $z_j(t) = 0$, and this condition reduces the sum to a single element for which $m = j$. As a result,

$$q_j(t + 1) = p_j(t) + [1 - p_j(t)]\, q_j(t).$$

Since the probabilities are stationary, $q_j(t + 1) = p_j + (1 - p_j)\, q_j(t)$, or $\Delta q_j(t + 1) = p_j\, (1 - q_j(t))$, or $1 - q_j(t + 1) = (1 - p_j)\, (1 - q_j(t + 1))$. According to (8), the experience criterion is

$$\begin{aligned} L(t) &= 1 - \sum_{k=1}^{K} p_k\, (1 - q_k(t)) = 1 - \sum_{k=1}^{K} p_k\, (1 - p_k) \times \\ &\times (1 - q_k(t-1)) = 1 - \sum_{k=1}^{K} p_k\, (1 - p_k)^t. \end{aligned} \tag{12}$$

Model 2. Consider a modification of Model 1 in which there is a unique UF state ($K = 1$), but the probability of mastering $w \in (0, 1]$ can be smaller than 1. Omitting the UF state subscript, by analogy with the expression (19) we obtain: $W(q(t)) = w$,

$$q_j(t + 1) = w + [1 - w]\, q_j(t),$$

or $1 - q_j(t + 1) = (1 - w)\, (1 - q_j(t + 1))$. From (11) it follows that the learning curve has the form

$$L(t) = 1 - (1 - w)^t. \tag{13}$$

(Also, see the expression (12) for $w = p_k = 1 / K$.)

Model 3. Consider a modification of Model 1 with the same stationary probability of mastering $w(q) \in (0, 1]$ for all UF states. By analogy with the expression (11) we obtain $W_j(q(t)) = w(q(t))\, p_j(t)$. Since the probabilities are stationary, from (5) it follows that $q_j(t + 1) = w(q(t))\, p_j + (1 - w(q(t))\, p_j)\, q_j(t)$, or $1 -$ $- q_j(t + 1) = (1 - w(q(t))\, p_j)\, (1 - q_j(t))$. Let the probability of mastering be a known function $g(\cdot)$ of the current experience criterion value, i.e.,

$w(q(t)) = g(L(q(t)))$. Then the equality $1 - q_j(t + 1) = (1 - g(L(t))\, p_j)\, (1 - q_j(t))$ implies

$$1 - q_j(t) = \prod_{\tau=0}^{t-1} (1 - p_j g(L(\tau))\,.$$

Denoting $b_j(t) = 1 - q_j(t)$, we write $b_j(t + 1) = (1 - g(L(t))\, p_j)\, b_j(t)$ and, consequently,

$$L(t) = 1 - \sum_{k=1}^{K} p_k b_k(t) = 1 - \sum_{k=1}^{K} p_k \prod_{\tau=0}^{t-1} (1 - p_k\; g(L(\tau))\,,$$

$$\Delta L(t) = g(L(t)) \sum_{k=1}^{K} p_k^2 \prod_{\tau=0}^{t-2} (1 - p_k\; g(L(\tau))\,.$$

In the uniform distribution case ($p_j = 1 / K,\ i = \overline{1, n}$), we have $b_j(t) = b(t)$ and

$$L(t) = 1 - \sum_{k=1}^{K} p_k b_k(t) = 1 - \sum_{k=1}^{K} \frac{1}{K} b_k(t) = 1 - b(t) = q(t).$$

Therefore,

$$\Delta L(t) = \sum_{k=1}^{K} p_k\; (b_k(t-1) - b_k(t)) =$$

$$= \sum_{k=1}^{K} \frac{1}{K} (b_k(t-1) - (1 - \frac{1}{K} g(L(t-1)))\, b_k(t-1)) =$$

$$= \sum_{k=1}^{K} \frac{1}{K} (1 - 1 + \frac{1}{K} g(L(t-1))\, b(t-1)) =$$

$$= \sum_{k=1}^{K} \frac{1}{K} g(L(t-1))\, b(t-1)) =$$

$$= \frac{1}{K}\; g(L(t-1))\, b(t-1)) =$$

$$= \frac{1}{K}\; g(L(t-1))\, (1 - L(t-1)),$$

which gives

$$\Delta L(t) = \frac{1}{K}\; g(L(t-1))\, (1 - L(t-1)). \tag{14}$$

Depending on $g(\cdot)$, the solution of the difference equation (14) can be an exponential, power, or logistic curve; see the models of different learning curves and a survey in [14]:

Probability of mastering $g(\cdot)$	Difference equation	Learning curve
$g(L) = \gamma K$	$\Delta L(t) = \gamma\, (1 - L(t-1))$	Exponential
$g(L) = \mu K L$	$\Delta L(t) = \mu\, L(t-1)\, (1 - L(t-1))$	Logistic
$g(L) = \eta K (1 - L)^a$	$\Delta L(t) = \eta\, (1 - L(t-1))^{a+1}$	Power

Model 4 ([18], subsection 3.3.4]) is the intersection of particular cases for the nine classification bases above. It differs from Model 1 in the presence of stationary probabilities of mastering and forgetting, which are generally not equal to 1 and 0, respectively.

Suppose that during the first realization of UF state k, the corresponding experience is formed with a known probability $0 \le w_k \le 1$, where w_k is *the probability of mastering*, and is not with the probability $(1 - w_k)$. After forming component k of the experience, in each next period, it changes as follows:

– If the UF state realized differs from k, then the result of mastering state k remains the same.

– If UF state k is realized again, then component k of the experience is "lost" with *the probability of forgetting* $0 \le u_k \le 1$ and remains the same with the probability $(1 - u_k)$.

We construct the vectors of the probabilities of mastering and forgetting: $W = (w_1, \ldots, w_K)$ and $U = (u_1, \ldots, u_K)$. Generally speaking, these vectors do not satisfy the normalization condition.

By analogy with the expression (11), we obtain: $W_j(q(t)) = w_j\ p_j$, $U_j(q(t)) = u_j\ p_j$. Substituting this result into (5) yields

$$q_j(t + 1) = p_j\, w_j + (1 - p_j\, (w_j + u_j))\, q_j(t). \tag{15}$$

Let the initial conditions be $q_j(0) = \alpha_j \in [0, 1]$. Then, using the recursive formula (15), we find

$$q_i(t) = \frac{w_j}{w_j + u_j} (1 - (1 - p_j(w_j + u_j))^t) + \\ + (1 - p_j(w_j + u_j))^t \alpha_j. \tag{16}$$

Substituting the sum (16) into (7), we finally arrive at

$$L(P, W, U, t) = \sum_{k=1}^{K} p_k \frac{w_k}{w_k + u_k} \Big(1 - \big(1 - p_k\,(w_k + u_k)\big)^t\Big) + \\ + \sum_{k=1}^{K} \alpha_k \big(1 - p_k\,(w_k + u_k)\big)^t = \sum_{k=1}^{K} p_k \frac{w_k}{w_k + u_k} + \\ + \sum_{k=1}^{K} \left(\alpha_k - p_k \frac{w_k}{w_k + u_k}\right) \big(1 - p_k\,(w_k + u_k)\big)^t. \tag{17}$$

Suppose that the AE obtains a reward (payoff) h_k for successfully forming component k of his experience in a certain period. Then over T_0 periods, his total expected payoff from forming the experience during his work is given by

$$F(P, W, U,\ T) = \sum_{t=1}^{T_0} \sum_{k=1}^{K} p_k h_k \frac{w_k}{w_k + u_k} \times \\ \times \Big(1 - \big(1 - p_k\,(w_k + u_k)\big)^t\Big).$$

Calculating the sum of this geometric progression in time, we obtain

$$F(P,W,U,T_0)=\sum_{k=1}^{K}\frac{p_k h_k w_k}{w_k+u_k}\Big(T_0-\big(1-p_k\left(w_k+u_k\right)\big)\times$$
$$\times\frac{1-\big(1-p_k\left(w_k+u_k\right)\big)^{T_0}}{p_k\left(w_k+u_k\right)}\Bigg).$$

In the *homogeneous* case (for all states, the probabilities of mastering and forgetting are $w_k = w$ and $u_k = u$), from the expression (17) it follows that

$$L(P,\,w,\,u,\,t)=\frac{w}{w+u}\left(1-\sum_{k=1}^{K}p_k\big(1-p_k\left(w+u\right)\big)^{t}\right),\quad t = 0,\,1,\,2,\,\ldots \tag{18}$$

In a particular homogeneous case ($w = 1$ and $u = 1$ under the uniform probability distribution), $L(t)=\frac{1}{2}-\frac{1}{2}\left(1-\frac{2}{K}\right)^{t}$. (The asymptote 0.5 means that the fact of forgetting is discovered during the repeated realization of the mastered UF state.)

Let us introduce the following assumption, known as *the learnability condition* [18]: the probabilities P, W, and U are such that

$$p_k\,(w_k + u_k) < 1,\ k=\overline{1,\,K}. \tag{19}$$

As was demonstrated in [18], condition (19) can be violated at most for one UF state. Moreover, under assumptions I–VI:

– The initial value of the experience criterion is 0.

– The experience curve is not decreasing and asymptotically tends to $\sum_{k=1}^{K}p_k\frac{w_k}{w_k+u_k}$; in addition, its growth rate is monotonically decreasing.

Introducing a *threshold* $\rho\in[0,1/K]$, we denote by $P_{\rho,K} = \{P = (p_1, \ldots, p_K) \mid \sum_{k=1}^{K}p_k=1,\ p_k \geq \rho,\ k=\overline{1,K}\}$ the set of K-dimensional probability distributions whose values are all not smaller that ρ.

As was established in [14], an analog of the expression (17) achieves maximum over all possible probability distributions $P \in P_{\rho,K}$ at the uniform distribution. We present a similar result for the model under consideration.

Proposition 1. *If*

$$w_k \in (0, 1],\ u_k \in[0,1),\ k=\overline{1,\,K}, \tag{20}$$

then $\forall\rho \in (0, 1/K]$ $\exists\, t(\rho, W, U) = \dfrac{2}{\rho\min\limits_{k=\overline{1,K}}\{w_k+u_k\}}-1$ *such that* $\forall\,\tau > t(\rho, W, U)$ *the function* (17) *is strictly concave in* $\{p_k\} \in P_{\rho,K}$.

Proof. Denoting

$$\alpha_k=\frac{w_k}{w_k+u_k}\in(0,1],\ \beta_k=w_k+u_k\in(0,\ 2),\ k=\overline{1,\,K},$$

we write the expression (25) as

$$x_t(P,W,U)=\sum_{k=1}^{K}\alpha_k p_k\left[1-\left(1-\beta_k p_k\right)^{t}\right]. \tag{21}$$

Differentiating the expression (21) twice, we easily check that the condition

$$t>\frac{2}{\rho\min\limits_{k=\overline{1,\,K}}\{\beta_k\}}-1$$

guarantees the strict concavity of the function (21) in all variables $\{p_k\} \in P_{\rho,K}$. ♦

Corollary. *If*

$$u_k = \sqrt{w_k}-w_k,\ k=\overline{1,K}, \tag{22}$$

then the uniform probability distribution $p_k = 1/K$, $k=\overline{1,K}$ *is a unique solution of the problem*

$$x_\tau(P,W,U)\ \to\max_{P\in P_{\rho,K}}. \tag{23}$$

This fact follows from Proposition 1 and the symmetry of the function (17) in all variables $\{p_k\} \in P_{\rho,K}$ under condition (22). (Also, see the proof of Proposition 4 in [14]). Note that condition (22) implies the learnability condition (19).

Thus, in the presence of forgetting and the nonunitary probabilities of mastering, the uniform probability distribution is generally not optimal in the problem (23); a sufficient condition for its optimality is given by (22), where $w_k\in(0,1]$.

In the homogeneous case, for the uniform probability distribution to be optimal in the problem (23), it suffices to satisfy the relation $w + u = 1$, under which condition (19) always holds and a particular case of which is the basic model with $w = 1$ and $u = 0$.

Substituting the uniform probability distribution $p_k = 1/K$, $k=\overline{1,K}$, into the expression (18), we obtain

$$x_t(K,\,w,\,u)=\frac{w}{w+u}\left(1-\left(1-\frac{w+u}{K}\right)^{t}\right)=$$
$$=\frac{w}{w+u}[1-\exp\big(-\gamma(K,\,w,u)\ t\big)],$$

where $\gamma(K, w, u) = \ln(1 + 1/(K - (u + w)))$ is *the rate of forming experience*. Since $(w+u)\in(0,2)$, the learnability condition (19) will be satisfied if $K \geq 2$.

Model 5 (learning and productive activity). Assume that the AE has a foresight horizon T_0. In this horizon, the first $T\in\{0,\,1,...,\,T_0\}$ periods are occupied by learning. In the initial period, the subject chooses an allocation $X = (X_1, \ldots, X_K)$ of his time (the same for all T future periods) among K possible activity types, where X_k is a share of his time for forming experience in activity type k and $X\in\Delta^K$ $=\{s\in\Re_+^K\mid\sum_{k=1}^{K}s_k=1\}$.

Suppose that there is no forgetting. Then from the expression (16) we obtain the following expectation that component k of the AE experience is successfully formed after period t:

$$q_k(X_k, t) = 1-(1-w_k X_k)^t,\ k=\overline{1,K}. \qquad (24)$$

The vector $q = (q_1, \ldots, q_K)$ in learning models will be called the AE's *qualification*.

Upon completing the learning process, the AE proceeds to productive activity. (By assumption, there is no learning during productive activity.) In each period of this activity, UF state k is realized with a probability p_k, which forces the AE to perform the complex activity of type k. If the experience corresponding to activity type k is formed by the given period, the AE performs this type of activity and obtains a reward h_k; otherwise, he obtains nothing.

Thus, in each period $t \in \{T+1, ..., T_0\}$ of his productive activity of type k, the AE obtains the expected "income" $h_k\, q_k(T)$, which is the expectation of obtaining the reward h_k in the case of successfully achieving the result of activity k. (If the corresponding experience is formed and not forgotten by the AE, the result is achieved.) The AE's objective function in period t has the value $f(X, t) = \sum_{k=1}^{K} h_k p_k q_k(X_k, T)$, $t \in \{T+1, ..., T_0\}$. (There is no learning during productive activity, and hence the probabilities of successfully achieving the positive results are determined by the learning results achieved by the end of learning.)

Consider *the AE's time allocation problem:* maximize the expected "income" per unit time of productive activity,

$$\sum_{k=1}^{K} h_k p_k q_k(X_k,\ T) \to \max_{X\in\Delta^K}. \qquad (25)$$

Substituting the difference (24) into (25), we write this problem as

$$\sum_{k=1}^{K} h_k p_k (1-w_k X_k)^T \to \min_{X\in\Delta^K}. \qquad (26)$$

$$h_k p_k w_k T(1-w_k X_k)^{T-1} = \lambda;$$

$$X_k = \frac{1}{w_k}\left(1-\left(\frac{\lambda}{h_k p_k w_k T}\right)^{\frac{1}{T-1}}\right);$$

$$\sum_{k=1}^{K} X_k = \sum_{k=1}^{K}\frac{1}{w_k} - \lambda^{\frac{1}{T-1}}\sum_{k=1}^{K}\frac{1}{w_k}\left(\frac{1}{h_k p_k w_k T}\right)^{\frac{1}{T-1}} = 1;$$

$$\lambda^{\frac{1}{T-1}} = \frac{\sum_{k=1}^{K}\frac{1}{w_k}-1}{\sum_{k=1}^{K}\frac{1}{w_k}\left(\frac{1}{h_k p_k w_k T}\right)^{\frac{1}{T-1}}}.$$

Solving the constrained optimization problem (26), we find the AE's optimal time allocation for learning:

$$X_k^* = \frac{1}{w_k}\left(1-\frac{\sum_{j=1}^{K}\frac{1}{w_j}-1}{(h_k p_k w_k)^{\frac{1}{T-1}}\sum_{j=1}^{K}\frac{1}{w_j}\left(h_j p_j w_j\right)^{\frac{1}{1-T}}}\right),$$

$$k=\overline{1,K}. \qquad (27)$$

In a particular case (the unitary probabilities of mastering and the same "incomes" from different activity types) we have

$$X_k^* = 1-\frac{K-1}{1+\left(p_k\right)^{\frac{1}{T-1}}\sum_{\substack{j=1\\ j\neq k}}^{K}\left(p_j\right)^{\frac{1}{1-T}}}. \qquad (28)$$

The solution (27) and (28) of the problem (26) with a fixed value T being available, we can formulate *the AE's optimal learning time problem* as follows. If in each period of learning the AE bears fixed costs $c \geq 0$, then the problem is to choose a period to terminate the learning process by maximizing the difference between the expected income and costs:

$$f(X^*, T+1)\ (T_0 - T) - c\,T \to \max_{T\in[0;\,T_0]}. \qquad (29)$$

Substituting the difference (28) into (29), we arrive in the scalar optimization problem

$$(T_0-T)\sum_{k=1}^{K} h_k p_k\left(1-\frac{\left(\sum_{j=1}^{K}\frac{1}{w_j}-1\right)^T}{(h_k p_k w_k)^{\frac{T}{T-1}}\left(\sum_{j=1}^{K}\frac{1}{w_j}\left(h_j p_j w_j\right)^{\frac{1}{1-T}}\right)^T}\right)$$

$$-\,c\,\overline{T} \to \max_{T\in[0;\,T_0]}. \qquad (30)$$

The solution of the problem (30) will give the AE's expected payoff under sequential learning and productive activity. An alternative is *learning during work*: during all T_0 periods some UF states are realized, and the AE forms the corresponding experience of practical activity, achieving a positive result (and obtaining a "reward" for it) through mastering. Assume that under learning during work, the AE bears costs c in each period. In the absence of forgetting, due to the expression (25), his total expected payoff will be

$$F(P, W,\ T_0) - c\,T_0 = \sum_{k=1}^{K} p_k h_k \times$$

$$\times\left[T_0 - \left(1-p_k w_k\right)\left(p_k w_k\right)^{T_0-1}\right] - cT_0. \qquad (31)$$

The expressions (30) and (31) can be compared in each particular case (for specific values of the model parameters) to answer the following question: which strategy – sequential learning and productive activity or learning during work – is more beneficial for the AE in terms of the total expected payoff?

Example 1. Let the probabilities of mastering be 1, different activity types yield the same "income," and the probability distribution be uniform. Then from the expression (28) it follows that $X_k^* = 1/K$. The optimization problem (30) takes the form $(T_0 - T)h\left(1-(1-1/K)^T\right) - c\,T \to \max_{T\in[0;\,T_0]}$; see the concave curve in (28), illustrating the dependence of this objective function on T.

In this case, the AE's total expected payoff under learning during work (formula (31)) is $h[T_0 - (1 - 1/K)\left(1/K\right)^{T_0-1}$ – cT_0; see the horizontal line in T.

Let $K = 10$, $T_0 = 100$, and $h = 4$. For $c = 2$, there exists an optimal learning time (18 periods) during which sequential learning and productive activity yield a higher total expected payoff (approximately 242.8) than learning during work (approximately 200.0). If the costs per unit time decrease (e.g., $c = 1$), the optimal choice is learning during work; see the optimal choice is learning during work; see Fig. 6*a* vs. Fig. 6*b*. ♦

A similar model can be constructed in the case of nonzero initial conditions for $\{q_j\}$, in which the optimal solution will depend on the AE's initial experience.

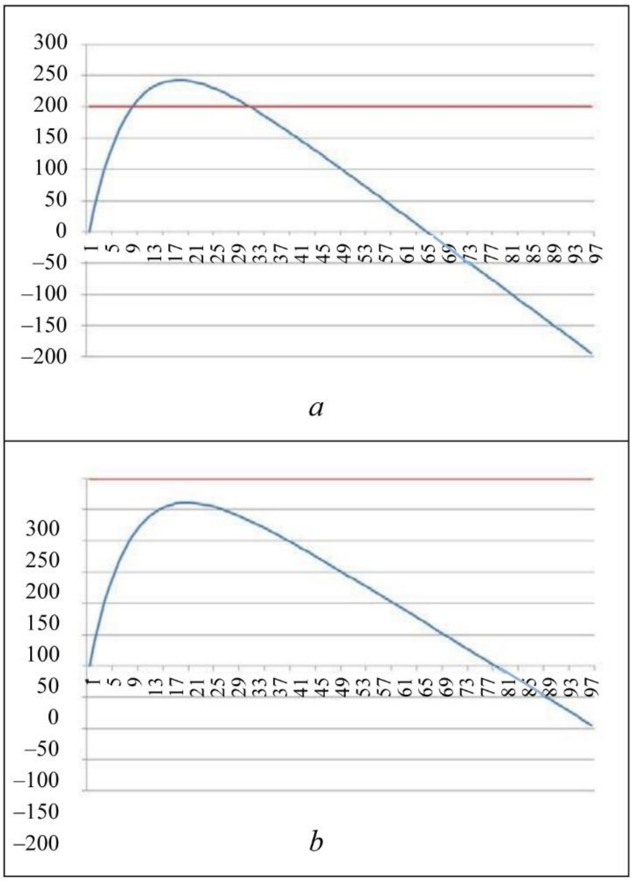

Fig. 6. AE's total expected payoffs in Example 1:
a) c = 2 and b) c = 1 (horizontal line corresponds to learning time T).

Model 6 (deterministic model with one subject). Consider a modification of Model 1 in which there is a unique UF state ($K = 1$) realized with the unitary probability, and the probability of mastering $w(q, t) \in [0,1]$ does not explicitly depend on the history. By analogy with (11), omitting the UF state subscript, we obtain $W(q(t), t) = w(q(t), t)$. The expression (5) gives the difference equation

$$q(t + 1) = w(q(t), t) + (1 - w(q(t), t))\, q(t).$$

The corresponding differential equation (see (10)) has the form

$$\dot{q}(t) = w(q, t)\,(1 - q). \qquad (32)$$

The family of differential equations (32) with an initial condition $q(0) \in [0,1]$ and a Lipschitz function $w(\cdot,\cdot) \in [0,1]$ as the parameter possesses the following properties:

– The solution of equations (32) exists and is unique.

– The experience curve $q(t)$ is strictly monotonically increasing and $\forall t \ge 0\ \dot{q}(t) \le 1$ (its growth rate is bounded).

– The experience curve $q(t)$ is *slowly asymptotic*, i.e., $\lim_{t\to+\infty} q(t) = 1$ and $\lim_{t\to+\infty} \dot{q}(t) = 0$.

Allowing the effect of forgetting, we obtain the family of differential equations with an initial condition $q(0) \in [0,1]$ and two parameters – Lipschitz functions $w(\cdot,\cdot) \in [0,1]$ and $u(\cdot,\cdot) \in [0,1]$:

$$\dot{q}(t) = w(q, t)\,(1 - q) - u(q, t)\, q. \qquad (33)$$

Let us analyze the differential equations (33), characterizing the family of solutions. The following question is of particular interest: for which time-varying functions $q(t) \in [0,1]$ is it possible to find Lipschitz functions $w(\cdot,\cdot) \in [0,1]$ and $u(\cdot,\cdot) \in [0,1]$ so that the function $q : [0, +\infty) \to [0, 1]$ will be the solution of (33)?

Proposition 2[2]. *A continuously differentiable function* $q : [0, +\infty) \to [0, 1]$ *with a Lipschitz derivative* $\dot{q}$ *is the solution of equations* (33) *under some Lipschitz functions* $w(\cdot,\cdot) \in [0, 1]$ *and* $u(\cdot,\cdot) \in [0, 1]$ *if and only if*

$$\forall t \ge 0 \quad -q(t) \le \dot{q}(t) \le 1 - q(t). \qquad (34)$$

Proof. Conditions (34) are immediate from $q(t) \in [0,1]$ and the constraints imposed on $w(\cdot,\cdot)$ and $u(\cdot,\cdot)$. Conversely, let a function $g(t)$ satisfying the hypotheses of this proposition be the solution of equations (33). Choosing

$$w(t) := \dot{q}(t) + q(t), \quad u(t) := 1 - \dot{q}(t) - q(t), \quad t \ge 0, \qquad (35)$$

we have $w(t)(1 - q(t)) - u(t)q(t) \equiv (\dot{q}(t) + q(t))(1 - q(t)) - (1 - \dot{q}(t) - q(t))q(t) \equiv \dot{q}(t)$. Moreover, from the relations (35)

[2] This result was established by S.E. Zhukovskiy, Dr. Sci. (Phys.–Math.).

it follows that $w(t) \in [0,1]$ and $u(t) \in [0,1]$ for all $t \ge 0$. ♦

Let us find possible equilibria: the right-hand side of (33) vanishes for

$$q(t) = \frac{w(q(t),t)}{w(q(t),t) + u(q(t),t)} .$$

According to (35), in the case $q(0) = 0$, the unique experience curve with a stationary (time-invariant) probability of mastering $\gamma > 0$ is the exponential curve $q(t) = \gamma\,(1 - \exp(-t))$.

4. MODELS OF COLLECTIVE AND SOCIAL EXPERIENCE

Model 7 (mastering social experience; see arrow 1 in Fig. 4). The mastering of social experience by a subject can be described by modifying the general model from Section 1. In the absence of forgetting, assume that: "social experience" contains all the necessary information about optimal actions for any UF values and "guides" the subject's learning; the subject sequentially encounters the UF states (for convenience, in accordance with their numbering); the same state repeats until the probability of forming the corresponding experience component (see formula (13))

$$q_k^1(t) = 1 - (1 - w_k)^t, \ t = 0, 1, 2, \ldots,$$

achieves a given threshold q^*. The time (the expected number of repetitions) required is

$$t_k^1(q^*) = \frac{\ln(1-q^*)}{\ln(1-w_k)} .$$

Hence, for all K possible UF values, the threshold q^* will be achieved in the time $t^1(q^*) = \sum_{k=1}^{K} \frac{\ln(1-q^*)}{\ln(1-w_k)}$. In the homogeneous case, $t^1(q^*) = \frac{\ln(1-q^*)^K}{\ln(1-w)}$.

Model 8 (forming individual experience; see arrow 2 in Fig. 4). Generally speaking, Models 1–6 all describe the formation of individual experience. We will consider a particular case: no forgetting and the uniform probability distribution ($p_k = 1 / K$) of various UF states. In this case, experience is formed by the rule

$$L_t^2(W) = 1 - \frac{1}{K} \sum_{k=1}^{K} \left(1 - \frac{w_k}{K}\right)^t, \ t = 0, 1, 2, \ldots,$$

representing a particular case of formulas (16) and (17).

In the homogeneous case, $L_t^2 = 1 - \left(1 - \frac{w}{K}\right)^t$. Hence, the threshold q^* will be achieved in the time $t^2(q^*) = \frac{\ln(1-q^*)}{\ln(1-\frac{w}{K})}$.

Proposition 3. *The ratio* $\frac{t^2(q^*)}{t^1(q^*)} = \frac{\ln(1-w)}{\ln(1-\frac{w}{K})^K} \ge 1$, *characterizing the relative effectiveness of mastering social experience compared to forming individual experience, is independent of* q^* *and monotonic in w and K.*

Model 9 (mastering collective experience; see arrow 3 in Fig. 4). Joint activity of subjects within collectives implies the possibility of exchanging their experience acquired during the process of activity. (*A team* is a particular case of collectives [19].)

In the absence of forgetting, assume that: a team includes n AEs; for each AE, a certain UF state (same for all subjects) is realized with a given probability distribution P in each period; the subjects form their experience of activity for this state independently within the model (17); after that, the subjects completely exchange their information with each other (i.e., all team members will form their experience for a certain UF state if at least one team member does). The elements of the matrix $\mathbf{W} = \|w_{ik}\|$ can be interpreted as the effectiveness of "learning by one's own and someone else's experience" for different subjects under different UF states.

After t periods, team member i will not master UF state k with the probability $(1 - p_k\, w_{ik})^t$, and all team members will not master it with the probability $\prod_{i=1}^{n} (1 - p_k w_{ik})^t$. We obtain the following experience curve for the entire team and each team member (the probability that none of the team members will encounter a new UF state for the entire team):

$$L_t^3(P, \mathbf{W}) = 1 - \sum_{k=1}^{K} p_k \prod_{i=1}^{n} (1 - p_k w_{ik})^t, \ t = 0, 1, 2, \ldots \quad (36)$$

In the case of homogeneous AEs and the uniform probability distribution, the expression (38) takes the form $L_t^3 = 1 - \left(1 - \frac{w}{K}\right)^{nt}$. The threshold q^* will be achieved in the time $t^3(y^*) = \ln(1-q^*) \big/ n \ln(1 - \frac{w}{K})$. Therefore, $t^3(q^*) = \frac{1}{n} t^2(q^*)$.

Proposition 4. *The complete exchange of experience between the subjects reduces the time for forming their individual experience proportionally to the number of AEs participating in this exchange.*

This conclusion is valid under a constant probability of mastering w. The decreasing dependence of the probability of mastering $w(n)$ on the number of interacting subjects seems to be more realistic. A promising line is to consider models with the coefficients w_{ij} depending not on the UF states but on the pairs of interacting AEs (subject i acquiring experience from subject j).

Model 10 (forming collective experience; see arrow 4 in Fig. 4). Assume that for each of n AEs, a certain UF state (same for all subjects) is realized with a given probability distribution P in each period. The effect of forgetting will be described by the matrix $\mathbf{U} = ||u_{ik}||$. We denote by π_{ikt} the probability that team member i will master UF state k after t periods. According to the expression (16), $\pi_{ikt} = \frac{w_{ik}}{w_{ik}+u_{ik}}\left(1-\left(1-p_k\left(w_{ik}+u_{ik}\right)\right)^t\right)$. Hence, team member i will not master UF state k after t periods with the probability

$$1-\pi_{ikt} = 1-\frac{w_{ik}}{w_{ik}+u_{ik}}\left(1-\left(1-p_k\left(w_{ik}+u_{ik}\right)\right)^t\right) =$$

$$= \frac{u_{ik}}{w_{ik}+u_{ik}} + \frac{w_{ik}}{w_{ik}+u_{ik}}\left(1-p_k\left(w_{ik}+u_{ik}\right)\right)^t.$$

None of the team members will master this state with the probability

$$\prod_{i=1}^{n}(1-\pi_{ikt}) = \prod_{i=1}^{n}\left(\frac{u_{ik}}{w_{ik}+u_{ik}} + \frac{w_{ik}}{w_{ik}+u_{ik}}\left(1-p_k\left(w_{ik}+u_{ik}\right)\right)^t\right).$$

We obtain the following experience curve (the probability that at least one team member will form the experience for a new UF state realized in period $(t+1)$; see formula (7)):

$$L_{\max}(P, \mathbf{W}, U, t) = 1 -$$

$$L_{\max}(P, \mathbf{W}, U, t) = 1 - \sum_{k=1}^{K} p_k \prod_{i=1}^{n}\left(\frac{u_{ik}}{w_{ik}+u_{ik}} + \right.$$

$$\left. + \frac{w_{ik}}{w_{ik}+u_{ik}}\left(1-p_k\left(w_{ik}+u_{ik}\right)\right)^t\right), \ t = 0, 1, 2, \ldots \quad (37)$$

As noted above, the group/collective experience criterion can be either the probability $L_{\max}(t)$ (**at least one** of the AEs will encounter a previously known, successfully mastered, and not forgotten UF state; see the expression (37)), or the probability $L_{\min}(t)$ (**each** of the AEs will do so; see the expression (8)):

$$L_{\min}(P, \mathbf{W}, U, t) = \sum_{k=1}^{K} p_k \prod_{i=1}^{n}\left(\frac{w_{ik}}{w_{ik}+u_{ik}} \times \right.$$

$$\left. \times\left(1-\left(1-p_k\left(w_{ik}+u_{ik}\right)\right)^t\right)\right), \ t = 0, 1, 2, \ldots \quad (38)$$

Substantially, the formation of social/collective experience differs from the formation of individual experience in that, in order to "consolidate" the methods of effective activity under a certain UF state, many subjects must encounter this state many times. In the course of modeling, this feature can be reflected, e.g., by a low probability of mastering; see Model 12 below.

Consider the homogeneous case of identical AEs ($u_{ik} = 0$ and $w_{ik} = w$) and the uniform probability distribution. Assuming that the probability of mastering is less than 1 and there is no forgetting, we reduce the expressions (37) and (38) to the following form (also, see (36)):

$$L_{\max}(n, w, t) = 1 - \left(1-\frac{w}{K}\right)^{nt}, \ t = 0, 1, 2, \ldots,$$

$$L_{\min}(n, w, t) = \left(1-\left(1-\frac{w}{K}\right)^t\right)^n, \ t = 0, 1, 2, \ldots$$

Based on the results for Model 8, we obtain the following experience curve of one AE with the unitary probability of mastering for all UF states in the absence of forgetting (also, see (13)):

$$L_t^2(w=1) = 1 - \left(1-\frac{1}{K}\right)^t, \ t = 0, 1, 2, \ldots$$

Applying trivial transformations to the condition $L_{\max}(n, w, t) = L_t^2$, we establish the following fact.

Proposition 5. *In the case of no forgetting and* $w(n) = K - K\left(1-\frac{1}{K}\right)^{\frac{1}{n}}$, *forming collective experience with the probability of mastering* $w(n)$ *is equivalent to forming individual experience by an AE with the unitary probability of mastering*.

An alternative version of Proposition 5 is as follows: in the case of no forgetting and

$$n(w) = \frac{\ln(1-\frac{1}{K})}{\ln(1-\frac{w}{K})},$$

forming individual experience by an AE with the unitary probability of mastering is equivalent to forming collective experience by $n(w)$ AEs with the same probability of mastering w.

Model 11 (deterministic model with several interacting subjects).

Assume that: there is a single UF state ($K = 1$) realized with the unitary probability; the probabilities of mastering and forgetting do not explicitly depend on the history (also, see Model 6). Then from the expression (1) we obtain the system of difference equations

$$q_i(t+1) = W_i(\boldsymbol{q}(t), t) + [1 - W_i(\boldsymbol{q}(t), t) - U_i(\boldsymbol{q}(t), t)]\, q_i(t), \ i = \overline{1, n}, \quad (39)$$

and the corresponding system of differential equations

$$\dot{q}_i(t) = w_i(\boldsymbol{q}, t)\,(1 - q_i) - u_i(\boldsymbol{q}, t)\, q_i, \ i = \overline{1, n},$$

with an initial condition $\mathbf{y}(0) \in [0,1)^n$ and two parameters – Lipschitz vector functions $\mathbf{w}: [0;1]^n \times \Re^1_+ \to [0;1]^n$ and $\mathbf{u}: [0;1]^n \times \Re^1_+ \to [0;1]^n$.

The difference equation (39) has a specific structure with given constraints on the functions in the right-hand side. Therefore, we cannot write in its terms, e.g., *the linear models*

$$\Delta q_i(t+1) = \sum_{j \in N_i(t)} a_{ij}(q_j(t) - q_i(t)), \qquad (40)$$

where $\sum_{j \in N_i(t)} a_{ij} \le 1$ (see a survey in Section 3.2 of the book [10] and the work [200]) or *the threshold behavior models* [10]

$$q_i(t+1) = I\left(\frac{1}{|N_i(t)|}\sum_{j \in N_i(t)} q_j(t) \ge \delta_i(t)\right), \qquad (41)$$

where $N_i(t) \in 2^N$ denotes the set of "*neighbors*" of AE i in period t, and $\delta_i(t) \in [0, 1]$ is his "*threshold*."

Therefore, we will proceed by considering the influence of other AEs not on the state of a given AE (the expected level of his experience q) but his probability of mastering. Let the probability of mastering be represented as the sum of two functions,

$$W_i(\boldsymbol{q}(t), t) = d_i(q_i) + D_i(q_{-i}),\ i = \overline{1, n}, \qquad (42)$$

taking values from the range [0, 1] but in the sum not exceeding 1, where $q_{-i} = (q_1, \ldots, q_{i-1}, q_{i+1}, \ldots, q_n)$ is the opponents' experience profile for AE i.

Some practical examples of such dependencies are "the linear model" (with the superscript L)

$$W^L{}_i(\boldsymbol{q}(t)) = \alpha_i + \beta_i \sum_{j \in N_i(t)} a_{ij} q_j(t), \qquad (43)$$

which can be compared with (40), and "the threshold model" (with the superscript T)

$$W^T{}_i(\boldsymbol{q}(t)) = \alpha_i + \beta_i\, I\left(\frac{1}{|N_i(t)|}\sum_{j \in N_i(t)} q_j(t) \ge \delta_i(t)\right) \qquad (44)$$

with constants $\alpha_i + \beta_i \le 1,\ i = \overline{1, n}$, which can be compared with (41).

The first term in the right-hand sides of (42), (43), and (44) can be interpreted as reflecting an *explicit experience* (directly transferred to the subject and mastered by him), whereas the second term as a *tacit experience* (acquired and mastered by the subject indirectly, through interactions with other subjects).

In this class of models, *natural selection* (competition) can be considered, e.g., by letting $W_i(\boldsymbol{q}(t), t) \to 0$ as $q_i(t) << \frac{1}{|N_i(t)|}\sum_{j \in N_i(t)} q_j(t)$.

Example 2. Consider two active elements with the same stationary probability of forgetting u and the probabilities of mastering $W_i(\boldsymbol{q}(t), t) = \gamma_i \frac{q_i(t)}{q_i(t) + q_{3-i}(t)}$, $i = \overline{1, 2}$. In a practical interpretation, the AEs compete for a constant amount of a resource for each period, distributed between them proportionally to their experience. The probability of mastering in each period is proportional to the amount of the resource received in the past period. The coefficients of proportionality γ_i can be treated as the individual learning aptitudes of the AEs.

Assume that the AEs differ in the initial values of their experience: $q_1(0) = 0.1$ and $q_2(0) = 0.2$; see the solid and dashed lines in Fig. 7, corresponding to the first and second AEs, respectively. However, despite the worse "starting position," the first AE has a higher learning aptitude: $\gamma_1 = 0.2$ and $\gamma_2 = 0.1$.

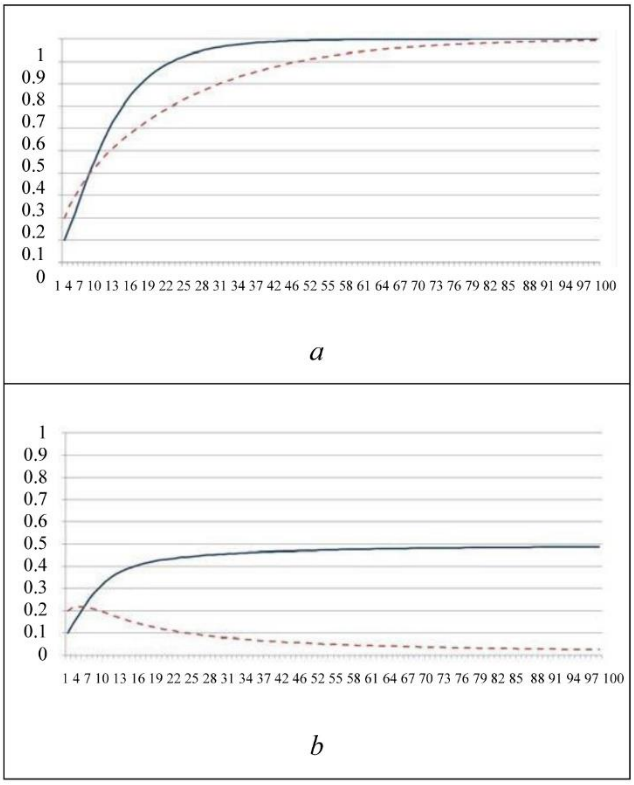

Fig. 7. Dynamics of AE's experience in Example 2:
a) u = 0 and b) u = 0.2 (horizontal line corresponds to time).

In the absence of forgetting, both AEs successfully learn and will equally share the resource on a sufficiently long horizon (see Fig. 7*a*). In the presence of forgetting (u = 0,2), the first AE wins the competition and will obtain the entire resource on a sufficiently long horizon (see Fig. 7*b*). ♦

Model 12 (forming social experience; see arrow 4 in Fig. 4). Within Model 10, collective experience is formed if at least one of the team members forms it. Model 12 rests on the assumption that social experience is formed only if all team members form it.

This approach makes the experience criterion multiplicative over subsets: in the homogeneous case of Model 10 (the identical AEs and the uniform probability distribution), if the society consists of two subsets: $N = N_1 \cup N_2,\ N_1 \cap N_2 = \varnothing,\ |N_1| = n_1,\ |N_2| = n_2$, then

$$L_{\min}(n, w, t) = L_{\min}(n_1, w, t) \quad L_{\min}(n_2, w, t) .$$

The expressions $L_{\max}(n, w, t) = 1 - (1 - w/K)^{nt}$ and $L_{\min}(n, w, t) = \left(1 - \left(1 - \frac{w}{K}\right)^t\right)^n$, $t = 0, 1, 2, \ldots$, can be used for estimating the expected time required for achieving a given collective experience level q^*, depending on the number of AEs:

$$t(n, q^*) \sim \frac{\ln(1 - (q^*)^{1/n})}{\ln(1 - \frac{w}{K})} . \tag{45}$$

Clearly, the time (45) grows rather slowly with an increase in the number of AEs (approximately linearly in the logarithm of this number). Since collective experience grows linearly in the number of AE, we arrive at the following result.

Proposition 6. *Social experience is formed approximately by* $n \ln(n)$ *times slower than the collective one*.

Applying trivial transformations to $L_{\min}(n, w, t) = L_t^2$, we establish the following fact.

Proposition 7. *In the case of no forgetting and* $w(n,t) = K\left\{1 - \left[1 - \left(1 - \left(1 - \frac{1}{K}\right)^t\right)^n\right]^{\frac{1}{t}}\right\}$, *forming social experience with the unitary probability of mastering is equivalent to forming individual experience by one AE with the probability of mastering* $w(n, t)$.

Due to Proposition 7, social experience can be viewed as the experience of one integral and virtual subject.

CONCLUSIONS

The results of Sections 3 and 4 show that the well-known learning models [11, 14, 18], including the learning curves (20)–(26), correspond to particular cases of the general experience model from Section 2. As was shown in [14, 18], the learning models, in turn, generalize the following models: testing of complex systems and checking their characteristics; increasing the efficiency of mass production during mastering (the model of T.P. Wright and his followers); software testing; dissemination of knowledge (ideas, theories, concepts) in society; knowledge management and extraction/acquisition; machine learning; iterative learning and testing of knowledge in pedagogy, psychology, and physiology of humans and animals. The surveys [14, 18] referred to numerous works on the experiment-based characterization of nontrivial processes that underlie learning models.

Let us outline some promising lines for further research:

- The formation of experience is an essential component of any activity. Therefore, it seems strategically important to develop a general mathematical model of complex activity (with operational decomposition and aggregation of particular models) within MCA, reflecting the active choice of subjects and considering the processes of forming their experience.

- The system of classification bases for experience models (see Section 2) yields various particular models of forming and mastering individual and collective experience. The development and study of such models are a well-founded “tactical” step. Here, some of the promising areas are as follows: exploring the joint formation of experience during work, optimizing the duration of learning before the transition to productive activity, analyzing the impact of forgetting and the history length, identifying the role of the time dependence of the uncertainty factor states, revealing the influence of the experience structure (the logical connections between its components), and optimizing and managing the experience formation process.

- The experience model proposed above is rich enough for describing many phenomena and processes:

– *personnel management* (recruitment, placement, *development*, promotion, and dismissal) and *human capital models*;

– *risk management* and *information security management*;

– *evolution* (including *adaptation*, *competition*, and *natural selection*) in biological systems (involving the conventional mathematical apparatus for this range of problems – inite automata[3] [21, 22], differential equations [23–25], evolutionary games [26, 27], etc.);

– different characteristic times and hierarchy for the translated (explicit and tacit) and nontranslated experience, which are considered within *psychological* and *sociological* approaches;

– selection, formation, mastering, consolidation, and transmission of social experience, considered as *cultural phenomena*.

[3] The areas of research mentioned here are very extensive. Without claiming to be exhaustive, we therefore refer to several classical monographs and/or modern surveys.

REFERENCES

1. Burkov, V.N., *Osnovy matematicheskoi teorii aktivnykh sistem* (Foundations of the Mathematical Theory of Active Systems), Moscow: Nauka, 1977. (in Russian)
2. Goubko, M., Burkov, V., Kondrat'ev, V., et al., *Mechanism Design and Management: Mathematical Methods for Smart Organizations*, New York: Nova Science Publishers, 2013.
3. *INCOSE Systems Engineering Handbook*, Haskins, C., Ed., San Diego: INCOSE, 2012.
4. *Systems Engineering Guide*, Bedford: MITRE Corporation, 2014.
5. Belov, M.V. and Novikov, D.A., *Methodology of Complex Activity: Foundations of Understanding and Modelling*, Springer, 2020.
6. Novikov, D.A., *Upravlenie, deyatel'nost', lichnost'* (Control, Activity, and Personality), Moscow: Institute of Control Sciences RAS, 2020. (in Russian)
7. Novikov, A.M., *Osnovaniya pedagogiki* (Foundations of Pedagogy), Moscow: Egves, 2010. (in Russian)
8. Ozhegov, S.I., *Slovar' russkogo yazyka* (Dictionary of the Russian Language), 23 ed., Moscow: Russkii Yazyk, 1991. (in Russian)
9. Platonov, K.K., *Kratkii slovar' sistemy psikhologicheskikh ponyatii* (Concise Dictionary of the System of Psychological Concepts), Moscow: Vysshaya Shkola, 1984. (in Russian)
10. Breer, V.V., Novikov, D.A., and Rogatkin, A.D., *Mob Control: Models of Threshold Collective Behavior*, Springer, 2017.
11. Novikov, D., *Regularities of Iterative Learning*, Moscow: Institute of Control Sciences RAS, 2019.
12. Aleksandrov, Yu.I. and Aleksandrova, N.L., *Sub"ektivnyi opyt, kul'tura i sotsial'nye predstavleniya* (Subjective Experience, Culture, and Social Views), Moscow: Institute of Psychology RAS, 2009. (in Russian)
13. *Teoriya upravleniya*: *dopolnitel'nye glavy* (Control Theory: Supplement Chapters), Moscow: Lenand, 2019. (in Russian)
14. Belov, M.V. and Novikov, D.A., *Models of Technologies*, Springer, 2020.
15. *Filosofskii entsiklopedicheskii slovar'* (Philosophical Encyclopedic Dictionary), Moscow: Sovetskaya Entsiklopediya, 1983. (in Russian)
16. Pelipenko, A.A. and Yakovenko, I.G., *Kul'tura kak sistema* (Culture as a System), Moscow: Yazyki Russkoi Kul'tury, 1998. (in Russian)
17. Reznik, Yu.M., *Vvedenie v sotsial'nuyu teoriyu* (Introduction to Social Theory), Moscow: Nauka, 2003. (in Russian)
18. Belov, M.V. and Novikov, D.A., *Upravlenie zhiznennymi tsiklami organizatsionno-tekhnicheskikh sistem* (Managing Life Cycles of Organizational and Technical Systems), Moscow: Lenand, 2020. (in Russian)
19. Novikov, D.A., *Matematicheskie modeli formirovaniya i funktsionirovaniya komand* (Mathematical Models of Team Formation and Functioning), Moscow: Fizmatlit, 2012. (in Russian)
20. Chkhartishvili, A., Gubanov, D., and Novikov, D., *Social Networks: Models of Information Influence, Control and Confrontation*, Heidelberg: Springer, 2019.
21. Bukharaev, R.G., *Osnovy teorii veroyatnostnykh avtomatov* (Foundations of the Theory of Probabilistic Automata), Moscow: Nauka, 1985. (in Russian)
22. Tsetlin, M.L., *Issledovaniya po teorii avtomatov i modelirovaniyu biologicheskikh sistem* (Researches into the Theory of Automata and Modeling of Biological Systems), Moscow: Nauka, 1969. (in Russian)
23. Emel'yanov, V.V., Kureichik, V.V., and Kureichik, V.M., *Teoriya i praktika evolyutsionnogo modelirovaniya* (Theory and Practice of Evolutionary Modeling), Moscow: Fizmatlit, 2003. (in Russian)
24. Otsuka, J., *The Role of Mathematics in Evolutionary Theory*, Cambridge: Cambridge University Press, 2020.
25. Schuster, P., Mathematical Modeling of Evolution. Solved and Open Problems, *Theory Biosci.*, 2011, vol. 130, pp. 71–89.
26. Vasin, A.A., *Nekooperativnye igry v prirode i obshchestve* (Noncooperative Games in Nature and Society), Moscow: Maks Press, 2005. (in Russian)
27. Weibull, J., *Evolutionary Game Theory*, Cambridge: MIT Press, 1997.

This paper was recommended for publication by A.A. Voronin, a member of the Editorial Board.

Received November 6, 2020, and revised December 11, 2020.
Accepted December 11, 2020.

Author information

Belov, Mikhail Valentinovich. Dr. Sci. (Eng.), Skolkovo Institute of Science and Technology, Moscow, ✉ mbelov59@mail.ru

Novikov, Dmitry Aleksandrovich. Corresponding Member, Russian Academy of Sciences, V.A. Trapeznikov Institute of Control Sciences, Russian Academy of Sciences, Moscow, ✉ novikov@ipu.ru

Cite this article

Belov, M.V., Novikov, D.A. Models of Experience. *Control Sciences* **1,** 37–52 (2021). http://doi.org/10.25728/cs.2021.1.5