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CONSTRUCTING POWER-EXPONENTIAL AND LINEAR-LOGARITHMIC REGRESSION MODELS

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Abstract. When using nonlinear regression models, the estimates of the resulting dependence are often difficult or even impossible to interpret. This paper develops nonlinear regression specifications in which any estimated parameter, except the free term, can always be given some practical interpretation. A multiplicative power-exponential regression generalizing the Cobb–Douglas production function and an additive linear-logarithmic regression are constructed. Three construction strategies are formulated for each of them, and the issues of interpreting their estimates are considered in detail. The construction strategies based on the least absolute deviations method are formalized as linear and partially Boolean linear programming problems. The mathematical apparatus developed in this paper is illustrated by modeling rail freight traffic in Irkutsk oblast.

Keywords: regression model, interpretation, multiplicative power-exponential regression, linear-logarithmic regression, feature selection, least absolute deviations, rail freight traffic.

INTRODUCTION

Regression analysis is a worldwide recognized tool for mathematical modeling based on statistics [1, 2]. One of the first (and, perhaps, most important) stages in constructing a regression model is specification, i.e., choosing an appropriate composition of the variables and a mathematical relationship among them. A significant number of such specifications have been developed to date, and most of them can be found in [3–6]. The simplest specification is the multiple linear regression model:

$$y_i = \alpha_0 + \sum_{j=1}^{l} \alpha_j x_{ij} + \varepsilon_i , \quad i = \overline{1,n} , \tag{1}$$

where y_i, $i = \overline{1,n}$, are the observed values of the independent (output) variable y; x_{ij}, $i = \overline{1,n}$, $j = \overline{1,l}$, are the observed values of the explanatory (input) variables x_1, x_2, …, x_l; ε_i, $i = \overline{1,n}$, are approximation errors; α_0, α_1, α_2, …, α_l are unknown parameters.

The linear regression (1) is easily estimated, e.g., using the least squares method (LSM). Let the estimated equation have the form

$$\tilde{y} = \tilde{\alpha}_0 + \sum_{j=1}^{l} \tilde{\alpha}_j x_j , \tag{2}$$

where $\tilde{y}$ is the model value of the independent variable; $\tilde{\alpha}_0, \tilde{\alpha}_1, \tilde{\alpha}_2, ..., \tilde{\alpha}_l$ are the estimates of the unknown parameters.

The coefficient $\tilde{\alpha}_s$ at the explanatory variable x_s in equation (2) is interpreted in the following way: if the value of the explanatory variable x_s varies by 1, then the value of the independent variable y varies by $\tilde{\alpha}_s$ on average.

Note that the development of new specifications for regression models continues to the present time. For example, a linear multiplicative regression (LMR) and a regression contrary to the Leontief production function were proposed in [7] and [8], respectively. Later on, they were combined in [9]. Another specification is an index regression introduced in [10].

For solving the specification problem, a technology to organize a “competition” of regression models was developed; for details, see the monograph [6]. The competition is intended to form a set of alternative regressions and select the best one among them.

The following algorithm for forming alternatives was considered in [6]. First, the set of original explanatory variables is enlarged using some transformations, e.g., the elementary functions $\ln x$, e^x, x^{-1}, x^2, x^3, $\sqrt{x}$, etc. Then, by a complete enumeration of all combinations, m features are selected [11]. Unfortunately, the resulting regression equation often turns out to be significantly nonlinear, making it difficult (or even impossible) to interpret the estimates found.

This paper develops nonlinear regression specifications in which any estimated parameter, except the free term, can always be given some practical interpretation during the competition of regression models.

1. MULTIPLICATIVE POWER-EXPONENTIAL REGRESSION

The exponential regression with one explanatory variable [12, 13] has the form

$$y_i = \alpha_0 \cdot e^{\alpha_1 x_i} \varepsilon_i , \ i = \overline{1, n}. \tag{3}$$

The model (3) is nonlinear in the estimated parameters but can be linearized by taking the logarithm:

$$\ln y_i = c_0 + \alpha_1 x_i + u_i , \ \ i = \overline{1, n} , \tag{4}$$

where $c_0 = \ln \alpha_0$ and $u_i = \ln \varepsilon_i$.

The linear model (4) is called the semi-log (left-log, or log-linear) regression [13].

The book [13] suggested the following interpretation of the estimated coefficient $\tilde{\alpha}_1$ of the models (3) and (4): if the explanatory variable x changes by 1, then the independent variable y changes by $100\tilde{\alpha}_1$ % on average.

Unfortunately, as noted in [13], this interpretation of the coefficient $\tilde{\alpha}_1$ of the models (3) and (4) applies to small $\tilde{\alpha}_1$ only.

Consider a generalization of the model (3): the additive multiple exponential regression

$$y_i = \alpha_0 + \sum_{j=1}^{l} \alpha_j e^{\beta_j x_{ij}} + \varepsilon_i , \ \ i = \overline{1, n} , \tag{5}$$

where β_j, $j = \overline{1, l}$, are unknown parameters.

It seems impossible to linearize the model (5). Even if its estimates were found, it would be difficult to give them any practical interpretation. Therefore, consider the multiplicative multiple exponential regression

$$y_i = \alpha_0 \prod_{j=1}^{l} e^{\alpha_j x_{ij}} \varepsilon_i , \ i = \overline{1, n} . \tag{6}$$

The model (6) is linearized by taking the logarithm, and all its coefficients have the interpretation described above.

The regression (6) resembles by properties the Cobb–Douglas production function (the power regression)

$$y_i = \alpha_0 \prod_{j=1}^{l} x_{ij}^{\alpha_j} \ \varepsilon_i , \ i = \overline{1, n} . \tag{7}$$

The model (7) is also linearized by taking the logarithm, and the estimated coefficient $\tilde{\alpha}_s$ at the explanatory variable x_s is interpreted in the following way: if the explanatory variable x_s changes by 1%, then the independent variable y changes by $\tilde{\alpha}_s$ % on average. In other words, $\tilde{\alpha}_s$ gives the elasticity of the variable y in x_s.

We construct a multiplicative combination of the models (6) and (7):

$$y_i = \alpha_0 \prod_{j=1}^{l} x_{ij}^{\alpha_j} \prod_{j=1}^{l} e^{\beta_j x_{ij}} \varepsilon_i , \ i = \overline{1, n} . \tag{8}$$

The expression (8) will be called the multiplicative power-exponential regression (MPER).

Note that the power and exponential regressions were also combined previously. For example, a modification of the Cobb–Douglas production function was considered in the paper [14]: labor and capital were included as power functions and scientific and technical information as an exponential function. In addition, we mention Tinbergen's production function [6], representing the product of the power regression (7) and a factor $e^{\gamma t}$ describing the "neutral" technical progress effect. However, the MPER generalizes all these known modifications.

The logarithmized MPER (8) has the form

$$\ln y_i = c_0 + \sum_{j=1}^{l} \alpha_j \ln x_{ij} + \sum_{j=1}^{l} \beta_j x_{ij} + u_i , \ i = \overline{1, n} . \tag{9}$$

Clearly, the MPER is easily estimated. However, a problem arises with a practical interpretation of its coefficients: each explanatory variable enters into the model (9) both linearly and logarithmically. Therefore, for interpreting any coefficient of the MPER, we should perform feature selection in modeling.

For the further presentation, we introduce the following Boolean variables:

$$\sigma_j^{\text{pow}} = \begin{cases} 1 \text{ if } x_j \text{ enters into the MPER} \\ \quad \text{via the power function,} \\ 0 \text{ otherwise,} \end{cases}$$

$$\sigma_j^{\text{exp}} = \begin{cases} 1 \text{ if } x_j \text{ enters into the MPER exponentially,} \\ 0 \text{ otherwise.} \end{cases}$$

Then linear constraints can be imposed on the coefficients of the models (8) and (9):

$$-M\sigma_j^{\text{pow}} \le \alpha_j \le M\sigma_j^{\text{pow}} , \ \ j = \overline{1, l} , \tag{10}$$

$$-M\sigma_j^{\exp} \le \beta_j \le M\sigma_j^{\exp},\ j=\overline{1,l}, \tag{11}$$

where M is a large positive number.

If $\sigma_j^{\text{pow}}=1$ and $\sigma_j^{\exp}=0$, $j=\overline{1,l}$, then the MPER (8) is transformed to the power regression (7); if $\sigma_j^{\exp}=0$ and $\sigma_j^{\exp}=1$, $j=\overline{1,l}$, to the exponential regression (6).

We formulate three strategies to construct the MPER:

• Strategy 1. There are no restrictions on how the variables enter into the model. In this case, we need to estimate the linear regression (9) with $(2l+1)$ parameters and pass to the MPER (8). The estimated equation can be used for prediction, but the coefficients cannot be interpreted.

• Strategy 2. Each explanatory variable enters into the model either via the power function or exponentially. This strategy is formally described by

$$\sigma_j^{\text{pow}}+\sigma_j^{\exp}=1,\quad j=\overline{1,l}. \tag{12}$$

In this case, we need to estimate 2^l linear regressions (9) with $(l+1)$ parameters, select the best one, and pass to the MPER (8). In the resulting equation, any coefficient (possibly except the free term) can always be given a practical interpretation if its sign corresponds to the problem's sense. Also, the resulting equation can be used for prediction. But if the number of variables l is large, then the problem arises with selecting a given number of the most informative ones.

• Strategy 3. Each explanatory variable enters into the model either via the power function or exponentially, and the total number of linear features is *m*. This strategy is formally described by

$$\sigma_j^{\text{pow}}+\sigma_j^{\exp}\le 1,\ j=\overline{1,l}, \tag{13}$$

$$\sum_{j=1}^{l}\left(\sigma_j^{\text{pow}}+\sigma_j^{\exp}\right)=m. \tag{14}$$

In this case, we need to estimate $C_l^m\cdot 2^m$ linear regressions of the form (9) with $(m+1)$ parameters, select the best one, and pass to the MPER (8). The resulting equation can be used for prediction and interpretation as well.

2. LINEAR-LOGARITHMIC REGRESSION

The logarithmic [12] (right-log, log-linear) regression with one explanatory variable has the form

$$y_i=\alpha_0+\alpha_1\ln x_i+\varepsilon_i,\ \ i=\overline{1,n}. \tag{15}$$

According to [15], the estimated coefficient $\tilde{\alpha}_1$ of the model (15) is interpreted in the following way: if the explanatory variable x changes by 1%, then the independent variable y changes by $\tilde{\alpha}_1/100$ on average.

As we believe, the estimate $\tilde{\alpha}_1$ of the logarithmic model (15) can be also given another interpretation: if the explanatory variable x changes by e times, then the independent variable y changes by $\tilde{\alpha}_1$ on average.

Consider a generalization of the model (15): the additive multiple logarithmic regression

$$y_i=\alpha_0+\sum_{j=1}^{l}\alpha_j\ln x_{ij}+\varepsilon_i,\ i=\overline{1,n}. \tag{16}$$

The model (16) is linear in the parameters, and any estimated coefficient at the logarithm of an explanatory variable can be interpreted as mentioned above.

Note that there is no sense to use logarithms with different bases in (16). For example, consider the model with two explanatory variables

$$y_i=\alpha_0+\alpha_1\log_2 x_1+\alpha_2\log_3 x_2+\varepsilon_i,\ \ i=\overline{1,n}.$$

With the well-known logarithmic relation $\log_a x=\dfrac{\log_c x}{\log_c a}$, this model is written as

$$y_i=\alpha_0+\alpha_1\frac{\ln x_1}{\ln 2}+\alpha_2\frac{\ln x_2}{\ln 3}+\varepsilon_i,\ i=\overline{1,n}.$$

Redenoting $\alpha_1=\dfrac{\alpha_1}{\ln 2}$ and $\alpha_2=\dfrac{\alpha_2}{\ln 3}$, we obtain a particular case of the regression (16).

Now we construct an additive combination of the models (1) and (16):

$$y_i=\gamma_0+\sum_{j=1}^{l}\gamma_j x_{ij}+\sum_{j=1}^{l}\delta_j\ln x_{ij}+\varepsilon_i,\ i=\overline{1,n}, \tag{17}$$

The expression (17) will be called the linear-logarithmic regression (LLR).

Trying to interpret the LLR, we face the same problem as for the MPER: each explanatory variable enters into equation (17) both linearly and logarithmically.

Let us introduce the following Boolean variables:

$$\sigma_j^{\text{lin}}=\begin{cases}1 \text{ if } x_j \text{ enters into the LLR linearly,}\\ 0 \text{ otherwise,}\end{cases}$$

$$\sigma_j^{\log}=\begin{cases}1 \text{ if } x_j \text{ enters into the LLR logarithmically,}\\ 0 \text{ otherwise.}\end{cases}$$

Then linear constraints can be imposed on the coefficients of the model (17):

$$-M\sigma_j^{\text{lin}}\le\gamma_j\le M\sigma_j^{\text{lin}},\ \ j=\overline{1,l}, \tag{18}$$

$$-M\sigma_j^{\log}\le\delta_j\le M\sigma_j^{\log},\ \ j=\overline{1,l}. \tag{19}$$

If $\sigma_j^{\text{lin}} = 1$ and $\sigma_j^{\log} = 0$, $j = \overline{1, l}$, then the LLR (17) is transformed to the linear regression (1); if $\sigma_j^{\text{lin}} = 0$ and $\sigma_j^{\log} = 1$, $j = \overline{1, l}$, to the logarithmic regression (16).

By analogy with the MPER, we formulate three strategies to construct the LLR:

• Strategy 1. There are no restrictions on how the variables enter into the model. In this case, we need to estimate the linear regression (17) with $(2l+1)$ parameters. The estimated equation can be used for prediction, but the coefficients cannot be interpreted.

• Strategy 2. Each explanatory variable enters into the model either linearly or logarithmically. This strategy is formally described by

$$\sigma_j^{\text{lin}} + \sigma_j^{\log} = 1, \ j = \overline{1, l}. \tag{20}$$

In this case, we need to estimate 2^l linear regressions of the form (17) with $(l+1)$ parameters and select the best one. It can be used for prediction and interpretation.

• Strategy 3. Each explanatory variable enters into the model either linearly or logarithmically, and the total number of features is *m*. This strategy is formally described by

$$\sigma_j^{\text{lin}} + \sigma_j^{\log} \le 1, \quad j = \overline{1, l}, \tag{21}$$

$$\sum_{j=1}^{l} \left(\sigma_j^{\text{lin}} + \sigma_j^{\log} \right) = m. \tag{22}$$

In this case, we need to estimate $C_l^m \cdot 2^m$ linear regressions of the form (17) with $(m+1)$ parameters and select the best one.

3. CONSTRUCTION OF MPER AND LLR USING PARTIALLY BOOLEAN LINEAR PROGRAMMING

Mathematical programming is widely used in regression analysis; for example, see [16–18].

Let the logarithmized MPER (9) be estimated using the least absolute deviations (LAD) method. As shown in the monograph [6], the LAD estimates of this regression can be obtained by solving the linear programming (LP) problem

$$\lambda_i^+ + \lambda_i^- \to \min, \tag{23}$$

$$v_i = c_0 + \sum_{j=1}^{l} \alpha_j z_{ij} + \sum_{j=1}^{l} \beta_j x_{ij} + \lambda_i^+ - \lambda_i^-, \ i = \overline{1, n}, \tag{24}$$

$$\lambda_i^+, \lambda_i^- \ge 0, \tag{25}$$

where $v_i = \ln y_i$, $z_{ij} = \ln x_{ij}$,

$$\lambda_i^+ = \begin{cases} v_i - c_0 - \sum_{j=1}^{l} \alpha_j z_{ij} - \sum_{j=1}^{l} \beta_j x_{ij} \\ \text{if } v_i - c_0 - \sum_{j=1}^{l} \alpha_j z_{ij} - \sum_{j=1}^{l} \beta_j x_{ij} > 0, \\ 0 \text{ otherwise,} \end{cases}$$

$$\lambda_i^- = \begin{cases} c_0 + \sum_{j=1}^{l} \alpha_j z_{ij} + \sum_{j=1}^{l} \beta_j x_{ij} - v_i \\ \text{if } c_0 + \sum_{j=1}^{l} \alpha_j z_{ij} + \sum_{j=1}^{l} \beta_j x_{ij} - v_i > 0, \\ 0 \text{ otherwise.} \end{cases}$$

Then one of the following problems should be solved depending on the strategy to construct the MPER:

• for strategy 1, the LP problem with the objective function (23) and the linear constraints (24) and (25);

• for strategy 2, the partially Boolean linear programming (PBLP) problem with the objective function (23) and the linear constraints (24), (25), and (10)–(12);

• for strategy 3, the PBLP problem with the objective function (23) and the linear constraints (24), (25), (10), (11), (13), and (14).

The problem of constructing the LLR is formalized by analogy. The LAD estimates of the LLR (17) are found by solving the LP problem

$$\theta_i^+ + \theta_i^- \to \min, \tag{26}$$

$$y_i = \gamma_0 + \sum_{j=1}^{l} \gamma_j x_{ij} + \sum_{j=1}^{l} \delta_j z_{ij} + \theta_i^+ - \theta_i^-, \ i = \overline{1, n}, \tag{27}$$

$$\theta_i^+, \ \theta_i^- \ge 0, \tag{28}$$

where $z_{ij} = \ln x_{ij}$,

$$\theta_i^+ = \begin{cases} y_i - \gamma_0 - \sum_{j=1}^{l} \gamma_j x_{ij} - \sum_{j=1}^{l} \delta_j z_{ij} \\ \text{if } y_i - \gamma_0 - \sum_{j=1}^{l} \gamma_j x_{ij} - \sum_{j=1}^{l} \delta_j z_{ij} > 0, \\ 0 \text{ otherwise,} \end{cases}$$

$$\theta_i^- = \begin{cases} \gamma_0 + \sum_{j=1}^{l} \gamma_j x_{ij} + \sum_{j=1}^{l} \delta_j z_{ij} - y_i \\ \text{if } \gamma_0 + \sum_{j=1}^{l} \gamma_j x_{ij} + \sum_{j=1}^{l} \delta_j z_{ij} - y_i > 0, \\ 0 \text{ otherwise.} \end{cases}$$

Then one of the following problems should be solved depending on the strategy to construct the LLR:

- for strategy 1, the LP problem with the objective function (26) and the linear constraints (27) and (28);
- for strategy 2, the PBLP problem with the objective function (26) and the linear constraints (27), (28), and (18)–(20);
- for strategy 3, the PBLP problem with the objective function (26) and the linear constraints (27), (28), (18), (19), (21), and (22).

4. MODELING RAIL FREIGHT TRAFFIC IN IRKUTSK OBLAST

Nowadays, a topical problem is to model rail freight traffic; for example, see [19, 20]. To demonstrate the mathematical apparatus proposed above, we considered this problem for Irkutsk oblast. Models were constructed based on the annual data of the Federal State Statistics Service for 2000–2018, available at the official website, with the following indicators:

– freight forward by public railway transport, y (million tons);

– labor force, x_3 (thousand people);

– gross regional product, x_{14} (million rubles);

– the number of enterprises and organizations, x_{18};

– industrial output (million rubles);

– electricity production, x_{22} (billion kWh);

– the average annual nominal wage in the mining industry, x_{23} (rubles);

– the average annual nominal wage in the manufacturing industry, x_{24} (rubles);

– agricultural output, x_{25} (million rubles);

– the average annual nominal wage in agriculture, hunting, and forestry (rubles);

– the number of active construction organizations;

– retail trade turnover, x_{31} (million rubles).

A special script was written in the hansl language of Gretl (an open-source statistical package for econometrics) to construct the MPER and LLR.

First, the MPER was constructed based on the initial data using strategy 3. The problem was solved by enumeration, and the estimates were obtained by the least squares method with m = 3 features. The complete enumeration of $C_{11}^3 \cdot 2^3 = 1320$ alternatives yielded the best one in terms of the coefficient of determination R^2. The resulting regression has the prologarithmic form

$$\begin{gathered}\ln \tilde{y} = -1.2502 + \underset{(2.972)}{8.431 \cdot 10^{-6}}\, x_{23} - \\ -\underset{(-6.098)}{3.388 \cdot 10^{-5}}\, x_{25} + \underset{(11.13)}{0.5176} \ln x_{31}\end{gathered}, \qquad (29)$$

and the coefficient of determination is $R^2 = 0.9334$. In equation (29), the values of the Student's t-test are indicated under the coefficients of the explanatory variables. According to these values, all coefficients are significant for the significance level $\alpha = 0.05$.

Unfortunately, due to the multicollinearity effect, the coefficient at the variable x_{25} changed its sign. Therefore, an attempt to interpret equation (29) leads to an absurd conclusion: we should reduce agricultural output for increasing rail freight traffic. Hence, when enumerating the models, we should check whether the signs of the regression equation coefficients agree with the practical interpretation of the variables. If at least one coefficient does not agree with its interpretation, then such a model is eliminated from further consideration. This recommendation can be found in the monograph [6]. Therefore, the MPER was rebuilt: an expert group determined that all explanatory variables should affect y with the "+" sign. The script was modified and launched with the same settings. As it turned out, among the 1320 alternatives, only 64 ones match the practical interpretation. The best of them is the logarithmic form model

$$\begin{gathered}\ln \tilde{y} = -6.4889 + \underset{(2.812)}{0.00127}\, x_3 + \\ + \underset{(3.555)}{0.533} \ln x_{18} + \underset{(3.447)}{0.754} \ln x_{22}\end{gathered}, \qquad (30)$$

where all coefficients of the explanatory variables are significant and $R^2 = 0.7437$.

The MPER corresponding to equation (30) is

$$\tilde{y} = 0.00152 \cdot e^{0.00127 x_3} \cdot x_{18}^{0.533} \cdot x_{22}^{0.754}. \qquad (31)$$

The sum of the squared residuals for the model (31) is 229.598.

The model (31) is interpreted in the following way: with an increase in the labor force x_3 by 1 thousand people, the freight forward y raises by 0.127% on average; with an increase in the number of enterprises and organizations x_{18} by 1%, the freight forward y raises by 0.533% on average; with an increase in the electricity production x_{22} by 1%, the freight forward y raises by 0.754% on average.

Then, the LLR was constructed based on the initial data using strategy 3. The script settings were the same as for the MPER. The enumeration of the 1320 alternatives yielded the model

$$\begin{gathered}\tilde{y} = -267.173 + \underset{(2.55)}{0.000639}\, x_{24} - \\ - \underset{(-6.185)}{0.00193}\, x_{25} + \underset{(12.87)}{29.124} \ln x_{14},\end{gathered} \qquad (32)$$

where all coefficients at the features are significant and $R^2 = 0.9328$.

In the model (32), the coefficient at the variable x_{25} again does not match the practical interpretation of the problem. Therefore, this model was rebuilt by checking the signs of the coefficients. As it turned out, among the 1320 alternatives, only 64 ones match the practical interpretation. The best of them is the regression

$$\tilde{y} = -552.38 + \underset{(2.764)}{0.0746} x_3 + \\ + \underset{(3.489)}{31.1352} \ln x_{18} + \underset{(3.282)}{42.7013} \ln x_{22}, \quad (33)$$

where all coefficients of the explanatory variables are significant, $R^2 = 0.7312$, and the sum of the squared residuals is 233.236.

Clearly, in terms of the sum of the squared residuals, the LLR (33) is somewhat worse than the MPER (31). Note that the LLR (33) includes the same features as the MPER (31).

The model (33) is interpreted in the following way: with an increase in the labor force x_3 by 1 thousand people, the freight forward y raises by 0.0746 million tons on average; with an increase in the number of enterprises and organizations x_{18} by 1%, the freight forward y raises by 0.3113 million tons on average; with an increase in the electricity production x_{22} by 1%, the freight forward y raises by 0.427 million tons on average. In addition: if the number of enterprises and organizations x_{18} increases by e times, the freight forward y will grow by 31.1352 million tons on average; if the electricity production x_{22} increases by e times, the freight forward y will grow by 42.7013 million tons on average.

Thus, if the researcher needs to predict the freight forward y, he should apply the models (29) and (32). If the researcher is also interested in interpreting the effect of different features on y, he should choose the MPER (31) and LLR (33), approximately of the same quality but with different meanings.

CONCLUSIONS

This paper has introduced two new specifications for regression models: the multiplicative power-exponential regression (MPER) and the linear logarithmic regression (LLR). The issues of their estimation and practical interpretation have been considered. The main advantage of these specifications is that each regression coefficient, except the free term, can always be given some practical interpretation. The MPER and LLR specifications allow identifying and studying new nonlinear regularities of processes or objects. Generally speaking, these specifications increase the usefulness of regression analysis.

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