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TRACKING SYSTEM DESIGN FOR A SINGLE-LINK SENSORLESS MANIPULATOR UNDER NONSMOOTH DISTURBANCES[1]

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Abstract. The controlled plant is a single-link manipulator elastically jointed to a DC motor and operating under uncertainty and incomplete measurements. The problem is to design a discontinuous feedback control for tracking a given reference signal of the plant's angular position. The angular position and velocity of the manipulator are not available for measurements; the sensors are located only on the drive; parametric and exogenous disturbances affecting the manipulator are nonsmooth and cannot be directly suppressed by control applied to the actuator. Within the block approach, a decomposition procedure is developed to design a nonlinear local feedback control. This control ensures the controlled variable's invariance with respect to uncertainties unmatched with the control action. A state observer of reduced order is constructed to estimate the angular position and velocity of the manipulator required for feedback design. The state variables in this observer are estimated using the principle of restoring exogenous disturbances by their action on the controlled plant. With this principle, a dynamic model of exogenous disturbances is not needed. In both problems (control and observation), *S*-shaped bounded continuous local feedback laws are used (smooth (sigmoid) and nonsmooth (piecewise linear) local feedback, respectively). These local feedback laws suppress bounded disturbances acting with them through the same channel. The algorithms developed below do not require real-time identification of parametric and exogenous disturbances. However, they stabilize the observation and tracking errors with some accuracy. The effectiveness of the dynamic feedback is validated by the results of numerical simulation.

Keywords: electromechanical system, tracking, invariance, block approach, state observer of reduced order, *S*-shaped functions.

INTRODUCTION

This paper considers a simple electromechanical system: a single-link sensorless manipulator elastically jointed to a DC motor and operating under parametric and exogenous disturbances. The basic problem is to control the angular position of the manipulator: stabilize it at a given level or track an admissible reference signal. Despite the seeming simplicity, this plant has all attributes of a complex automatic control system. Namely, it is described by a fifth-order dynamic model with nonlinearity and uncertain parameters, has an incomplete set of sensors, and is affected by exogenous disturbances. At present, many efficient control algorithms have been developed for mechanical and electromechanical systems within various approaches; for example, see [1–5]. However, when solving such problems, a specific type of uncertainties (parametric uncertainties, or exogenous disturbances of a certain class, or incomplete measurements) is often considered. In many studies, the mathematical model consists of the mechanical system only (the dynamics of actuators are neglected), and the suppression of unmatched disturbances acting through different channels with control remains an open problem [6].

In the previous publication [7], we considered a single-link sensorless manipulator elastically jointed to a DC motor under the assumption that its reference

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signal, parametric and exogenous disturbances are smooth functions of time. As a result, the controlled plant model was written in the canonical input-output form in mixed variables (linear combinations of the state variables, exogenous actions, and their derivatives), and the observation and tracking problems were successfully solved based on the form. The unmeasured controlled variable necessary for feedback design in the mixed variables observer was estimated using an additional loop in the observation subsystem. However, different nonsmooth disturbances (shock loads and dry friction forces) [8] often act on a mechanical plant during operation. These disturbances cannot be differentiated and cannot be directly suppressed or compensated by applying control to the actuator. Below, we consider such nonsmooth and unmatched exogenous and parametric disturbances and piecewise discontinuous reference signals, which are an obstacle to applying typical control methods, particularly the feedback linearization method [9, 10]. The controlled plant and the problem statement are described in Section 1.

The block approach seems reasonable to design the tracking system under such conditions. According to this approach, the state variables are used as fictitious controls from a certain class of smooth functions [11, 12]. In this case, the disturbances are matched with the fictitious controls and can be suppressed with a given accuracy. The original system is reduced to another one with respect to the tracking error and the residuals between the real and generated fictitious controls (invariant local feedback laws), and the exogenous signals are not differentiated. The true discontinuous control applied to the actuator ensures the sequential convergence of the residuals to given neighborhoods of zero, thereby stabilizing the tracking error (achieving the goal of control). Stabilizing fictitious controls are constructed as smooth and bounded sigmoid functions to avoid, at the beginning of transients, the bursts (an overshoot of the state variables) inherent in systems with linear high-gain feedback laws [11, 13]. (They are traditionally used to suppress exogenous disturbances.) The paper [12] presented a local sigmoid feedback design procedure for a nonlinear single-channel plant under the assumption that the functions of the state variables on the right-hand sides of the differential system dynamics equations are bounded everywhere. The scientific novelty of this study consists in developing a block parametric design procedure for sigmoid fictitious controls for an almost linear fifth-order system with nonsmooth exogenous disturbances (whose derivatives have a discontinuity). The plant has the following peculiarity: from the theoretical point of view, the linear combinations of state variables in the system equations are not bounded. To achieve the goal of control, the values of internal variables in the control process must belong to given ranges. The design procedure of the basic control law considering these features of the plant is given in Section 2.

Section 3 solves the observation problem in the case where the angular position and velocity of the manipulator required for feedback design cannot be measured (e.g., due to an aggressive environment, vibration, etc. [14]) and the sensors are mounted on the actuator only. A reduced-order state observer is constructed. It estimates the state variables by restoring exogenous disturbances by their effect on the controlled plant without any dynamic model of the signals [15, 16]. According to this restoration principle, the variable is treated as an exogenous disturbance and estimated using a feedback law in the observer (a corrective action). Following this approach, we design a robust state observer without the system equations with uncertain parameters. Also, we develop a decomposition procedure for designing piecewise linear feedback laws to solve the observation problem with a given accuracy under the parametric and exogenous disturbances affecting the mechanical subsystem. Due to the *S*-shaped invariant feedback laws (smooth in tracking, and piecewise-smooth in observation), there is no need to identify the uncertain parameters and exogenous disturbances in the observation and control processes: it suffices to know their ranges. The controller's parameters require no retuning when uncertainties change arbitrarily within the admissible ranges. This conclusion is illustrated by the numerical simulation results in Section 4.

1. DESCRIPTION OF THE CONTROLLED PLANT. PROBLEM STATEMENT

Consider a single-link rigid manipulator with a rotating joint elastically connected to the shaft of a DC motor. Its mathematical model is described by the differential equations [7, 17]

$$\dot{x}_1 = x_2,\ \dot{x}_2 = -a_{21}x_1 - a_2\sin(x_1) + b_2x_3 + f(t), \quad (1)$$

$$\dot{x}_3 = x_4,\ \dot{x}_4 = -a_{41}x_1 - a_{43}x_3 - a_{44}x_4 + b_4x_5, \qquad \dot{x}_5 = -a_{54}x_4 - a_{55}x_5 + b_5u. \quad (2)$$

Equations (1) correspond to the manipulator dynamics whereas equations (2) to the dynamics of the DC motor with permanent magnets [6]. In addition, $a_{41} = -\ a_{43} < 0$, and the other design factors are positive:

$$b_2 = a_{21} = k_l / J_l,\ a_2 = \bar{m}\bar{g}h / J_l,\ a_{43} = k_l / J_m,$$
$$a_{44} = d / J_m,\ b_4 = k_m / J_m,$$
$$a_{54} = c / L,\ a_{55} = R / L,\ b_5 = 1 / L.$$

The variables $x = (x_1, ..., x_5)^{\mathrm{T}}$ and parameters of system (1), (2) are described in Table 1.

Table 1

The variables and parameters of the controlled plant

Notation	Description, measurement unit	Notation	Description, measurement unit
x_1	The angular position of manipulator's link, rad	$\bar{g}$	Acceleration of gravity, 9.8 m/s^2
x_2	The angular velocity of manipulator's link, rad/s	k_l	Gear rigidity, N·m/rad
x_3	The angular position of DC motor's shaft, rad	J_l	The manipulator's moment of inertia, kg·m^2
x_4	The angular velocity of DC motor's shaft, rad/s	k_m	Gain, N·m/A
x_5	The drive armature current of DC motor, A	J_m	The moment of inertia of DC motor, kg·m^2
$f(t)$	Uncontrolled disturbance, N/(kg·m)	D	Damping coefficient, kg·m^2/s
u	The drive armature voltage of DC motor, V	c	The counter emf coefficient of DC motor, V·s/rad
h	Manipulator's link length, m	L	The drive armature inductance of DC motor, H
$\bar{m}$	Manipulator's link mass, kg	R	The drive armature resistance of DC motor, Ω

In system (1), (2), the output controlled variable is the angular position $x_1(t)$ of the manipulator's link; the drive armature voltage u of the DC motor is a discontinuous control. The problem is to design a dynamic feedback control under which the output variable $x_1(t)$ will track a given admissible signal $g(t)$ under the following assumptions:

- The reference point $x_1(t) = 0$ is the low vertical position of the manipulator's link, which is stable; the maximum angular velocity of the manipulator's link is bounded:

$$|x_1(t)| \le \pi, \ |x_2(t)| \le X_2, \ t \ge 0, \ X_2 = \text{const} > 0. \quad (3)$$

- The initial values of the state variables belong to given ranges:

$$|x_i(0)| \le X_{i,0} = \text{const} > 0, i = \overline{1,5}. \quad (4)$$

- The sensors are located only on the actuator. The variables $x_1(t)$ and $x_2(t)$ are unmeasured, whereas the variables $x_3(t)$, $x_4(t)$, and $x_5(t)$ are measured without noise.
- The current values of the reference signal $g(t)$ are known; its derivative $\dot{g}(t)$ is assumed to be a nonsmooth unknown function of time, bounded by a given constant:

$$|g(t)| \le G_0 < \pi; \ |\dot{g}(t)| \le G_1, \quad t \ge 0; \ G_0, G_1 = \text{const} > 0. \quad (5)$$

- The values of the parameters k_l , J_m, d, and k_m (hence, a_{43}, a_{44}, and b_4) are known. The parameters $\bar{m}$, h, J_l, c, R, and L (hence, $b_2 = a_{21}$, a_2, a_{54}, a_{55}, and b_5) are uncertain but belong to given ranges:

$$a_{21,\min} \le a_{21}(t) \le a_{21,\max}, a_{2,\min} \le a_2(t) \le a_{2,\max};$$

$$a_{5j,\min} \le a_{5j}(t) \le a_{5j,\max}, \ j = 4,5;$$

$$b_{5,\min} \le b_5(t) \le b_{5,\max}, t \ge 0; \quad (6)$$

- The time-varying function $f(t)$ is unknown, nonsmooth, and bounded by a given constant:

$$|f(t)| \le F = \text{const} > 0, \ t \ge 0. \quad (7)$$

The feedback loop involves only an observer of the unmeasured state variables: identifiers of the unknown parameters and generators of the exogenous actions are not introduced. Under these conditions, the tracking error $e_1(t) = x_1(t) - g(t) \in R$ can be stabilized only with some accuracy. The goal of control is to ensure the condition

$$|e_1(t)| \le \Delta_1, \ t \ge t_1, \quad (8)$$

in the closed loop system, where $\Delta_1 > 0$ and $t_1 > 0$ are a given stabilization accuracy and a given time to reach it, respectively.

2. THE BASIC CONTROL LAW

First, we form a control law in system (1), (2) using a given reference signal $g(t)$ and all state variables. Then, we construct an observer to estimate the unmeasured state variables. To design the feedback, we apply the block control principle [11, 12].

System (1), (2) is a block controllability form [11]. This means that the true control appears additively with a nonzero factor only in the last equation; the right-hand side of each ith equation (block), $i = \overline{1,4}$, contains functions only of the state variables $x_1,..., x_i$, and the variable of the next $(i+1)$th equation appears additively with a nonzero factor. Due to such a form, the variable x_{i+1} in each ith equation can be treated as a fictitious control, and the local feedback laws can be sequentially obtained in each equation (top-to-bottom). In the last block, the local feedback laws are provided by the true control. Since the first equation is written in the controlled variable, system (1), (2) can be considered a triangular input-output form (by the composition of the arguments of functions in each equation except fictitious controls). Therefore, we solve the tracking problem based on this form.

To suppress parametric and exogenous disturbances on the same channels with fictitious controls, we construct local feedback laws as bounded *S*-shaped sigmoid functions with two tuned parameters [12]:

$$x_i^* = -m_{i-1}\sigma(k_{i-1}e_{i-1}),\ k_{i-1}, m_{i-1} = \text{const} > 0,\ i = \overline{2,5},\quad (9)$$

where $\sigma(k_{i-1}e_{i-1}) = 2/(1+\exp(-k_{i-1}e_{i-1})) - 1$ is an odd and bounded sigmoid function, $|\sigma(k_{i-1}e_{i-1})| < 1$, and $e_i \in R$ ($i = \overline{2,5}$) are the residuals between the variables x_i and the desired fictitious controls x_i^* (9):

$$e_i = x_i - x_i^* = x_i + m_{i-1}\sigma(k_{i-1}e_{i-1}),\ i = \overline{2,5}.\quad (10)$$

The true control (the drive armature voltage of the DC motor) is naturally taken [6] as the discontinuous function

$$u = -m_5 \text{sign}(e_5),\ m_5 = \text{const} > 0,$$
$$\text{sign}(e_5) = \begin{cases} +1, & e_5 > 0, \\ -1, & e_5 < 0. \end{cases}\quad (11)$$

For $e_5 = 0$, the value of the sign function is undefined but restricted to the interval $[-1, 1]$. The closed loop system (1), (2), (11), written in the tracking error and the residuals (10), has the form

$$\begin{aligned} &\dot{e}_1 = e_2 - m_1\sigma(k_1e_1) - \dot{g}, \\ &\dot{e}_i = b_i(e_{i+1} - m_i\sigma(k_ie_i)) - \\ &\sum_{j=1}^{i} a_{ij}e_j + f_i + \Lambda_{i-1}, i = 2,3,4, \\ &\dot{e}_5 = -a_{54}e_4 - a_{55}e_5 + f_5 + \Lambda_4 - b_5m_5\text{sign}(e_5), \end{aligned}\quad (12)$$

where $b_3 = 1$, the elements a_{ij} figuring in formula (12) but absent in system (1), (2), are zero, and Λ_{i-1} are the full derivatives of the fictitious controls (9):

$$\begin{aligned} &\Lambda_{i-1} = \tfrac{d}{dt} m_{i-1}\sigma(k_{i-1}e_{i-1}) = \\ &0{,}5m_{i-1}k_{i-1}(1-\sigma^2(k_{i-1}e_{i-1}))\dot{e}_{i-1},\ i = \overline{2,\ 5}; \\ &f_2 = -a_{21}g - a_2\sin(e_1 + g) + f(t),\ f_3 = 0, \\ &f_4 = a_{43}(m_2\sigma(k_2e_2) + g) + a_{44}m_3\sigma(k_3e_3), \\ &f_5 = a_{54}m_3\sigma(k_3e_3) + a_{55}m_4\sigma(k_4e_4). \end{aligned}\quad (13)$$

Due to formulas (5)–(7), the values of f_i are bounded, and their estimates depend on the amplitudes of fictitious controls:

$$\begin{aligned} &|f_2(t)| \le a_{21,\max}G_0 + a_{2,\max} + F = F_2, \\ &|f_4(t)| \le a_{43}(m_2 + G_0) + a_{44}m_3 = F_4, \\ &|f_5(t)| \le a_{54,\max}m_3 + a_{55,\max}m_4 = F_5. \end{aligned}\quad (14)$$

The original problem (8) is reduced to the stabilization of the closed loop system (12). In this case, the control design for a single-channel system of the fifth order is decomposed into five elementary design subproblems solved sequentially: choosing the parameters of true and fictitious controls that ensure invariance with respect to the existing uncertainties with a given accuracy. The amplitude of the discontinuous control (11) is chosen to ensure the occurrence of a sliding mode on the surface $e_5 = 0$ in system (12) in a finite time $0 < t_5 < t_1$. According to the block control principle, the parameters of fictitious controls (9) are chosen to ensure the sequential convergence of the residuals to some neighborhoods of zero:

$$\begin{aligned} &|e_5(t)| \le \Delta_5, t \ge t_5 > 0 \Rightarrow |e_4(t)| \le \Delta_4, \\ &t \ge t_4 > t_5 \Rightarrow ... \Rightarrow |e_1(t)| \le \Delta_1,\ t \ge t_1 > t_2, \end{aligned}\quad (15)$$

where Δ_1 and t_1 are given by (8), and Δ_i = const > 0, $i = \overline{2,5}$, are assigned arbitrarily. The first inequality in (15) reflects the following fact: due to various imperfections, the sliding mode in real systems occurs in some boundary layer of the switching surface [6].

The sigmoid function can be estimated from below by the piecewise linear function

$$\begin{aligned} &0.8|\text{sat}(k_ie_i)| \le |\sigma(k_ie_i)| < 1, \\ &\text{sat}(k_ie_i) = \begin{cases} \text{sign}(e_i), & |e_i| > 2.2/k_i, \\ k_ie_i/2.2, & |e_i| \le 2.2/k_i,\ i = \overline{1,\ 4}, \end{cases} \end{aligned}\quad (16)$$

where $\sigma(\pm 2.2) \approx \pm 0.8$ and $e_i = \pm 2.2/k_i$ are the points separating $\sigma(k_ie_i)$ into almost linear and almost constant functions [12]. The choice of the value k_i (the gain in the argument of the sigmoid function) determines the stabilization accuracy of the corresponding residual. We fix an inversely proportional relation between the gains and the stabilization accuracy of the residuals: $|e_i| \le 2.2/k_i = \Delta_i, i = \overline{1,4}.$ Under the gains

$$k_i \ge 2.2/\Delta_i, i = \overline{1,4},\quad (17)$$

yielding the desired stabilization accuracy, the design problem is reduced to choosing the amplitudes m_i, $i = \overline{1,\ 5}$, that ensure the sequential convergence of the residuals to the corresponding neighborhoods of zero (15). Sufficient conditions for $|e_i| \le \Delta_i$ have the form $e_i\dot{e}_i < 0$ for $|e_i| > \Delta_i, i = \overline{1,5}$ [6, 12]. From these inequalities we obtain a lower estimate for the amplitude in the *i*th block ($i = \overline{1,4}$) provided that in all subsequent blocks $j = i + 1, i + 2, ..., 5$, the residuals have already converged to the given neighborhoods of zero $|e_j| \le \Delta_j$ (15). Given formulas (5), (14)–(17), we have:

$$
\begin{gathered}
0.8m_1 > G_1 + \Delta_2 \Rightarrow e_1\dot{e}_1 = e_1(e_2 - m_1\sigma(k_1e_1) - \dot{g}) \le \\
|e_1|(\Delta_2 + G_1 - 0.8m_1) < 0, \\
0.8b_{i,\min}m_i > b_{i,\min}\Delta_{i+1} + \sum_{j=1}^{i-1} a_{ij,\max}|e_j| + \\
F_i + |\Lambda_{i-1}| \Rightarrow e_i\dot{e}_i = e_i(b_i(e_{i+1} - m_i\sigma(k_ie_i)) - \\
\sum_{j=1}^{i} a_{ij}e_j + f_i + \Lambda_{i-1}) \le \\
|e_i|(b_{i,\min}(\Delta_{i+1} - 0.8m_i) - a_{ij,\min}|e_i| + \\
\sum_{j=1}^{i-1} a_{ij,\max}|e_j| + F_i + |\Lambda_{i-1}|) < 0,\ i = 2,\ 3,\ 4; \\
b_{5,\min}m_5 > a_{54,\max}|e_4| + F_5 + |\Lambda_4| \Rightarrow e_5\dot{e}_5 = \\
e_5(-a_{54}e_4 - a_{55}e_5 + f_5 + \Lambda_4 - b_5m_5\mathrm{sign}(e_5)) \le \\
|e_5|(a_{54,\max}|e_4| - a_{55,\min}|e_5| F_5 + |\Lambda_4| - b_{5,\min}m_5) < 0.
\end{gathered}
\tag{18}
$$

To implement the sufficient conditions (18), we estimate the ranges of the variables of system (12) and their derivatives in the control process with a given time to stabilize the tracking error (8). The corresponding procedure and the resulting inequalities for choosing the amplitudes m_i ($i = \overline{1,5}$) are presented in the Appendix. The procedure rests on conservative estimates and proves the existence of solutions of inequalities (18). The values of the gains (17) and amplitudes can be decreased based on simulation results.

For the system with an incomplete set of sensors, the control law (11) involves the measured variables $g(t)$, $x_3(t)$, $x_4(t)$, and $x_5(t)$ together with the estimates $\hat{x}_1(t)$ and $\hat{x}_2(t)$ of the unmeasured state variables $x_1(t)$ and $x_2(t)$. Section 3 considers the design problem of a state observer that ensures a given accuracy and a given estimation time under the parametric uncertainty of the controlled plant:

$$
|x_i(t) - \hat{x}_i(t)| \le \delta_i,\ i = 1,2,\ t \ge T,\ 0 < T < t_5. \tag{19}
$$

In the dynamic feedback system, the control law (11) takes the form

$$
u = -m_5\mathrm{sign}(\hat{e}_5), \tag{20}
$$

where the tracking error and the residuals (10) are constructed by the measured and estimated signals:

$$
\hat{e}_1 = \hat{x}_1 - g,\ \hat{e}_2 = \hat{x}_2 + m_1\sigma(k_1\hat{e}_1),\ \hat{e}_3 = x_3 + m_2\sigma(k_2\hat{e}_2),
$$
$$
\hat{e}_4 = x_4 + m_3\sigma(k_3\hat{e}_3),\ \hat{e}_5 = x_5 + m_4\sigma(k_4\hat{e}_4).
$$

In the closed loop system (1), (2) with the dynamic feedback (20), the estimation errors (19) act as imperfections. As a result, the boundary layer

$$
|\hat{e}_5 - e_5| = m_4|\sigma(k_4\hat{e}_4) - \sigma(k_4e_4)| \le \Delta_5. \tag{21}
$$

appears in the sliding mode. Due to (21) and the *S*-shaped form of the sigmoid function, the greatest deviation is achieved in the vicinity of zero, and the estimation errors have an almost negligible effect at infinity. Using the first approximation $\sigma(kx) \underset{x\to 0}{\sim} 0.5kx$, we estimate the deviation (21) as follows:

$$
\begin{gathered}
|\sigma(k_4\hat{e}_4) - \sigma(k_4e_4)| \approx 0.5k_4|\hat{e}_4 - e_4| \approx \\
0.5k_4m_3|\sigma(k_3\hat{e}_3) - \sigma(k_3e_3)| \approx 0.5^2k_4m_3k_3|\hat{e}_3 - e_3| \approx \\
0.5^2k_4m_3k_3m_2|\sigma(k_2\hat{e}_2) - \sigma(k_2e_2)| \approx \\
0.5^3k_4m_3k_3m_2k_2|\hat{e}_2 - e_2| \approx \\
0.5^3k_4m_3k_3m_2k_2|\delta_2 + m_1|\sigma(k_1\hat{e}_1) - \sigma(k_1e_1)|| \approx \\
0.5^3k_4m_3k_3m_2k_2(\delta_2 + 0.5m_1k_1\delta_1).
\end{gathered}
$$

With the accepted values of the controller parameters and the boundary layer Δ_5, we finally arrive at the following constraint on the estimation errors in the observation problem:

$$
\delta_2 + 0.5m_1k_1\delta_1 \le \frac{8\Delta_5}{m_4k_4m_3k_3m_2k_2}. \tag{22}
$$

3. A STATE OBSERVER TO ESTIMATE THE VARIABLES OF THE MECHANICAL SUBSYSTEM

System (1), (2) is observable with respect to the measurements $x_3(t)$, $x_4(t)$, and $x_5(t)$, but the full-order state observer cannot be designed due to the parametric uncertainties in the model and the exogenous disturbances affecting the plant. To restore the values of the unmeasured variables $x_1(t)$ and $x_2(t)$, we will use the procedure for estimating exogenous disturbances without a dynamic disturbance generator [7, 15, 16] and construct a reduced-order observer. In this case, the desired estimates $\hat{x}_1$ and $\hat{x}_2$ can be obtained only with the given accuracy (19), (22).

According to this procedure, we construct a reduced-order observer based on the fourth and first equations of the original system (1), (2), which do not depend on the uncertain parameters and include the unmeasured variables with nonzero factors:

$$
\dot{x}_4 = -a_{43}(x_3 - x_1) - a_{44}x_4 + a_{45}x_5,\ \dot{x}_1 = x_2. \tag{23}
$$

The reduced-order observer is constructed for system (23) with the measured signals

$$
\dot{z}_1 = -a_{43}(x_3 - z_2) - a_{44}x_4 + a_{45}x_5 + v_1,\ \dot{z}_2 = v_2, \tag{24}
$$

where z_1 and z_2 are the observer's state variables, and v_1 and v_2 are its corrective actions.

We introduce the observation errors $\varepsilon_1 = x_4 - z_1$ and $\varepsilon_2 = x_1 - z_2$. Considering formulas (23) and (24), we obtain the system

$$
\dot{\varepsilon}_1 = a_{43}\varepsilon_2 - v_1,\ \dot{\varepsilon}_2 = x_2 - v_2. \tag{25}
$$

Due to the available measurements of the parameter x_4, the current errors $\varepsilon_1(t)$ are known, whereas the errors $\varepsilon_2(t)$ are not. Let us assign the following initial conditions for the observer (24) and, accordingly, for the virtual system (25):

$$z_1(0) = x_4(0) \Rightarrow \varepsilon_1(0) = 0;$$
$$z_2(0) = 0 \Rightarrow \varepsilon_2(0) = x_1(0), \ |\varepsilon_2(0)| \le \pi. \qquad (26)$$

The goal is to form corrective actions v_1 and v_2 to stabilize the observation errors and their derivatives with the given accuracy and in the given time (19), (22). We use piecewise linear corrective actions [7, 15, 16]:

$$v_1 = p_1 \text{sat}(l_1\varepsilon_1), \ v_2 = p_2 \text{sat}(l_2 v_1), \ l_i, \ p_i = \text{const} > 0,$$
$$\text{sat}(l_1\varepsilon_1) = \begin{cases} \text{sign}(\varepsilon_1), & |\varepsilon_1| > 1/l_1, \\ l_1\varepsilon_1, & |\varepsilon_1| \le 1/l_1. \end{cases} \qquad (27)$$

Like the sigmoid function, the functions (27) are *S*-shaped and have two tunable parameters. They are easier to implement but nonsmooth. However, the latter property is not critical: in the observation problem, the corrective actions have no physical sense and may be nonsmooth. (In contrast, smoothness is required for fictitious controls—the state variables (10).)

Lemma. *Let the external signal* $x_2(t)$ *in system* (25)–(27) *be bounded by condition* (3). *Then for any* $\delta_1 > 0$, $\delta_2 > 0$, *and* $T > 0$, *there exist positive real numbers* $\bar{p}_i$ *and* $\bar{l}_i$ *such that*

$$|\varepsilon_2(t)| = |x_1(t) - z_2(t)| \le \delta_1,$$
$$|x_2(t) - v_2(t)| \le \delta_2, \ t \ge T, \qquad (28)$$

for any $p_i > \bar{p}_i$ and $l_i \ge \bar{l}_i$, $i = 1, 2$.

The proof of this lemma is given in the Appendix. It follows from (19) and (28) that the state variable and the corrective action of the second equation of the reduced-order observer (24) are the desired estimates of the unmeasured variables: $\hat{x}_1(t) = z_2(t)$, $\hat{x}_2(t) = v_2(t)$.

4. SIMULATION RESULTS

To check the effectiveness of this dynamic feedback design method, we simulated the closed loop system (1), (2), (20), (24) with the initial conditions $x_i(0) = 0, \ i = \overline{1, 5}$, in MATLAB-Simulink. The goal of control was to ensure condition (8) with

$$\Delta_1 = 0.04 \text{ rad}, \ t_1 = 2 \text{ s}. \qquad (29)$$

The following values and ranges of the model parameters and exogenous actions were selected:

$$k_l = 0.2, \ J_m = 0.01, \ d = 0.045, \ k_m = 0.3;$$
$$\bar{m} \in [0.18, \ 0.25], \ h \in [0.2, \ 0.3],$$
$$J_l \in [0.0072, \ 0.0225], \ c \in [0.25, \ 0.33], \qquad (30)$$
$$R \in [3.8, \ 4.2], \ L \in [0.006, \ 0.013];$$
$$|g^{(i)}(t)| \le 0.2, \ i = \overline{0,1}; \ |f(t)| \le 0.1.$$

Based on inequalities (17) and (A.9)–(A.13) (see the Appendix), we took the following gains:

$$k_1 = 80, \ k_2 = 25, \ k_3 = 5, \ k_4 = 8;$$
$$m_1 = 0.3, \ m_2 = 0.7, \ m_3 = 10, \ m_4 = 40, \ m_5 = 90. \qquad (31)$$

Due to the expressions (19) and (22), the target values for the observation problem were determined as $\delta_1 = 0.0008$ rad, $\delta_2 = 0.002$ rad/s, and $T = 0.1$ s. The following corrective coefficients (27) were adopted in the observer (24) based on inequalities (A.22) and (A.26) with the constraint $|x_2(t)| \le 5 = X_2$ rad/s:

$$l_1 = 155, \ l_2 = 150; \ p_1 = 60, \ p_2 = 40. \qquad (32)$$

Two numerical experiments were performed with the same controller (31) and observer (32) gains but different plant's parameters and exogenous actions from the ranges (30). In experiment 1, the left limits of the ranges (30) and the exogenous actions $g(t) = 0.05|\sin t| + 0.15\cos(0.5t)$ and $f = 0.05$ were taken as the plant's parameters. In experiment 2, the right limits of the ranges (30) and the exogenous actions $g(t) = 0.18|\cos(t)|$ and $f(t) = 0.05t, \ t \in [0, 2)$ (the sawtooth function with the main period of 2 s) were taken as the plant's parameters. Note that in both experiments, the reference signals are piecewise smooth functions whose derivatives have discontinuities of the first kind. For comparison, we also simulated the system with the static feedback law (11) under the assumption that all state variables are measurable and the system with the dynamic feedback law (20) with the unmeasured variables $\hat{x}_1(t)$ and $\hat{x}_2(t)$ estimated using the observer (24), (27). The integration was carried out by the Euler method with a constant step of 10^{-5}.

Figures 1–4 show the simulation results of experiment 1. For system (1), (2) with the dynamic feedback law (20), (24), (27), the following graphs are presented: the reference signal $g(t)$ and the manipulator's angular position $x_1(t)$ (Fig. 1); the tracking error $e_{1d}(t) = x_1(t) - g(t)$ (Fig. 2); the estimation errors $\alpha_1(t) = x_1(t) - \hat{x}_1(t)$ and $\alpha_2(t) = x_2(t) - \hat{x}_2(t)$, i.e., the deviations of the estimates $\hat{x}_1(t) = z_2(t)$ and $\hat{x}_2(t) = v_2(t)$ yielded by the observer (24), (27) from the variables $x_1(t)$ and $x_2(t)$ (Fig. 3); $e_{1d}(t) - e_{1s}(t)$, i.e., the deviation of $e_{1d}(t)$ from the tracking error $e_{1s}(t)$ in the system with the static feedback law (11). Figures 5–8 show similar graphs for experiment 2.

Table 2 provides several control performance indices in both experiments: the settling time t^*: $|e_1(t)| \le 0.04$, $t \ge t^*$; the overshoot $e_{1\max} \ge |e_1(t)|$, $t \ge 0$; the tracking accuracy $\bar{\Delta}_1 \ge |x_1(t) - g(t)|$ in the steady-state mode for $t \ge 10$ s.

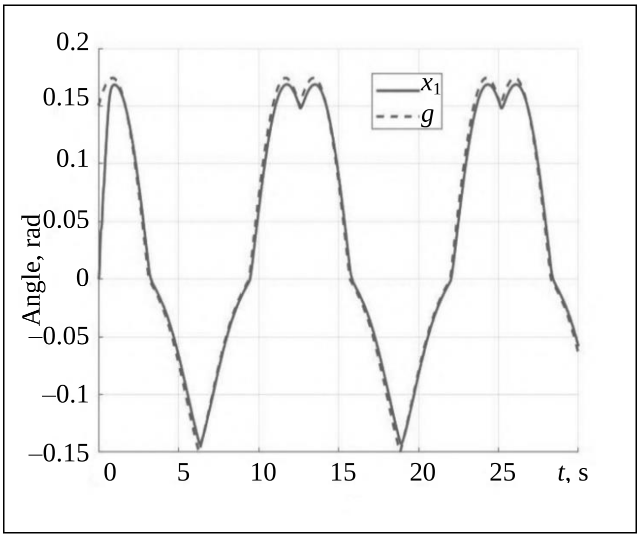

Fig. 1. The graphs of g(t) and x1(t) in experiment 1.

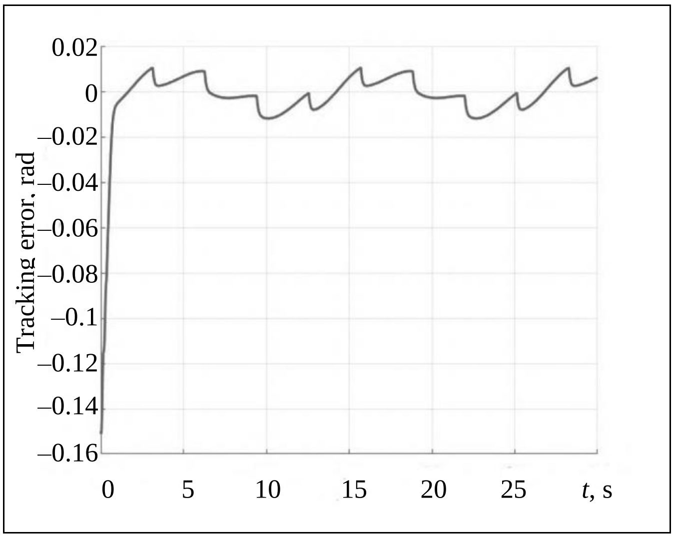

Fig. 2. The graph of tracking error $e_{1d}(t)$ in experiment 2.

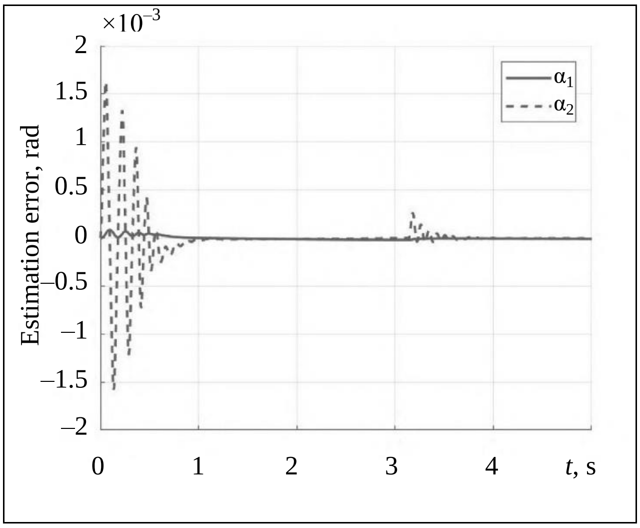

Fig. 3. The graphs of estimation errors $\alpha_1(t) = x_1(t) - \hat{x}_1(t)$ and $\alpha_2(t) = x_2(t) - \hat{x}_2(t)$ in experiment 1.

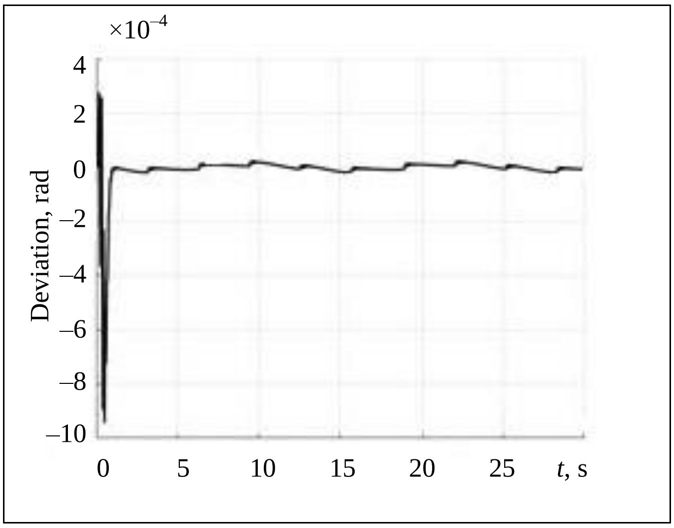

Fig. 4. The graph of deviation $e_{1d}(t) - e_{1s}(t)$ in experiment 1.

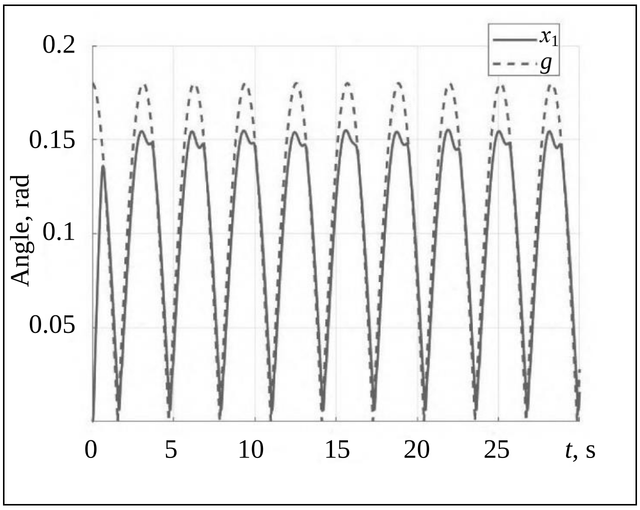

Fig. 5. The graphs of $g(t)$ and $x_1(t)$ in experiment 2.

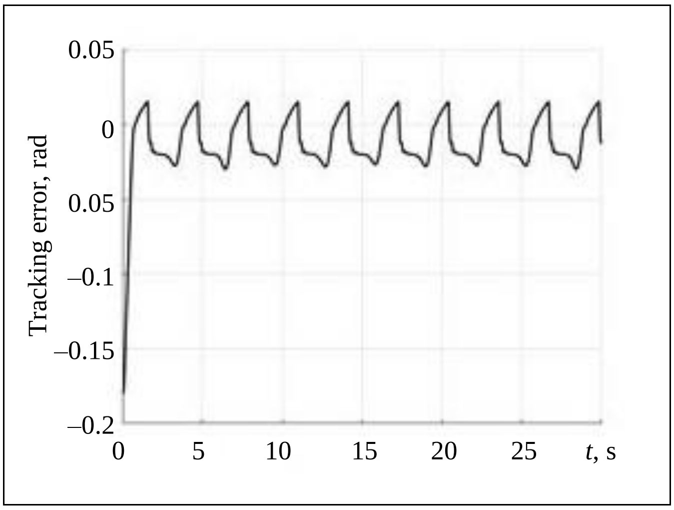

Fig. 6. The graph of tracking error $e_{1d}(t)$ in experiment 2.

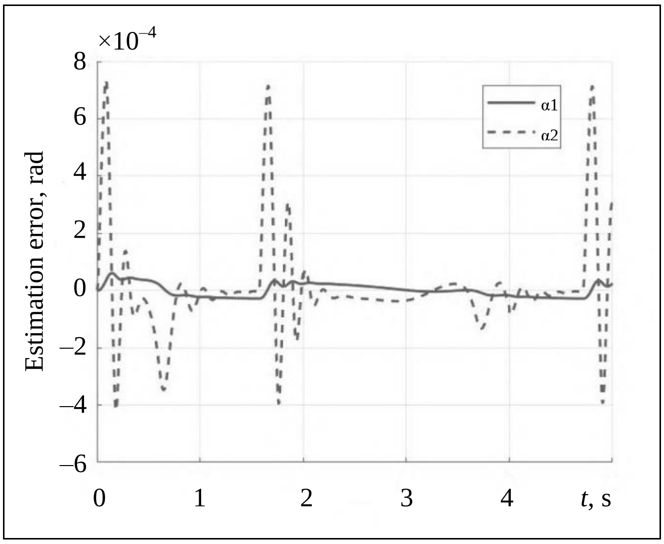

Fig. 7. The graphs of estimation errors $\alpha_1(t) = x_1(t) - \hat{x}_1(t)$ and $\alpha_2(t) = x_2(t) - \hat{x}_2(t)$ in experiment 2.

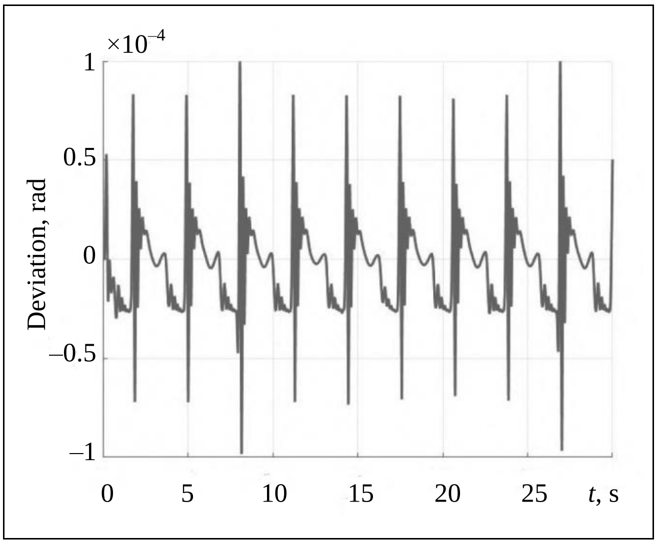

Fig. 8. The graph of deviation $e_{1d}(t)$ - $e_{1s}(t)$ in experiment 2.

Table 2

Control performance indices

Index, measurement unit	Static feedback (11)	Dynamic feedback (20), (24), (27)
Experiment 1		
t^*, s	0.5380	0.5408
$e_{1\max}$, rad	0.1510	0.1510
$\bar{\Delta}_1$, rad	0.0119	0.0119
Experiment 2		
t^*, s	0.5146	0.5147
$e_{1\max}$, rad	0.18	0.18
$\bar{\Delta}_1$, rad	0.0299	0.0299

According to Figs. 1–8 and Table 2, the target values of the performance indices (29) were achieved in all experiments. Introducing the reduced-order observer (24), (27) into the feedback loop caused no significant deterioration of the system performance.

CONCLUSIONS

This paper has considered the following problem: the angular position of a single-link manipulator should track a given reference signal under nonsmooth exogenous disturbances in the case where the sensors are mounted only on the actuator. This problem has been solved by applying a block feedback design procedure with sigmoid fictitious controls tuned for the worst-case admissible values of uncertain parameters and exogenous disturbances. A state observer of reduced order has been constructed to estimate the angular position and velocity of the manipulator. This observer needs no precise knowledge of the parameters of the mechanical subsystem. The results of numerical simulation have confirmed the effectiveness of the developed tracking and observation systems. As shown, the performance indices of the closed loop system with the dynamic feedback based on the reduced-order observer have comparable values with those of the system with full measurements.

Further research will aim at extending the block design procedure of sigmoid fictitious controls to linear plants with several inputs and outputs. Also, the performance of the reduced-order observer will be studied under noisy measurements.

APPENDIX

P r o c e d u r e (choosing the amplitudes of the true (11) and fictitious (9) controls). We begin with estimating the state variables of system (12). Due to formulas (3)-(5) and (10), we have the following estimates for the initial values:

$$|e_1(0)| \le X_{1,0} + G_0, |e_i(0)| \le X_{i,0} + m_{i-1}, i = \overline{2,5}. \quad \text{(A.1)}$$

In the general case $|e_i(0)| > \Delta_i, i = \overline{1,5},$ the monotonic transients are guaranteed in system (12) only for the last block variable $e_5(t)$. In the worst case, the variables $e_i(t)$, $i = \overline{1,4}$, will grow by absolute value until all variables $e_j(t)$, $j = 5,4,\ldots,i+1$, reach the given neighborhoods (15), i.e.,

$$|e_5(t)| \le |e_5(0)| = e_{5,\max}, |e_i(t)| \le |e_i(t_{i+1})| = e_{i,\max}, i = \overline{4,1}. \quad \text{(A.2)}$$

In system (1), (2) and, consequently, in system (12), we have $a_{ii} = 0, i = \overline{1,3}$, and $a_{44} > 0$. Due to the expressions

(18), (A.1), and (A.2), the state variables and their derivatives can be estimated as

$$\begin{gathered} e_{1,\max} \le X_{1,0} + G_0 + (e_{2,\max} - \Delta_2)t_2, \\ e_{i,\max} \le X_{i,0} + m_{i-1} + b_{i,\max}(e_{i+1,\max} - \Delta_{i+1})t_{i+1}, i = 2,3, \\ e_{4,\max} \le X_{4,0} + m_3 + b_4(e_{5,\max} - \Delta_5)(1-\exp(-a_{44}t_5))/a_{44}, \\ e_{5,\max} \le X_{5,0} + m_4; \end{gathered} \quad \text{(A.3)}$$

$$\begin{gathered} |\dot{e}_i(t)| < b_{i,\max}(2m_i + e_{i+1,\max} - \Delta_{i+1}), t \in [0, t_{i+1}), \\ |\dot{e}_i(t)| < 2b_{i,\max}m_i,\ t \ge t_{i+1},\ i = \overline{1,3},\ b_{1,\max} = b_{3,\max} = 1; \\ |\dot{e}_4(t)| < b_4(2m_4 + e_{5,\max} - \Delta_5) + a_{44}e_{4,\max},\ t \in [0,\ t_5), \\ |\dot{e}_4(t)| < 2b_4m_4 + a_{44}e_{4,\max},\ t \in [t_5,\ t_4), \\ |\dot{e}_4(t)| < 2b_4m_4 + a_{44}\Delta_4,\ t \ge t_4. \end{gathered} \quad \text{(A.4)}$$

Considering the separation (16), the derivative of the sigmoid function $\sigma'(k_ie_i) = 0.5k_i(1-\sigma^2(k_ie_i))$ satisfies the inequalities

$$\begin{gathered} 0 < \sigma'(k_ie_i) < 0.18k_i,\ t \in [0, t_i),\ |e_i(t)| > 2.2/k_i, \\ 0.18k_i \le \sigma'(k_ie_i) \le 0.5k_i,\ t \ge t_i,\ |e_i(t)| \le 2.2/k_i, \\ i = \overline{1,4}. \end{gathered} \quad \text{(A.5)}$$

Let us restrict the maximum absolute values of the time-dependent residuals (A.3) by

$$e_{1,\max} \le 2\pi, e_{i+1,\max} \le 3m_i + \Delta_{i+1}, i = \overline{1,3}. \quad \text{(A.6)}$$

Then the expressions (A.3) take the form

$$\begin{gathered} e_{1,\max} \le X_{1,0} + G_0 + 3m_1t_2 \le 2\pi, \\ e_{i,\max} \le X_{i,0} + m_{i-1} + 3b_{i,\max}m_it_{i+1} \le 3m_{i-1} + \Delta_i,\ i = 2,\ 3, \\ e_{4,\max} \le X_{4,0} + m_3 + b_4(X_{5,0} + m_4 - \Delta_5)\times \\ (1-\exp(-a_{44}t_5))/a_{44} \le 3m_3 + \Delta_4. \end{gathered} \quad \text{(A.7)}$$

Using formulas (A.4)–(A.7), we obtain the following estimates for the derivatives of fictitious controls (13):

$$\begin{gathered} |\Lambda_i(t)| = m_i\,\frac{k_i(1-\sigma^2(k_ie_i))}{2}|\dot{e}_i| \le k_im_i^2b_{i,\max}, i = \overline{1,3}; \\ |\Lambda_4(t)| \le k_4m_4^2b_4 + 0.5k_4m_4a_{44}e_{4,\max}, t \ge 0. \end{gathered} \quad \text{(A.8)}$$

To satisfy inequalities (A.6), the amplitudes m_i, $i = \overline{1,4}$, must be bounded from above. Assume for convenience that $0 < \Delta_i = X_{i,0}$, $i = \overline{2,5}$. Then the right-hand sides of inequalities (A.7) yield

$$\begin{gathered} m_1 \le m_{1,\max} = \frac{2\pi - X_{1,0} + G_0}{3t_2},\ m_i \le m_{i,\max} = \frac{2m_{i-1}}{3b_{i,\max}t_{i+1}}, \\ i = 2,\ 3,\ m_4 \le m_{4,\max} = \frac{2m_3a_{44}}{b_4(1-\exp(-a_{44}t_5))}. \end{gathered} \quad \text{(A.9)}$$

Obviously, the upper bounds on the amplitudes (A.9) can be made arbitrarily large by decreasing t_i, $i = \overline{2,5}$, within the hierarchy $0 < t_5 < t_4 < ... < t_2 < t_1$.

Now we formalize the sequential choice procedure for the amplitudes m_i, $i = \overline{1,5}$, to satisfy conditions (15) in the closed loop system (12). (As a result, the goal of control (8) will be achieved under the given values Δ_1, t_1, $\Delta_i = X_{i,0}$, $i = \overline{2,5}$, and the corresponding gains (17)). The amplitudes should be chosen to ensure the convergence of the residuals $e_i(t)$, $i = \overline{5,1}$, on the intervals $[t_{i+1}, t_i]$, $t_6 = 0$, to the given neighborhoods of zero, considering the expressions (18) and (A.7)–(A.9). The parameters varied are the time instants t_i, $i = \overline{2,5}$.

Step 1. Given (A.7) and the convergence interval $[t_2, t_1]$, the first inequality (18) takes the form

$$\begin{gathered} 0.8m_1 \ge \frac{X_{1,0} + G_0 + 3m_1t_2 - \Delta_1}{t_1 - t_2} + G_1 + X_{2,0} \Rightarrow \\ m_1 \ge m_{1,\min} = \frac{X_{1,0} + G_0 - \Delta_1 + (G_1 + X_{2,0})(t_1 - t_2)}{0.8t_1 - 3.8t_2}. \end{gathered} \quad \text{(A.10)}$$

From inequality (A.10) it follows that $0.8t_1 - 3.8t_2 > 0 \Rightarrow t_2 < 0.2t_1$, where $0 < t_2 < t_1$. Fixing the values t_2^* and m_1^* by

$$\begin{gathered} 0 < t_2^* < 0.2t_1 : m_{1,\min}(t_2^*) < m_{1,\max}(t_2^*), \\ m_1^* \in [m_{1,\min}(t_2^*),\ m_{1,\max}(t_2^*)], \end{gathered}$$

we move to the next step of the procedure.

Step i (i = 2, 3). Given (A.7), (A.8), and the convergence interval $[t_{i+1},\ t_i^*]$, the ith inequality in (18) takes the form

$$\begin{gathered} 0.8b_{i,\min}m_i \ge \frac{m_{i-1}^* + 3b_{i,\max}m_it_{i+1}}{t_i^* - t_{i+1}} + b_{i,\min}X_{i+1,0} + \\ \sum_{j=1}^{i-1} a_{ij,\max}e_{j,\max}(m_j^*) + F_i + k_{i-1}(m_{i-1}^*)^2b_{i-1,\max} \Rightarrow \\ m_i \ge m_{i,\min} = \frac{m_{i-1}^* + (b_{i,\min}X_{i+1,0} + \sum_{j=1}^{i-1} a_{ij,\max}e_{j,\max}(m_j^*))}{0.8b_{i,\min}t_i^* - (0.8b_{i,\min} + 3b_{i,\max})t_{i+1}} + \\ \frac{F_i + k_{i-1}(m_{i-1}^*)^2b_{i-1,\max})(t_i^* - t_{i+1})}{0.8b_{i,\min}t_i^* - (0.8b_{i,\min} + 3b_{i,\max})t_{i+1}},\ i = 2,\ 3. \end{gathered} \quad \text{(A.11)}$$

From inequality (A.11) it follows that $0.8b_{i,\min}t_i^* - (0.8b_{i,\min} + 3b_{i,\max})t_{i+1} > 0 \Rightarrow$ $t_{i+1} < 0.2b_{i,\min}t_i^*/b_{i,\max}$, i = 2, 3, $b_{3,\min} = b_{3,\max} = 1$. Fixing the values t_{i+1}^* and m_i^* by

$$\begin{gathered} t_{i+1}^* < 0.2b_{i,\min}t_i^*/b_{i,\max} : m_{i,\min}(t_{i+1}^*) < m_{i,\max}(t_{i+1}^*), \\ m_i^* \in [m_{i,\min}(t_{i+1}^*), m_{i,\max}(t_{i+1}^*)], \end{gathered}$$

we move to the next step of the procedure.

Step 4. Given (A.7), (A.8), and the convergence interval $[t_5, t_4^*]$, the fourth inequality in (18) takes the form

$$\begin{gathered} 0.8b_4m_4 \ge \frac{m_3^* + b_4m_4(1-\exp(-a_{44}t_5))/a_{44}}{t_4^* - t_5} + \\ +b_4X_{5,0} + a_{41}e_{1,\max} + a_{43}e_{3,\max} + F_4 + k_3(m_3^*)^2 \Rightarrow \\ m_4 \ge m_{4,\min} = \frac{m_3^* + [b_4X_{5,0} + a_{41}(X_{1,0} + G_0 + 3m_1^*t_2^*)}{b_4(0.8(t_4^* - t_5) - (1-\exp(-a_{44}t_5))/a_{44})} + \\ \frac{a_{43}(X_{3,0} + m_2^* + 3m_3t_4^*) + F_4 + k_3(m_3^*)^2](t_4^* - t_5)}{b_4(0.8(t_4^* - t_5) - (1-\exp(-a_{44}t_5))/a_{44})}. \end{gathered} \quad \text{(A.12)}$$

From inequality (A.12) it follows that $0.8t_5 + (1-\exp(-a_{44}t_5))/a_{44} < 0.8t_4^*$. Fixing the values t_5^* and m_4^* by

$$t_5^* + 1.25(1-\exp(-a_{44}t_5^*))/a_{44} < t_4^* : m_{4,\min}(t_5^*) < m_{4,\max}(t_5^*),$$
$$m_4^* \in [m_{4,\min}(t_5^*), m_{4,\max}(t_5^*)],$$

we move to the last step of the procedure.

Step 5. Given (14), (A.7), (A.8), and the convergence interval $[0, t_5^*]$, the last inequality in (18) takes the form

$$\begin{gathered} b_{5,\min} m_5 \ge \frac{X_{5,0} + m_4^*}{t_5^*} + a_{54,\max} m_3^* + a_{55,\max} m_4^* + \\ k_4 (m_4^*)^2 b_4 + (a_{54,\max} + 0.5 k_4 m_4^* a_{44}) \times \\ (X_{4,0} + m_3^* + b_4 m_4^*)(1-\exp(-a_{44}t_5^*))/a_{44}). \end{gathered} \tag{A.13}$$

The amplitude choice procedure is complete. ♦

P r o o f of the lemma. When solving the control problem, the variables of the closed loop system (12) converge to some neighborhood of zero sequentially bottom-to-top (15): first, the convergence of e_5 is ensured, then that of e_4, and so on until achieving the goal of control (the convergence of e_1). When solving the observation problem in system (25), on the contrary, the order of convergence of the observation errors and their derivatives in the neighborhood of zero is "top-to-bottom":

$$|\varepsilon_1(t)| \le 1/l_1,\ t \ge 0; \tag{A.14}$$
$$|a_{43}\varepsilon_2(t) - v_1(t)| = |\gamma_1(t)| \le a_{43}\beta_2,\ t \ge t_{01}; \tag{A.15}$$
$$|\varepsilon_2(t)| \le \beta_2 + 1/(a_{43}l_2),\ t \ge t_{02},\ \ 0 < t_{01} < t_{02} < T < t_5, \tag{A.16}$$

where $\beta_2 = \text{const} > 0$.

Inequalities (A.14), (A.16) and the time when the arguments of the corrective actions (27) fall in the neighborhood of zero, where the correcting actions are described by linear functions without saturation (hereinafter, the linear zones), are ensured by choosing the corresponding amplitudes p_1 and p_2. Inequality (A.15), the second inequality in (28), and the given estimation accuracy (19) are ensured by choosing the gains l_1 and l_2. Let us formalize sufficient conditions for choosing the parameters of the corrective actions (27) that satisfy these requirements.

First, we tune the amplitudes. Due to the expression (26), $\varepsilon_1(0) = 0 \le 1/l_1$, i.e., the variable $\varepsilon_1(t)$ is initially in the linear zone. Inside this zone, the first equation of system (25), (27) has the form $\dot\varepsilon_1 = a_{43}\varepsilon_2 - p_1 l_1 \varepsilon_1$. Based on its form outside the linear zone, $\dot\varepsilon_1 = a_{43}\varepsilon_2 - p_1 \text{sign}(\varepsilon_1)$, we obtain sufficient conditions for choosing the value p_1 to satisfy inequality (A.14):

$$\begin{gathered} p_1 > a_{43}|\varepsilon_2| \Rightarrow \varepsilon_1\dot\varepsilon_1 = \varepsilon_1(a_{43}\varepsilon_2 - p_1\text{sign}(\varepsilon_1)) \le \\ |\varepsilon_1|(a_{43}|\varepsilon_2| - p_1) < 0. \end{gathered} \tag{A.17}$$

In the second equation of system (25), (27), the equality $\text{sign}(v_2(t)) = \text{sign}(\varepsilon_2(t))$ holds outside the domain $|\varepsilon_2(t)| \le \beta_2$ for $t \ge t_{01}$ under condition (A.15). For the worst case, the expressions (A.15) and (A.16) yield

$$\dot\varepsilon_2 = \begin{cases} x_2 + p_2\text{sign}(\varepsilon_2),\ t \in [0, t_{01}), \\ x_2 - p_2\text{sign}(\varepsilon_2),\ t \in [t_{01}, t_{02}), \\ x_2 - p_2 l_2 (a_{43}\varepsilon_2(t) \pm \gamma_1),\ t \ge t_{02}. \end{cases} \tag{A.18}$$

Given (3), the maximum absolute value of the variable $\varepsilon_2(t)$ is reached at $t = t_{01}$:

$$|\varepsilon_2(t)| \le |\varepsilon_2(t_1)| \le \pi + (X_2 + p_2)t_{01} = E_2,\ t \ge 0. \tag{A.19}$$

Sufficient conditions for choosing the value p_2 are similar to (A.17). Due to (A.19), the convergence to the linear zone (A.16) on the interval $[t_{01}, t_{02})$ is ensured if

$$\begin{gathered} p_2 \ge \frac{\pi + (X_2 + p_2)t_{01} - \delta_1}{t_{02} - t_{01}} + X_2 \Rightarrow \\ p_2 \ge \frac{\pi + X_2 t_{02} - \delta_1}{t_{02} - 2t_{01}}. \end{gathered} \tag{A.20}$$

From the expression (A.20) it follows that $t_{02} > 2t_{01}$. This constraint must be considered when assigning the time intervals. For example, let

$$t_{02} - 2t_{01} = T - t_{02} = t_{01} \Rightarrow t_{01} = T/4. \tag{A.21}$$

Combining (A.17) and (A.19)–(A.21), we obtain the final inequalities for choosing the amplitudes of the corrective actions (27) sequentially to satisfy conditions (A.14) and (A.16) in the given time:

$$\begin{gathered} p_2 \ge \bar p_2 = \frac{4(\pi - \delta_1)}{T} + 3X_2, \\ p_1 > \bar p_1 = a_{43}(\pi + (X_2 + p_2)T/4). \end{gathered} \tag{A.22}$$

Next, we tune the gains of the corrective actions (27) to satisfy conditions (A.15) and (28). For this purpose, we estimate the solutions of system (25), (27) in linear zones (the first equation on the interval [0, t_{01}], and the second equation on the interval [t_{02}, $t_{02} + t_{01} = T$]). Based on the third equation in (A.18) and (A.19) and (A.21) we have:

$$\begin{gathered} |\varepsilon_1(t_1)| \le \frac{a_{43}E_2}{p_1 l_1} + \frac{p_1 - a_{43}E_2}{p_1 l_1} e^{-p_1 l_1 t_{01}} \Rightarrow \\ p_1 l_1 |\varepsilon_1(t_1)| - a_{43}E_2 \le (p_1 - a_{43}E_2) e^{-p_1 l_1 t_{01}}, \\ |a_{43}\varepsilon_2(t) - v_1(t)| \le a_{43}\beta_2,\ t \ge t_{01} \Leftrightarrow \\ (p_1 - a_{43}E_2) e^{-p_1 l_1 T/4} \le a_{43}\beta_2; \end{gathered} \tag{A.23}$$

$$\begin{gathered} |\varepsilon_2(T)| \le \frac{X_2}{p_2 l_2 a_{43}} + \beta_2 + \frac{p_2 - X_2}{p_2 l_2 a_{43}} e^{-p_2 l_2 a_{43} t_{01}} \le \delta_1, \\ p_2 l_2 a_{43}(|\varepsilon_2(T)| - \beta_2) - X_2 \le (p_2 - X_2) e^{-p_2 l_2 a_{43} t_{01}}, \\ |x_2(t) - v_2(t)| \le \delta_2, t \ge T \Leftrightarrow (p_2 - X_2) e^{-p_2 l_2 a_{43} T/4} \le \delta_2. \end{gathered} \tag{A.24}$$

According to (A.23) and (A.24), the observation errors for $t \ge T$ converge to the following neighborhood of zero:

$$|\varepsilon_1(t)| \le \frac{a_{43}(E_2 + \beta_2)}{p_1 l_1};\ |\varepsilon_2(t)| \le \frac{X_2 + \delta_2}{p_2 l_2 a_{43}} + \beta_2 \le \delta_1. \tag{A.25}$$

For example, let $\beta_2 = \delta_1/2$. Considering (A.23)–(A.25), the given accuracy (19) is achieved if the gains with the fixed amplitudes (A.22) satisfy

$$\begin{gathered} l_1 \ge \bar l_1 = \frac{4}{p_1 T} \ln \frac{2(p_1 - a_{43}E_2)}{a_{43}\delta_1}; \\ l_2 \ge \bar l_2 = \frac{1}{p_2 a_{43}} \max\left\{ \frac{2(X_2 + \delta_2)}{\delta_1};\ \frac{4}{T}\ln\frac{p_2 - X_2}{\delta_2} \right\}. \end{gathered} \tag{A.26}$$

Thus, there exist $\bar p_i > 0$ (A.22) and $\bar l_i > 0$ (A.26) such that the lemma conditions (28) hold for any $p_i > \bar p_i$ and $l_i \ge \bar l_i$, $i = 1, 2$. ♦

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