OBSERVABILITY OF NONLINEAR CONTINUOUS–DISCRETE SYSTEMS AND THEIR STABILIZATION USING ASYMPTOTIC OBSERVERS

S. V. Akmanova

Nosov Magnitogorsk State Technical University, Magnitogorsk, Russia

✉ svet.akm_74@mail.ru

Abstract. This paper considers nonlinear dynamic systems described by a set of differential and difference equations and a nonlinear output equation. The states of such systems contain both continuous- and discrete-time components, so they are called continuous–discrete, or hybrid, dynamic control systems. For this class of systems, the issues of observability, design of full- and minimum-order observers, and stabilization with observers are investigated. A transition is performed from an original nonlinear hybrid system to a nonlinear discrete dynamic system equivalent in a natural sense. An observability criterion, sufficient conditions for the existence of full- and minimum-order asymptotic observers for nonlinear hybrid control systems, and sufficient conditions for the stabilization of a nonlinear hybrid system with designed asymptotic observers are established. Examples of designing asymptotic observers and stabilizing the augmented system with such observers are provided.

Keywords: continuous–discrete system, hybrid system, fully observable system, observer, stabilizing control.

INTRODUCTION

Nonlinear continuous–discrete (hybrid) dynamic systems serve as adequate mathematical models for real control systems with nonlinear functioning and heterogeneous processes. For example, these are control systems in aviation and aerospace, metallurgy, economics, and other fields [1–4]. To effectively control hybrid systems, it is necessary to have complete information about the state vector of the system. However, this condition is difficult to realize: in practice, some components of the system's state vector are often unmeasurable. In such cases, state-feedback control is preferable to program control [5], as it allows reconstructing the state vector components not available for measurement. This reconstruction is called *estimation*, and a device generating the system's state vector estimate is called an *observer*; it is a dynamic system estimating the state vector of an observed hybrid system based on the latter's measurable output.

By now, considerable experience has been gained in estimating states and stabilizing certain types of nonlinear hybrid systems. For instance, for control systems described by nonlinear stochastic differential equations, state estimation algorithms based on various modifications of Kalman filters and particle filters with noisy discrete measurements were proposed in [6, 7]; for a class of nonlinear deterministic continuous control systems with irregular discrete measurements, a pulse continuous–discrete observer was designed in [8]. A sliding-mode observer for estimating the continuous and discrete states of nonlinear switched control systems was described in [9]. For nonlinear continuous control systems with discrete measurements and uncertain disturbances, a control law based on a continuous observer with feedback was constructed in [10]. Methods for estimating the states of nonlinear continuous–discrete systems with uncertain disturbances and designing a discrete control law to suppress initial deviations and the impact of uncertain disturbances, based on the method of matrix comparison systems and the technique of differential-difference linear matrix inequalities (LMIs), were presented in [11]. Also, note research into the stabilization of nonlinear hybrid systems with a measurable output that involve no observers, provided that the systems are described by differential equations with various nonlinearities and are regulated by a discrete state- or output-feedback controller; for example, see [12–14].

The object of this study is nonlinear deterministic continuous–discrete systems with continuous output and a constant sampling step; their state is characterized by a set of two different vectors, one varying continuously in accordance with differential equations while the other discretely in accordance with difference equations containing a control term. For similar systems without output equations, previous research was aimed at finding optimal control (e.g., see [1, 4] and the references therein); furthermore, for the above systems without a measurable output, sufficient stabilization conditions and methods were provided by the author [15, 16]. However, the issues of state estimation and stabilization of such systems with continuous output require further investigation.

In this paper, we propose general approaches to designing full- and minimum-order asymptotic observers for nonlinear deterministic continuous–discrete systems with a continuous output and a constant sampling step, as well as to stabilizing these systems using asymptotic observers. The approaches are based on a transition from an original nonlinear continuous–discrete (hybrid) system to an equivalent nonlinear discrete system (in a natural sense) and involve D. Luenberger's [17, 18] ideas regarding observer design.

The contributions of this study include the following:

- an observability criterion for a nonlinear hybrid system with a continuous output and a necessary condition for the observability of an equivalent nonlinear discrete system with a discrete output;
- sufficient conditions for the existence of full- and minimum-order asymptotic observers for a nonlinear hybrid system with a continuous output; and new observer models considering the linear part of output equations and the nonlinearity of the equivalent discrete system;
- sufficient conditions for the stabilization of a nonlinear hybrid system, algorithms for constructing full- and minimum-order asymptotic observers for such a system, and algorithms for stabilizing such a system using the observers.

1. PROBLEM STATEMENT

Consider a nonlinear continuous–discrete (hybrid) system of the form

$$\begin{cases} x'(t) = f\left(x(t), y(t_k)\right), \ t_k \le t < t_{k+1}, \\ y(t_{k+1}) = g\left(x(t_{k+1}), y(t_k), u(t_k)\right) \\ w(t) = \varphi\left(x(t), y(t_k)\right), k = 0, 1, 2, \ldots, \end{cases} \tag{1}$$

where $x(t) \in R^n$ and $y(t_k) \in R^m$ are the state vectors of system (1); $u(t_k) \in R^q$ is the control variable (input); $w(t) \in R^p$ is the system output $(p \le n + m)$; the time instants t_k define a uniform grid on the set R with a constant step $h = t_{k+1} - t_k > 0$ and nodes $t_k = kh, \ k = 0, 1, 2, \ldots$; the functions $f(x, y)$, $g(x, y, u)$, and $\varphi(x, y)$ are continuously differentiable with respect to the aggregate of their variables, and

$$f(0, 0) = 0, \ g(0, 0, 0) = 0, \ \varphi(0, 0) = 0. \tag{2}$$

In other words, system (1) with $u = 0$ has the trivial equilibrium $x = 0$, $y = 0$.

Consider system (1) with initial conditions $x(t_0) = x_0, \ y(t_0) = y_0$. On each interval $[t_k, t_{k+1}]$, $k \in \bar{N} \ \left(\bar{N} = N \cup \{0\} = \{0, 1, 2, \ldots\}\right)$, for the chosen control u (a given sequence $u(t_k), \ k \in \bar{N}$), we can find a solution $x(t) = \psi_k(t)$ of the Cauchy problem $x'(t) = f\left(x(t), \ y(t_k)\right), \ x(t_k) = x_k$, as well as the vectors $x(t_{k+1}) = \psi_k(t_{k+1})$ and $y(t_{k+1}) = g\left(x(t_{k+1}), \ y(t_k), u(t_k)\right)$. Therefore, for the chosen control u, the state of system (1) at a time instant $t \in [t_k, \ t_{k+1})$ is defined as a pair $z(t) = \left(x(t), \ y(t_k)\right)$, where $x(t) = \psi_k(t)$, $k \in \bar{N}$.

(Full) observability is an important property of dynamic systems that ensures their safe and efficient functioning. Is system (1) observable? The answer depends on the grid step $h > 0$. In view of the results presented in [19], we introduce the following.

Definition 1. System (1) is said to be *observable* (*fully observable*) *for a given* $h > 0$ if there exists a number $l \in N \left(l \le n + m\right)$ such that, for any quadruples $\left(u_1(t_k), \ x_1(t), \ y_1(t_k), w_1(t)\right)$ and $\left(u_2(t_k), \ x_2(t), \ y_2(t_k), w_2(t)\right)$ satisfying system (1) on the interval $\left[t_k, \ t_{k+1}\right)$, the equality of the inputs and outputs:

$$u_1(t_k)=u_2(t_k),\ w_1(t)=w_2(t),\ t\in[t_k,\ t_{k+1}),$$
$$k=\overline{0,\ l},$$

implies the equality of the corresponding states:

$$x_1(t)=x_2(t),\ y_1(t_k)=y_2(t_k),$$
$$t\in\left[t_{k,}\ t_{k+1}\right),\ k=\overline{0,\ l}.\ \blacklozenge$$

Then the known values of the input $u(t_k)$ and output $w(t)$, $t\in[t_k,\ t_{k+1})$, $k=\overline{0,\ l}$, can be used to uniquely determine the state $z(t)$, $t\in[t_k,\ t_{k+1})$, $k=\overline{0,\ l}$, of system (1) and conclude on the processes occurring in this system.

System (1) may be unobservable for some $h>0$ if there exists an input leading to identical outputs for different states of this system; in other words, it is possible to find different initial conditions $z_0=(x_0,\ y_0)$, $\tilde{z}_0=(\tilde{x}_0,\ \tilde{y}_0)$ and an input $u^*(t_k)$, $k\in\bar{N}$, such that $w(z(t))\equiv w(\tilde{z}(t))$ for the corresponding states $z=z(t)$ and $\tilde{z}=\tilde{z}(t)$, $t\geq 0$.

The primary problems addressed in this paper are as follows: establish observability criteria for system (1); determine existence conditions for asymptotic observers and construct them for system (1); and stabilize system (1) using the constructed observers. To solve these problems, we make a transition from the original hybrid system (1) to an equivalent, in the natural sense, nonlinear discrete system with an output.

2. TRANSITION TO AN EQUIVALENT DISCRETE SYSTEM WITH AN OUTPUT

Based on the relations (2), the functions $f(x,\ y)$, $g(x,\ y,\ u)$, and $\varphi(x,\ y)$ in the neighborhoods of the points $x=0,\ y=0$ and $x=0,\ y=0,\ u=0$ can be expressed through the Maclaurin series expansion as

$$f(x,\ y)=A_1x+B_1y+a(x,\ y),$$
$$g(x,\ y,\ u)=A_2x+B_2y+Cu+b(x,\ y,\ u),$$
$$\varphi(x,\ y)=D_1x+D_2y+c(x,\ y),$$

where

$$A_1=f_x^{'}(0,\ 0), B_1=f_y^{'}(0,\ 0),$$
$$A_2=g_x^{'}(0,\ 0,\ 0),\ B_2=g_y^{'}(0,\ 0,\ 0),$$
$$C=g_u^{'}(0,\ 0,\ 0),\ D_1=\varphi_x^{'}(0,\ 0),$$
$$D_2=\varphi_y^{'}(0,\ 0),$$

and the smooth nonlinear functions $a(x,\ y)$, $b(x,\ y,\ u)$, and $c(x,\ y)$ have the form $a(x,\ y)=o(\|x\|+\|y\|)$, $c(x,\ y)=o(\|x\|+\|y\|)$ as $\|x\|+\|y\|\to 0$, and $b(x,\ y,\ u)=o(\|x\|+\|y\|+\|u\|)$ as $\|x\|+\|y\|+\|u\|\to 0$. In other words, they are infinitesimals of a higher order of smallness than $\|x\|+\|y\|$ and $\|x\|+\|y\|+\|u\|$ as $\|x\|+\|y\|\to 0$ and $\|x\|+\|y\|+\|u\|\to 0$, respectively.

Throughout this paper, $\|\cdot\|$ denotes the Euclidean norm of a vector in the corresponding space of real numbers, or the norm of a matrix.

Let $x(t_k)=x_k$, $y(t_k)=y_k$, and $u(t_k)=u_k$; then system (1) is equivalent to the hybrid one

$$\begin{cases} x'(t)=A_1x(t)+B_1y_k+a(x(t),y_k),\ t_k\leq t<t_{k+1}, \\ y_{k+1}=A_2x_{k+1}+B_2y_k+Cu_k+b(x_{k+1},\ y_k,\ u_k) \\ w(t)=D_1x(t)+D_2y_k+c(x(t),y_k),\ k\in\bar{N}. \end{cases} \quad (3)$$

Under the assumption $\det A_1\neq 0$, we apply the trajectory shift operator to the system $x'=f(x,\ y)$ on the time interval between $t=0$ and $t=h$ [20] to pass from the original system (3) to the discrete counterpart

$$\begin{cases} x_{k+1}=e^{A_1h}x_k+A_1^{-1}\left(e^{A_1h}-I\right)B_1y_k \\ \qquad +\varepsilon(x_k,\ y_k;\ h) \\ y_{k+1}=A_2e^{A_1h}x_k+\left(A_2A_1^{-1}\left(e^{A_1h}-I\right)B_1+B_2\right)y_k \\ \qquad +Cu_k+\delta(x_k,\ y_k,\ u_k;\ h) \\ w_k=D_1e^{A_1h}x_k+\left(D_1A_1^{-1}\left(e^{A_1h}-I\right)B_1+D_2\right)y_k \\ \qquad +\gamma(x_k,\ y_k;\ h),\ k\in\bar{N}, \end{cases} \quad (4)$$

where I is an identity matrix of compatible dimensions, and

$$\varepsilon(x_k,\ y_k;\ h)$$
$$=e^{(t_k+h)A_1}\int_{t_k}^{t_k+h}e^{-sA_1}a(p(s,\ x_k,\ y_k),y_k)ds,$$

where $x=p(t,x_k,\ y_k)$ is the solution of the Cauchy problem $x'=A_1x+B_1y_k+a(x,\ y_k)$, $x(t_k)=x_k$,

$$\delta(x_k,\ y_k,\ u_k;\ h)$$
$$=A_2\varepsilon(x_k,\ y_k;\ h)+b(x_{k+1},\ y_k,\ u_k), \quad (5)$$
$$\gamma(x_k,\ y_k;\ h)=D_1\varepsilon(x_k,\ y_k;\ h)+c(x_{k+1},y_k).$$

System (4) can be compactly written as

$$\begin{cases} z_{k+1} = A(h) z_k + Bu_k + \xi(z_k, u_k, h) \\ w_k = D(h) z_k + \gamma(z_k, h), \ k \in \bar{N}, \end{cases} \quad (6)$$

where

$$z_k = \begin{bmatrix} x_k \\ y_k \end{bmatrix}, \quad A(h) = \begin{bmatrix} e^{A_1 h} & A_1^{-1}\left(e^{A_1 h} - I\right) B_1 \\ A_2 e^{A_1 h} & A_2 A_1^{-1}\left(e^{A_1 h} - I\right) B_1 + B_2 \end{bmatrix},$$

$$B = \begin{bmatrix} O \\ C \end{bmatrix}, \ \xi(z_k, u_k, h) = \begin{bmatrix} \varepsilon(x_k, y_k; \ h) \\ \delta(x_k, y_k, u_k; h) \end{bmatrix},$$

$$D(h) = \begin{bmatrix} D_1 e^{A_1 h} & D_1 A_1^{-1}\left(e^{A_1 h} - I\right) B_1 + D_2 \end{bmatrix},$$

and O is the zero matrix of the corresponding dimensions.

Based on the results of [20] and equalities (5), it is straightforward to show that

$$\xi(z, u, h) = o(\|z\| + \|u\|) \ \text{as} \ \|z\| + \|u\| \to 0,$$

$$\gamma(z, h) = o(\|z\|) \ \text{as} \ \|z\| \to 0.$$

Systems (1) and (4) are equivalent in the natural sense: given the solution $\left(x(t), y_k\right)$ of system (1) with the output $w(t)$, one can find the solution $\left(x_k, y_k\right)$ of system (4) with the output w_k, and vice versa [15, 20]. Therefore, systems (1) and (6) are equivalent as well.

Definition 2. System (6) is said to be *observable* (*fully observable*) *for* a *given* $h > 0$ if there exists a number $l \in N(l \le n + m)$ such that, for any triples $\left(u_k^{(1)}, z_k^{(1)}, w_k^{(1)}\right)$, $\left(u_k^{(2)}, z_k^{(2)}, w_k^{(2)}\right)$, $k = \overline{0, l}$, satisfying system (6), the equality of the inputs and outputs:

$$u_k^{(1)} = u_k^{(2)}, \ w_k^{(1)} = w_k^{(2)}, \ \ k = \overline{0, l},$$

implies the equality of the corresponding states:

$$z_k^{(1)} = z_k^{(2)}, \ k = \overline{0, l}. \ ♦$$

Then we arrive at the following result.

Theorem 1. *The hybrid system* (1) *is observable for a given* $h > 0$ *if and only if the discrete system* (6) *is observable for* $h > 0$.

P r o o f. Let system (1) be observable for a given $h > 0$. Let us denote $u_i(t_k) = u_k^{(i)}$, $x_i(t_k) = x_k^{(i)}$, $w_i(t_k) = w_k^{(i)}$, and $y_i(t_k) = y_k^{(i)}$, where $i = 1, 2$; $k = \overline{0, l}$. If $w_1(t) = w_2(t)$ and $x_1(t) = x_2(t)$ for $t \in [t_k, t_{k+1})$, $k = \overline{0, l}$, then $w_k^{(1)} = w_k^{(2)}$ and $x_k^{(1)} = x_k^{(2)}$, $k = \overline{0, l}$. In this case, by Definition 1, $u_k^{(1)} = u_k^{(2)}$ and $w_k^{(1)} = w_k^{(2)}$ imply $x_k^{(1)} = x_k^{(2)}$ and $y_k^{(1)} = y_k^{(2)}$, i.e., $z_k^{(1)} = z_k^{(2)}$, $k = \overline{0, l}$. Hence, for any triples $\left(u_k^{(1)}, z_k^{(1)}, w_k^{(1)}\right)$ and $\left(u_k^{(2)}, z_k^{(2)}, w_k^{(2)}\right)$, $k = \overline{0, l}$, satisfying system (6) for a given $h > 0$, the equality of the inputs and outputs implies the equality of the corresponding states. In other words, system (6) is observable for $h > 0$.

Now, let system (6) be observable for $h > 0$. Then the equality $z_k^{(1)} = z_k^{(2)}$, $k = \overline{0, l}$, implies the equalities $x_k^{(1)} = x_k^{(2)}$ and $y_k^{(1)} = y_k^{(2)}$, $k = \overline{0, l}$, i.e., $x_1(t_k) = x_2(t_k)$ and $y_1(t_k) = y_2(t_k)$. Due to the unique solution of the Cauchy problem $x' = f(x, y_k)$, $x(t_k) = x_k$, $k = \overline{0, l}$, the equality $x_1(t) = x_2(t)$ holds for $t \in [t_k, t_{k+1})$, $k = \overline{0, l}$. Since $w_k^{(1)} = w_k^{(2)} \Leftrightarrow \varphi\left(x_k^{(1)}, y_k^{(1)}\right) = \varphi\left(x_k^{(2)}, y_k^{(2)}\right) \Leftrightarrow \varphi\left(x_1(t_k), y_1(t_k)\right) = \varphi\left(x_2(t_k), y_2(t_k)\right) \Leftrightarrow \varphi\left(x_1(t), y_1(t_k)\right) = \varphi\left(x_2(t), y_2(t_k)\right) \Leftrightarrow w_1(t) = w_2(t)$ for $t \in [t_k, t_{k+1})$, $k = \overline{0, l}$, from Definition 2 and the above considerations we establish that if $u_1(t_k) = u_2(t_k)$ and $w_1(t) = w_2(t)$, $t \in [t_k, t_{k+1})$, $k = \overline{0, l}$, then $x_k^{(1)} = x_k^{(2)}$ and $y_k^{(1)} = y_k^{(2)}$, $k = \overline{0, l}$. Consequently, $x_1(t) = x_2(t)$ and $y_1(t_k) = y_2(t_k)$, meaning that system (1) is observable for $h > 0$. ♦

3. THE EXISTENCE AND CONSTRUCTION OF A FULL-ORDER OBSERVER FOR A HYBRID SYSTEM

Hereinafter, we will consider systems (1) and (6) for some value $h = h_0 > 0$. Then the discrete system (6) has the first-order approximation of the form

$$\begin{cases} z_{k+1} = A_0 z_k + Bu_k \\ w_k = D_0 z_k, \ \ k \in \bar{N}, \end{cases} \quad (7)$$

where $A_0 = A(h_0)$ and $D_0 = D(h_0)$.

If the pair (A_0, D_0), or system (7), is observable, then by the rank observability criterion for linear time-invariant systems [19], we have

$$\operatorname{rank}\left[D_0^{\mathrm{T}}, \ A_0^{\mathrm{T}} D_0^{\mathrm{T}}, \left(A_0^{\mathrm{T}}\right)^2 D_0^{\mathrm{T}}, \ldots, \left(A_0^{\mathrm{T}}\right)^{n+m-1} D_0^{\mathrm{T}}\right] = n + m.$$

Therefore, the following result is true.

Theorem 2. *If system* (6) *is observable, then the pair* (A_0, D_0) *is observable as well.*

P r o o f. By the hypothesis of this theorem, the system

$$\begin{cases} z_{k+1} = A_0 z_k + Bu_k + \xi(z_k, u_k, h_0) \\ w_k = D_0 z_k + \gamma(z_k, h_0), k \in \bar{N}, \end{cases} \quad (8)$$

is observable. In view of Definition 2, we compile the following sequence of $l = n + m$ equalities from the relations (8):

$$\begin{cases} w_0 = D_0 z_0 + \gamma(z_0, h_0) \\ w_1 = D_0 A_0 z_0 + D_0 B u_0 + D_0 \xi(z_0, u_0, h_0) + \gamma(z_1, h_0) \\ w_2 = D_0 A_0^2 z_0 + D_0 A_0 B u_0 + D_0 A_0 \xi(z_0, u_0, h_0) + D_0 B u_1 + D_0 \xi(z_1, u_1, h_0) + \gamma(z_2, h_0) \\ \ldots \\ w_{n+m-1} = D_0 A_0^{n+m-1} z_0 + \sum_{i=0}^{n+m-2} D_0 A_0^{n+m-2-i} B u_i + \sum_{i=0}^{n+m-2} D_0 A_0^{n+m-2-i} \xi(z_i, u_i, h_0) + \gamma(z_{n+m-1}, h_0). \end{cases}$$

They can be written as

$$\begin{bmatrix} D_0 \\ D_0 A_0 \\ D_0 A_0^2 \\ \ldots \\ D_0 A_0^{n+m-1} \end{bmatrix} \cdot z_0 = \begin{bmatrix} w_0 - \gamma(z_0, h_0) \\ w_1 - D_0 B u_0 - D_0 \xi(z_0, u_0, h_0) - \gamma(z_1, h_0) \\ w_2 - D_0 A_0 B u_0 - D_0 A_0 \xi(z_0, u_0, h_0) - D_0 B u_1 - D_0 \xi(z_1, u_1, h_0) - \gamma(z_2, h_0) \\ \ldots \\ w_{n+m-1} - \sum_{i=0}^{n+m-2} D_0 A_0^{n+m-2-i} B u_i - \sum_{i=0}^{n+m-2} D_0 A_0^{n+m-2-i} \xi(z_i, u_i, h_0) - \gamma(z_{n+m-1}, h_0) \end{bmatrix}.$$

Due to the observability of system (8), the states $z_k \left(k = \overline{0,\ n+m-1}\right)$ are uniquely determined by the known values of the inputs $u_k \left(k = \overline{0,\ n+m-2}\right)$ and outputs $w_k \left(k = \overline{0,\ n+m-1}\right)$. In the resulting equality, the right-hand side is a constant vector, and the product on the left-hand side is a vector representing the linear combinations of the components of the state vector z_0. Thus, we obtain $(n+m)p$ equalities with respect to the components of the uniquely determined vector $z_0 \in R^{n+m}$, so only $n+m$ rows of the matrix

$$\left[D_0^{\mathrm{T}},\ A_0^{\mathrm{T}} D_0^{\mathrm{T}},\ \left(A_0^{\mathrm{T}}\right)^2 D_0^{\mathrm{T}}, \ldots, \left(A_0^{\mathrm{T}}\right)^{n+m-1} D_0^{\mathrm{T}} \right]$$

are linearly independent. Consequently,

$$\operatorname{rank}\left[D_0^{\mathrm{T}},\ A_0^{\mathrm{T}} D_0^{\mathrm{T}},\ \left(A_0^{\mathrm{T}}\right)^2 D_0^{\mathrm{T}}, \ldots, \left(A_0^{\mathrm{T}}\right)^{n+m-1} D_0^{\mathrm{T}} \right] = n+m,$$

i.e., the pair (A_0, D_0) is observable [19]. ♦

Corollary 1. *If system* (6) *is observable, then system* (7) *is observable as well*.

For brevity, the expression $|A| < 1$ below means that all eigenvalues λ of a square matrix A satisfy the condition $|\lambda| < 1$.

Based on the results of [15], Theorem 2, and D. Luenberger's approach to the design of full-order observers for linear systems, we establish the following fact.

Theorem 3. *Let the pair (A_0, D_0) be observable. Then there exists a matrix $K \in R^{(n+m)\times p}$ such that the solutions of system* (6) *and system*

$$\begin{aligned} v_{k+1} = (A_0 - K D_0) v_k + B u_k + K D_0 z_k \\ + \xi(z_k, u_k, h_0),\ k \in \bar{N}, \end{aligned} \tag{9}$$

satisfy the condition

$$\lim_{k\to\infty} \|z_k - v_k\| = 0 \ \ \forall u_k \in R^q;\ z_0, v_0 \in R^{n+m}.$$

P r o o f. Let $p_k = z_k - v_k$, $k \in \bar{N}$, where z_k and v_k are the solutions of systems (6) and (9), respectively, under the chosen control u, i.e., for a given sequence u_k, $k \in \bar{N}$. Then $p_{k+1} = z_{k+1} - v_{k+1}$, i.e., we have the system

$$p_{k+1} = (A_0 - K D_0) p_k,\ k \in \bar{N}. \tag{10}$$

Now we verify the existence of a matrix $K \in R^{(n+m)\times p}$ such that system (10) is asymptotically stable.

According to the hypothesis of the theorem, the pair (A_0, D_0) is observable; then, by Kalman's duality principle, the pair $\left(A_0^{\mathrm{T}}, D_0^{\mathrm{T}}\right)$ is fully controllable, which implies the full controllability of the pair $\left(A_0^{\mathrm{T}}, -D_0^{\mathrm{T}}\right)$ as well. In this case, the system

$$z_{k+1} = A_0^{\mathrm{T}} z_k - D_0^{\mathrm{T}} u_k$$

is stabilizable [15], i.e., there exists a matrix $K^{\mathrm{T}} \in R^{p\times(n+m)}$ such that $\left|A_0^{\mathrm{T}} - D_0^{\mathrm{T}} K^{\mathrm{T}}\right| < 1$. Then $|A_0 - K D_0| < 1$, where

$K \in R^{(n+m)\times p}$, and consequently, system (10) is asymptotically stable. Therefore, $\lim_{k\to\infty}\|p_k\| = 0$ $\forall u_k \in R^q;\ z_0,\ v_0 \in R^{n+m}$. ♦

Corollary 2. *If system* (6) *is observable, then there exists system* (9) *for it.*

System (9) will be called the *observer of system* (6); in this case, system (9) is a full-order asymptotic observer since the estimation vector $v_k \in R^{n+m}$ asymptotically converges to the state vector $z_k \in R^{n+m}$ of system (6). The dynamic system (9) is also a *full-order asymptotic observer* for the hybrid system (1) on the discrete time interval $[t_0, +\infty)$ because the states $v_k = v(t_k)$, $k \in \bar{N}$, of system (9) are treated as estimates for the corresponding states of system (6), and the states of systems (1) and (6) coincide at the time instants $t = t_k$, $k \in \bar{N}$.

4. STABILIZATION OF A SYSTEM WITH A FULL-ORDER OBSERVER

Consider the asymptotic stabilization problem for the trivial equilibrium of system (1) using dynamic output feedback, with the system state estimated by the asymptotic observer (9). Let us demonstrate the applicability of the separation principle to the above problem. According to this principle, a system is stabilized by separately constructing a stabilizing state-feedback control law and an observer to estimate the system state, followed by substituting the state estimate into the feedback control [21]. In view of the results presented in [15], we arrive at the following.

Theorem 4. *Let the observable system* (6) *have a stabilizing control law* $u_k = Fz_k,\ k \in \bar{N}$, *with a gain matrix* $F \in R^{q\times(n+m)}$, *and let system* (9) *be its observer. Then the solution* $x = 0,\ y = 0,\ v = 0$ *of the augmented system* (1), (9) *with the control law* $u(t_k) = Fv(t_k),\ k \in \bar{N}$, *is asymptotically stable.*

P r o o f. Let $u_k = Fv_k,\ k \in \bar{N}$. We write system (6) (without the output) and system (9) in terms of the variables z_k and p_k, where $p_k = z_k - v_k$, $k \in \bar{N}$, to obtain

$$\begin{cases} z_{k+1} = (A_0 + BF)z_k - BFp_k + \xi(z_k,\ F(z_k - p_k),\ h_0) \\ p_{k+1} = (A_0 - KD_0)\,p_k,\ \ k \in \bar{N}. \end{cases} \tag{11}$$

This system can be represented in matrix form as

$$\begin{bmatrix} z_{k+1} \\ p_{k+1} \end{bmatrix} = G \cdot \begin{bmatrix} z_k \\ p_k \end{bmatrix} + \begin{bmatrix} \xi(z_k, F(z_k - p_k), h_0) \\ O \end{bmatrix}, \quad k \in \bar{N}, \tag{12}$$

where $G = \begin{bmatrix} A_0 + BF & -BF \\ O & A_0 - KD_0 \end{bmatrix}$ is a block-triangular matrix of order $2(n+m)$.

Then $\sigma(G) = \sigma(A_0 + BF) \cup \sigma(A_0 - KD_0)$ is the spectrum of the matrix G. Since $u_k = Fz_k$, $k \in \bar{N}$, is a stabilizing control law of system (6), we have $|A_0 + BF| < 1$ [15]. Due to the observability of system (6), Theorem 2, and the proof of Theorem 3, it follows that $|A_0 - KD_0| < 1$ and consequently, $|G| < 1$.

Now, let $\bar{\xi}(z_k,\ p_k,\ h_0) = \begin{bmatrix} \xi(z_k,\ F(z_k - p_k),\ h_0) \\ O \end{bmatrix}$. Considering the relation $\xi(z, u, h) = o(\|z\| + \|u\|)$ as $\|z\| + \|u\| \to 0$, we establish that $\xi(z_k,\ F(z_k - p_k),\ h_0) = o(\|z_k\| + \|p_k\|)$ as $\|z_k\| + \|p_k\| \to 0$, which implies $\bar{\xi}(z_k,\ p_k,\ h_0) = o(\|z_k\| + \|p_k\|)$ as $\|z_k\| + \|p_k\| \to 0$. According to [20], system (12) (equivalently, system (11)) has the asymptotically stable trivial solution $z = 0$, $p = 0$. In this case, the augmented system (6), (9) with the control law $u_k = Fv_k$, $k \in \bar{N}$, also has the asymptotically stable trivial solution $z = 0, v = 0$. Therefore, the corresponding augmented system (1), (9) with the control law $u(t_k) = Fv(t_k)$, $k \in \bar{N}$, has the asymptotically stable trivial solution $x = 0,\ y = 0,\ v = 0$ [20]. ♦

Note that the observer-based stabilization of the dynamic system (1) has several advantages compared to the approach without an observer. One advantage is increased control reliability, as a property characterizing a system in terms of its ability to achieve a specified control objective [22]. All real-world systems, including those modeled by (1), incorporate internal noises [23]. These noises represent random disturbances that cannot be reflected in a mathematical model of the system but can significantly change its behavior if, e.g., the parameter *h* of system (1) is near a bifurcation point. Furthermore, if the output vector $w(t)$ has a dimension $p < n+m$, then some components of the state vector of system (1) are unavailable for measurement (i.e., only certain linear combinations of the components can be observed); in this case, a stabilizing control law based on incomplete data about the system's state may fail to achieve the desired result [21]. However, with an observer used to stabilize system (1), the former's state at a given time instant serves as an estimate for the latter's state at the same instant: the observer models the system in real time and estimates all components of the system's state vector, thereby providing high-quality feedback, especially if the observer is asymptotic. Such an observer ensures an asymptotically decreasing estimation error for the state of system (1); thus, the problem of stabilizing system (1) using an observer is successfully solved. Finally, observer-based stabilization allows reconstructing the unmeasurable state vector components without the need to install additional physical sensors in a real system, which improves the operational and cost characteristics of the control system [24].

5. THE EXISTENCE AND CONSTRUCTION OF A MINIMUM-ORDER OBSERVER FOR A HYBRID SYSTEM

The requirement for a reduced-order observer is a key concern in control theory: such an observer simplifies the analysis and design of a dynamic system, decreases the demands on a computing device for observer implementation, and accelerates the estimation procedure for the system's state vector [25].

The above design of a full-order observer for system (1) has taken into account the linear part of the output $w_k \in R^p$ and the nonlinear function $\xi(z_k, u_k, h_0)$ in system (6) (see Section 3). Let us apply the same idea to the design of a reduced-order observer. In this case, $w_k \in R^p$ and $z_k \in R^{n+m}$; then the linear part of the output $D_0 z_k$ consists of p linear combinations of $n+m$ state variables, so generally it is necessary to estimate at most $n+m-p$ state variables. This idea is realized when synthesizing a reduced-order observer; an observer of order $n+m-p$ is considered to be a minimum-order observer [18]. Below, a minimum-order observer will also be referred to as a minimal observer.

For linear dynamic systems without disturbances, minimal observers were constructed, e.g., in [17, 18, 25, 26]. We will utilize D. Luenberger's approach [26] and synthesize system (9) to construct a minimal observer for the nonlinear discrete system (6). Assume that the pair (A_0, D_0) is observable. Let us choose an appropriate matrix $M \in R^{r\times(n+m)}$, where $r \geq n+m-p$, so that

$$\operatorname{rank}\begin{bmatrix} M \\ D_0 \end{bmatrix} = n+m. \qquad (13)$$

With $s_k = Mz_k$, we have the equality

$$\begin{bmatrix} s_k \\ D_0 z_k \end{bmatrix} = \begin{bmatrix} M \\ D_0 \end{bmatrix} z_k; \qquad (14)$$

by the full rank of the matrix $\left[M^{\mathrm{T}}, D_0^{\mathrm{T}}\right]$, to reconstruct the vector z_k, it suffices to estimate the vector s_k. For simplicity, let $r = n+m-p$. From equation (14), it follows that

$$z_k = \begin{bmatrix} M \\ D_0 \end{bmatrix}^{-1} \begin{bmatrix} s_k \\ D_0 z_k \end{bmatrix},$$

and if $\bar{s}_k$ is an estimate of the vector s_k, then the state vector z_k of system (6) can be estimated as

$$\bar{z}_k = \begin{bmatrix} M \\ D_0 \end{bmatrix}^{-1} \begin{bmatrix} \bar{s}_k \\ D_0 z_k \end{bmatrix}. \qquad (15)$$

To estimate the vector s_k, we use an observer of order $n+m-p$ of the form

$$\bar{s}_{k+1} = L\bar{s}_k + ED_0 z_k + Tu_k + M\xi(z_k, u_k, h_0), \qquad (16)$$

where $L, E, T,$ and M are constant matrices (the parameters of the observer (16)) to be determined. The following result is true.

Theorem 5. *Let the pair* (A_0, D_0) *be observable. Then there exist matrices* $L \in R^{(n+m-p)\times(n+m-p)}$, $E \in R^{(n+m-p)\times p}$, $T \in R^{(n+m-p)\times q}$, and $M \in R^{(n+m-p)\times(n+m)}$ *such that the solutions of systems* (6) *and* (16) *satisfy the condition*

$$\lim_{k\to\infty}\|Mz_k - \bar{s}_k\| = 0 \ \forall\, u_k \in R^q,\ z_0 \in R^{n+m},\ \bar{s}_0 \in R^{n+m-p}.$$

P r o o f. The pair (A_0, D_0) is observable; therefore, there exists a matrix $M \in R^{(n+m-p)\times(n+m)}$ satisfying condition (13). Let $s_k = Mz_k$, where $z_k\left(k \in \bar{N}\right)$ is the solution of system (6); then the observation error $p_k = s_k - \bar{s}_k$ is the solution of the difference equation $p_{k+1} = s_{k+1} - \bar{s}_{k+1}$; i.e., in view of formula (16), we have

$$p_{k+1} = MA_0 z_k + MBu_k + M\xi(z_k, u_k, h_0)$$
$$-\left(L\bar{s}_k - Ls_k + LMz_k + ED_0 z_k + Tu_k + M\xi(z_k, u_k, h_0)\right),$$
$$p_{k+1} = Lp_k + (MA_0 - LM - ED_0)z_k + (MB - T)u_k.$$

For the observation error to vanish, we select the observation parameters from the conditions

$$\begin{cases} |L| < 1 \\ ED_0 = MA_0 - LM \\ T = MB. \end{cases}$$

Since the pair (A_0, D_0) is observable, the considerations similar to [17, 18] allow establishing the feasibility of these conditions under formula (13). In this case, the system

$p_{k+1}=Lp_k,\ k\in\bar{N},$ is asymptotically stable, meaning that $\lim_{k\to\infty}\|p_k\|=0\quad \forall u_k\in R^q,\ p_0\in R^{n+m-p},$ i.e., $\lim_{k\to\infty}\|Mz_k-\bar{s}_k\|=0\ \ \forall u_k\in R^q,\ z_0\in R^{n+m},\ \bar{s}_0\in R^{n+m-p}$. ♦

System (16) will be called *the minimum-order asymptotic observer of system* (6) (*system* (1)) or *the minimal asymptotic observer of system* (6) (*system* (1)) that estimates the vector Mz_k (the vector $Mz(t_k)$, respectively) on the discrete time interval $[t_0,+\infty)$, where $t_k=h_0k,\ k\in\bar{N}$.

The observer (16) can be designed using the following algorithm [26].

1. Find a matrix $K\in R^{(n+m)\times p}$ such that $|A_0-KD_0|<1,$ and construct the matrix $\Lambda=\mathrm{diag}\{\mu_1,\ldots,\mu_{n+m}\},$ where $\mu_1,\ldots,\mu_{n+m}$ are the eigenvalues of the matrix A_0-KD_0.

2. Find the matrix $P\in R^{(n+m)\times(n+m)}$ of left eigenvectors of the matrix A_0-KD_0, where $P(A_0-KD_0)=\Lambda P.$

3. Compile the matrix M by selecting $n+m-p$ rows of the matrix P so that $M(A_0-KD_0)=\Lambda_M M$ and condition (13) is satisfied, where Λ_M is the submatrix of order $n+m-p$ of the matrix Λ corresponding to the matrix *M*.

4. Find the matrices $L=\Lambda_M,\ E=MK,$ and $T=MB$, and construct the observer (16).

Remark. The parameters of the observer (16) can be obtained using a different algorithm: first, choose an arbitrary nonzero matrix $L\in R^{(n+m-p)\times(n+m-p)}$ such that $|L|<1,$ and a constant matrix $E\in R^{(n+m-p)\times p}$; then, from the condition $ED_0=MA_0-LM$, find the unknown matrix $M\in R^{(n+m-p)\times(n+m)}$ and verify condition (13); and finally, find the matrix $T=MB\in R^{(n+m-p)\times q}$.

We proceed to the problem of stabilizing system (1) with the observer (16).

6. STABILIZATION OF A HYBRID SYSTEM WITH A MINIMUM-ORDER OBSERVER

Considering the expression (15), we find the estimate $\bar{z}_k=\bar{z}(t_k)$ of the state vector of system (1) at time instants $t_k=h_0k,\ k\in\bar{N}$. Then the stabilization of system (1) with the observer (16) reduces to that of the augmented system (1), (16); according to the separation principle and the results presented in [15], we arrive at the following.

Theorem 6. *Let the observable system* (6) *have a stabilizing control law* $u_k=Fz_k,\ \ k\in\bar{N}$, *with a gain matrix* $F\in R^{q\times(n+m)}$, *and let system* (16) *be its minimal observer. Then the solution* $x=0,\ y=0,\ \bar{s}=0$ *of the augmented system* (1), (16) *with the control law* $u(t_k)=F\bar{z}(t_k),\ \ k\in\bar{N},$ *is asymptotically stable.*

P r o o f. By Theorem 2, the pair $(A_0,\ D_0)$ is observable, and system (16) is an observer of system (6); therefore, the matrix $M\in R^{(n+m-p)\times(n+m)}$ in (16) satisfies conditions (13) and (15). Consequently,

$$z_{k+1}-\bar{z}_{k+1}=\begin{bmatrix}M\\D_0\end{bmatrix}^{-1}\left(\begin{bmatrix}s_{k+1}\\D_0z_{k+1}\end{bmatrix}-\begin{bmatrix}\bar{s}_{k+1}\\D_0z_{k+1}\end{bmatrix}\right)$$
$$=\begin{bmatrix}M\\D_0\end{bmatrix}^{-1}\begin{bmatrix}I\\O\end{bmatrix}p_{k+1}=Sp_{k+1},$$

where $S=\begin{bmatrix}M\\D_0\end{bmatrix}^{-1}\begin{bmatrix}I\\O\end{bmatrix}$. Hence, it follows that $\bar{z}_k=z_k-Sp_k$, where $p_k=s_k-\bar{s}_k,\ \ k\in\bar{N}.$

Based on the proof of Theorem 5, we write system (6) (without the output) and system (16) in the variables z_k and p_k with $u_k=F\bar{z}_k,\quad k\in\bar{N},$ then $z_{k+1}=A_0z_k+BF(z_k-Sp_k)+\xi(z_k,F(z_k-Sp_k),h_0)=(A_0+BF)z_k-BFSp_k+\xi(z_k,F(z_k-Sp_k),h_0),$ and $p_{k+1}=Lp_k,\ k\in\bar{N},$ where $|L|<1$. Consider the system

$$\begin{cases}z_{k+1}=(A_0+BF)z_k-BFSp_k+\xi(z_k,\ F(z_k-Sp_k),\ h_0)\\ p_{k+1}=Lp_k,\ \ k\in\bar{N},\end{cases}$$

or, in matrix notation,

$$\begin{bmatrix}z_{k+1}\\p_{k+1}\end{bmatrix}=G\cdot\begin{bmatrix}z_k\\p_k\end{bmatrix}+\bar{\xi}(z_k,\ p_k,\ h_0),\ \ k\in\bar{N},$$

where

$G=\begin{bmatrix}A_0+BF & -BFS\\ O & L\end{bmatrix}$ is a matrix of order $2(n+m)-p,$

and $\bar{\xi}(z_k,\ p_k,\ h_0)=\begin{bmatrix}\xi(z_k,\ F(z_k-Sp_k),\ h_0)\\ O\end{bmatrix}.$

Since $u_k=Fz_k,\quad k\in\bar{N},$ is a stabilizing control law for system (6), we have $|A_0+BF|<1$ [15] and $|L|<1$, hence $|G|<1.$ Moreover, $\xi(z_k,\ F(z_k-Sp_k),\ h_0)=o(\|z_k\|+\|p_k\|)$ as $\|z_k\|+\|p_k\|\to 0$, so $\bar{\xi}(z_k,\ p_k,\ h_0)=o(\|z_k\|+\|p_k\|)$ as $\|z_k\|+\|p_k\|\to 0$. Consequently, the resulting system has the asymptotically stable trivial solution $z=0,\ p=0$ [20].

Then the augmented system (6), (16) with the control law $u_k = F\bar{z}_k,\ k \in \bar{N},$ also has the asymptotically stable trivial solution $z = 0,\ \bar{s} = 0.$ Thus, the augmented system (1), (16) with the control law $u_k = F\bar{z}(t_k),\ k \in \bar{N},$ has the asymptotically stable trivial solution $x = 0,\ y = 0,\ \bar{s} = 0$ as well [20]. ♦

7. A NUMERICAL EXAMPLE

Consider the problem of constructing full- and minimum-order asymptotic observers for a nonlinear hybrid system described by

$$\begin{cases} x'(t) = x(t) - y(t_k) + y^2(t_k),\ t_k \le t < t_{k+1}, \\ y(t_{k+1}) = -x(t_{k+1}) - y(t_k) + 2u(t_k) + u(t_k)y(t_k) \\ w(t) = 4x(t) + 3y(t_k) - 2x(t)y(t_k),\ k \in \bar{N}, \end{cases}$$
$$x,\ y,\ w \in R,$$

and the problem of stabilizing this system using the observers.

According to Theorems 3 and 5, the desired observers can be constructed if the pair (A_0, D_0) is observable, i.e., under the condition $\operatorname{rank}\left[D_0^{\mathrm{T}},\ A_0^{\mathrm{T}} D_0^{\mathrm{T}}\right] = 2.$ Note that for $h_0 = \ln\frac{3}{2}$, this system is equivalent to a discrete system of the form (8) with

$$A_0 = \begin{bmatrix} 3/2 & -1/2 \\ -3/2 & -1/2 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 2 \end{bmatrix},$$

$$\xi(z_k, u_k,\ h_0) = \begin{bmatrix} y_k^2/2 \\ -y_k^2/2 + u_k y_k \end{bmatrix},$$

$$D_0 = [6\quad 1],\ \gamma(z_k,\ h_0) = 3y_k^2 - 3x_k y_k - y_k^3.$$

Therefore, for $h_0 = \ln\frac{3}{2}$, the pair (A_0, D_0) is observable.

Letting $K = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$ gives $A_0 - KD_0 = \begin{bmatrix} 1.5 - 6k_1 & -0.5 - k_1 \\ -1.5 - 6k_2 & -0.5 - k_2 \end{bmatrix}$, and the characteristic equation of the resulting matrix can be written as

$$\mu^2 + (6k_1 + k_2 - 1)\mu + 1.5(k_1 - 3k_2 - 1) = 0.$$

Now we find appropriate values of k_1 and k_2 such that $|\mu_{1,2}| < 1.$ For example, if $\mu_1 = -\frac{1}{2}$ and $\mu_2 = \frac{1}{3}$, then $K = \begin{bmatrix} 79/342 \\ -25/114 \end{bmatrix}.$ Thus, we obtain the following second-order observer (9) for this system:

$$\begin{cases} v(t_{k+1,1}) = \frac{13}{114}v(t_{k,1}) - \frac{125}{171}v(t_{k,2}) + \frac{79}{57}x(t_k) \\ \qquad + \frac{79}{342}y(t_k) + \frac{1}{2}y^2(t_k) \\ v(t_{k+1,2}) = -\frac{7}{38}v(t_{k,1}) - \frac{16}{57}v(t_{k,2}) + 2u(t_k) - \frac{25}{19}x(t_k) \\ \qquad - \frac{25}{114}y(t_k) - \frac{1}{2}y^2(t_k) + u(t_k)y(t_k),\ k \in \bar{N}. \end{cases} \tag{17}$$

Using the results of [15] or [16], we will design a stabilizing control law for the system $z_{k+1} = A_0 z_k + Bu_k + \xi(z_k,\ u_k,\ h_0)$. For instance, with $g = [1\ 0],\ \lambda_1 = -\frac{1}{4},$ and $\lambda_2 = \frac{2}{3},$ the algorithm described in [16] yields the stabilizing control law

$$u_k = \frac{53}{24}x_k - \frac{7}{24}y_k,\ k \in \bar{N}.$$

By Theorem 4, the stabilizing control law for the system with the observer (17) is

$$u(t_k) = \frac{53}{24}v(t_{k,1}) - \frac{7}{24}v(t_{k,2}),\ k \in \bar{N}.$$

Indeed, the discrete system (6) with the observer (17) and the above stabilizing control, which is equivalent to this system for $h_0 = \ln\frac{3}{2}$, takes the form

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \\ v_{k+1,1} \\ v_{k+1,2} \end{bmatrix} = \bar{G} \cdot \begin{bmatrix} x_k \\ y_k \\ v_{k,1} \\ v_{k,2} \end{bmatrix} + \eta(x_k,\ y_k,\ v_{k,1},\ v_{k,2}),$$

where

$$\bar{G} = \begin{bmatrix} 3/2 & -1/2 & 0 & 0 \\ -3/2 & -1/2 & 53/12 & -7/12 \\ 79/57 & 79/342 & 13/114 & -125/171 \\ -25/19 & -25/114 & 965/228 & -197/228 \end{bmatrix} \text{ and}$$

$$\eta(x_k,\ y_k,\ v_{k,1},\ v_{k,2}) = \begin{bmatrix} \frac{y_k^2}{2} \\ -\frac{y_k^2}{2} + \frac{53}{24}v_{k,1}y_k - \frac{7}{24}v_{k,2}y_k \\ \frac{y_k^2}{2} \\ -\frac{y_k^2}{2} + \frac{53}{24}v_{k,1}y_k - \frac{7}{24}v_{k,2}y_k \end{bmatrix},$$

$\sigma(\bar{G}) = \left\{-\frac{1}{2}, -\frac{1}{4}, \frac{1}{3}, \frac{2}{3}\right\}$ and $\eta(x_k,\ y_k,\ v_{k,1},\ v_{k,2}) = o\left(|x_k| + |y_k| + |v_{k,1}| + |v_{k,2}|\right)$ as $|x_k| + |y_k| + |v_{k,1}| + |v_{k,2}| \to 0$.

Therefore, the trivial solution of the constructed augmented system with the observer (17) is asymptotically stable. Then, due to Theorem 4, this nonlinear hybrid system with the observer (17) and the control law $u(t_k)=\frac{53}{24}v(t_{k,1})-\frac{7}{24}v(t_{k,2})$, $k\in\bar{N}$, has the asymptotically stable trivial solution $x=0, y=0, v=0$.

Let us construct the minimal observer (16) of order $n+m-p=1$ for this system. To synthesize such an observer, we will use the algorithm from Section 5 and the matrix $K=\begin{bmatrix}79/342\\ -25/114\end{bmatrix}$. As a result, $\Lambda=\begin{bmatrix}-1/2 & 0\\ 0 & 1/3\end{bmatrix}$, $P=\begin{bmatrix}3 & 10\\ -21 & 25\end{bmatrix}$, $M=[3\ 10]$, $\Lambda_M=-\frac{1}{2}$, $E=-\frac{3}{2}$, $ED_0=\begin{bmatrix}-9 & -\frac{3}{2}\end{bmatrix}$, and $T=20$. Consequently, the desired observer for estimating the vector $s(t_k)=Mz(t_k)$ of this system is given by

$$\bar{s}_{k+1}=-\frac{1}{2}\bar{s}_k-9x_k-\frac{3}{2}y_k+20u_k-\frac{7}{2}y_k^2+10u_ky_k. \quad (18)$$

For the matrix $\begin{bmatrix}M\\ D_0\end{bmatrix}=\begin{bmatrix}3 & 10\\ 6 & 1\end{bmatrix}$, $\begin{bmatrix}M\\ D_0\end{bmatrix}^{-1}=\begin{bmatrix}-1/57 & 10/57\\ 2/19 & -1/19\end{bmatrix}$, and the estimate (15) of the state vector $z_k=z(t_k)$ of this system takes the form $\bar{z}_k=\begin{bmatrix}\bar{x}_k\\ \bar{y}_k\end{bmatrix}=\begin{bmatrix}-1/57 & 10/57\\ 2/19 & -1/19\end{bmatrix}\cdot\begin{bmatrix}\bar{s}_k\\ D_0z_k\end{bmatrix}$. In other words,

$$\begin{cases}\bar{x}_k=-\frac{1}{57}\bar{s}_k+\frac{20}{19}x_k+\frac{10}{57}y_k\\ \bar{y}_k=\frac{2}{19}\bar{s}_k-\frac{6}{19}x_k-\frac{1}{19}y_k.\end{cases}$$

Let $u_k=\frac{53}{24}x_k-\frac{7}{24}y_k$, $k\in\bar{N}$, be the stabilizing control law of the corresponding system (6) for $h_0=\ln\frac{3}{2}$. We construct the stabilizing control law $u_k=\frac{53}{24}\bar{x}_k-\frac{7}{24}\bar{y}_k$, or equivalently, $u_k=\frac{29}{12}x_k+\frac{29}{72}y_k-\frac{5}{72}\bar{s}_k$, $k\in\bar{N}$. Then, for $h_0=\ln\frac{3}{2}$ and this control law, the augmented system—the equivalent discrete system and the constructed observer (18)— takes the form

$$\begin{bmatrix}x_{k+1}\\ y_{k+1}\\ \bar{s}_{k+1}\end{bmatrix}=\bar{G}\cdot\begin{bmatrix}x_k\\ y_k\\ \bar{s}_k\end{bmatrix}+\eta(x_k, y_k, \bar{s}_k),$$

where

$$\bar{G}=\begin{bmatrix}\frac{3}{2} & -\frac{1}{2} & 0\\ \frac{10}{3} & \frac{11}{36} & -\frac{5}{36}\\ \frac{118}{3} & \frac{59}{9} & -\frac{17}{9}\end{bmatrix},$$

$$\eta(x_k, y_k, \bar{s}_k)=\begin{bmatrix}\frac{1}{2}y_k^2\\ -\frac{7}{72}y_k^2-\frac{5}{72}\bar{s}_ky_k+\frac{29}{12}x_ky_k\\ \frac{19}{36}y_k^2-\frac{25}{36}\bar{s}_ky_k+\frac{145}{6}x_ky_k\end{bmatrix},$$

$\sigma(\bar{G})=\left\{-\frac{1}{2}, -\frac{1}{4}, \frac{2}{3}\right\}$, and $\eta(x_k, y_k, \bar{s}_k)=o(|x_k|+|y_k|+|\bar{s}_k|)$ as $|x_k|+|y_k|+|\bar{s}_k|\to 0$. Hence, the constructed augmented system with the observer has the asymptotically stable trivial solution. Then, by Theorem 6, the original hybrid system for $h_0=\ln\frac{3}{2}$ with the constructed minimal observer (18) and the stabilizing control law $u_k=\frac{29}{12}x_k+\frac{29}{72}y_k-\frac{5}{72}\bar{s}_k$, $k\in\bar{N}$, also has the asymptotically stable trivial solution. ♦

CONCLUSIONS

In this paper, we have established an observability criterion for a nonlinear deterministic continuous–discrete (hybrid) system with a continuous output and a constant sampling step, as well as a necessary condition for the observability of an equivalent nonlinear discrete system with a discrete output. Mathematical models and sufficient conditions for the existence of full- and minimum-order asymptotic observers for a nonlinear hybrid system have been presented; in addition, sufficient conditions for the stabilization of a nonlinear hybrid system with the constructed observers have been provided. Algorithms for constructing asymptotic observers have been demonstrated on a numerical example, and the corresponding augmented nonlinear control systems with observers, which have asymptotically stable trivial solutions, have been obtained.

The results of this paper may be useful when addressing the issues of effective control of nonlinear deterministic hybrid dynamic systems with a constant sampling step, particularly control of such systems in a bifurcation regime [27].

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Author information

Akmanova, Svetlana Vladimirovna. Cand. Sci. (Ped.), Nosov Magnitogorsk State Technical University, Magnitogorsk, Russia
✉ svet.akm_74@mail.ru
ORCID iD: https://orcid.org/0000-0002-7211-1610

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Translated into English by *Alexander Yu. Mazurov*,
Cand. Sci. (Phys.–Math.),
Trapeznikov Institute of Control Sciences,
Russian Academy of Sciences, Moscow, Russia
✉ alexander.mazurov08@gmail.com