ANALYSIS OF A TIME-DELAY CONSENSUS PROTOCOL WITH COMPLEMENTARY LINKS

R. P. Agaev* and D. K. Khomutov**

*,**Trapeznikov Institute of Control Sciences, Russian Academy of Sciences, Moscow, Russia

*✉ agaraf3@gmail.com, **✉ homutov_dk@mail.ru

Abstract. This paper introduces a new consensus protocol in a multi-agent system (MAS) with complementary links and a time delay. According to this protocol, agents in an MAS receive information from some agents immediately, and from the others with a delay. This protocol is a special case of a more general model, commonly applied to analyze traffic flows, forecast price dynamics in commodity markets, and describe biological and epidemiological processes. A stability condition is derived under which the protocol converges regardless of the delay. Tsypkin's test is used to obtain the boundary delay, which depends on the spectrum of the Laplacian matrix. The frameworks of algebraic graph theory, control theory, and operational calculus are employed to establish the asymptotic behavior of a stable MAS for different initial conditions and, consequently, a condition for reaching consensus for any initial value vector. As demonstrated, if the influence digraph is balanced, the time-delay consensus protocol will converge to an average consensus.

Keywords: multi-agent system (MAS), time-delay consensus protocol, Laplacian matrix, Tsypkin's test, eigenprojector.

INTRODUCTION

Time-delay (delayed) systems, particularly multi-agent systems (MASs), are of great interest for research and arise in many fields. In such systems, delay is due to physical and/or biological constraints. In traffic flow models, delay is caused by the human factor, i.e., the driver's time to identify a change in the road situation, his reaction time, etc. [1–4]. Such models are applied in vehicle automation to ensure road safety and the efficient utilization of highways; for example, see [5].

Delays also occur in models describing price dynamics in commodity markets [6]. The price of a commodity depends on its supply and demand at a given time instant, which depend, in turn, on the information about past prices and production delays, respectively. The factor of production delays influences the stability of market models and may lead to instability, whereas delays in receiving price information in interconnected markets may have a stabilizing effect [7]. Delays in production processes arise from the time required for transporting materials, setting up equipment, and the production itself [8].

Delays are also common in models of biological processes [9–11]. When modeling cell growth, one shall consider the time required for cell division [12]. A similar principle is used in epidemiological models [13]. In population growth models, a new individual can produce offspring only after maturing and becoming reproductive [14]. The Volterra population growth model takes into account the decreasing growth rate when increasing population density. Therefore, the population growth rate depends on past population size dynamics; this feature is reflected, e.g., by the "predator–prey" model [15]. In genetic networks, delay arises due to the reaction time. Some reactions ongoing at a fixed time instant do not affect the current state of the system but will affect its future states [16]. In the respiratory process model, it is necessary to consider the delayed character of carbon dioxide circulation in the blood [17].

Most models with delay are based on the interaction between the rate of change of different variables and their values at previous time instants, which depend on the delay. Continuous-time dynamic models are described by differential and functional-differential equations with a deviating argument and by delayed systems, and their stability is of great interest. Since the 1950s, many classical results have appeared in this field; for example, see [18–21]. The stability of such systems is closely related to that of quasipolynomials, a topic of numerous publications as well, e.g., [22–25].

Multi-agent systems with informational interactions occupy a central place among all MASs. Agents receive data from their neighbors, and the data modification process takes place. In this way, agents coordinate their characteristics (reach a consensus) [26–29]. For such systems, a topical problem is to establish conditions under which agents will reach a consensus regardless of their initial state. It is equally important to find the resulting vector of the agent's characteristics. In particular, this problem reduces to finding the consensus value.

In this paper, we consider a consensus protocol in an MAS with informational influences and complementary links. Agents receive information from some agents immediately, and from the others with a delay. In the case of a continuous protocol, Laplacian matrices are constructed and used in a system of differential equations with delay. Section 1 presents the necessary conceptual background on MASs and the auxiliary results obtained earlier. In Section 2, we find the stability region of the MAS protocol. Section 3 is devoted to the asymptotic behavior of the MAS protocol. In particular, the problem of the protocol's convergence to consensus for any initial value vector is solved.

1. CONCEPTUAL BACKGROUND AND AUXILIARY RESULTS

Multi-agent systems with information influences and a set of agents $V=\{1,\dots,n\}$ can be conveniently represented as a weighted digraph $\Gamma=\langle V, E\rangle$. In this graph, $E\subset V\times V$ is the set of arcs. If agent j influences agent i with a weight a_{ij}, then there exists an arc from vertex j to vertex i with this weight. The matrix $A=\left(a_{ij}\right)$ is called the communication matrix.

An outgoing forest (out-forest) in a graph Γ is a subgraph of Γ in which there are no cycles, and each vertex has at most one incoming arc. Vertex j is the root of an out-forest if it has no incoming arcs. The set $\mathcal{F}_k(\Gamma)$ consists of all possible out-forests in a digraph Γ containing k arcs. The set $\mathcal{F}_k^{j\to i}(\Gamma)$ consists of all possible out-forests in a digraph Γ in which j is the root and vertex i is reachable from vertex j.

Definition 1. The weight of an out-forest is the product of the weights of all its arcs. The weight of the set $\mathcal{F}_k(\Gamma)$ is the sum of the weights of all possible out-forests in a digraph Γ containing k arcs; it is denoted by $\epsilon\left(\mathcal{F}_k(\Gamma)\right)$.

Definition 2. A base bicomponent of a digraph Γ is a subgraph K in which all vertices are mutually reachable, and there are no arcs from $\Gamma\setminus K$ to K.

Definition 3. The Laplacian matrix L corresponding to a digraph Γ is given by

$$l_{ij}=\begin{cases}-a_{ij}, & j\neq i,\\ \sum\limits_{k\neq i}a_{ik}, & j=i.\end{cases}\ \blacklozenge$$

Also, a digraph Γ is said to correspond to a Laplacian matrix $L\in\mathbb{R}_{n\times n}$ if this digraph is constructed on the vertex set $\{1,\dots,n\}$, and each arc $(j,\, i)$ with a weight $\left|l_{ij}\right|$ belongs to the arc set if and only if $l_{ij}<0$.

By definition, obviously, $L\mathbf{1}=\mathbf{0}$, where $\mathbf{1}$ and $\mathbf{0}$ stand for the n-dimensional column vectors composed of ones and zeros, respectively. Thus, 0 is an eigenvalue of a Laplacian matrix. Let $\sigma(L)=(\lambda_1,\dots,\lambda_n)$ denote the spectrum of a matrix L (the set of all its eigenvalues); it will be considered with due account of multiplicity. In addition, let $\sigma'(L)\subset\sigma(L)$ denote the spectrum of a matrix L excluding the zero eigenvalue.

Definition 4. The index v of a square matrix A is the size of the largest Jordan cell with zero diagonal, or the smallest number v such that $\operatorname{rank}\left(A^v\right)=\operatorname{rank}\left(A^{v+1}\right)$.

Proposition 1. [29] *The index of a Laplacian matrix* L *equals 1.*

Definition 5. [30] The eigenprojector of a square matrix A corresponding to the zero eigenvalue is an idempotent matrix $A^{\vdash}$ such that $\mathcal{R}\left(A^{\vdash}\right)=\mathcal{N}\left(A^v\right)$ and $\mathcal{R}\left(A^v\right)=\mathcal{N}\left(A^{\vdash}\right)$, where v is the index of A, and $\mathcal{R}(\cdot)$ and $\mathcal{N}(\cdot)$ indicate the image and kernel of an appropriate matrix. ♦

Let I denote an identity matrix of dimensions $n\times n$.

Theorem 1. [31] *Assume that a digraph Γ corresponds to a Laplacian matrix L. Then:*

1. The matrix $L^{\vdash}$ is stochastic and idempotent.

2. $\left(I+pL\right)^{-1} = \frac{1}{r_\Gamma\left(p\right)}\left(Q_0\left(\Gamma\right) + pQ_1(\Gamma) + ... + p^{n-l}Q_{n-l}(\Gamma)\right)$, *where* $p>0$, $r_\Gamma\left(p\right) = \sum_{k=0}^{n-l} p^k \epsilon\left(\mathcal{F}_k\left(\Gamma\right)\right)$, $Q_k = \left(q_{ij}^{(k)}\right)$, $q_{ij}^{(k)} = \epsilon\left(\mathcal{F}_k^{j\to i}\left(\Gamma\right)\right)$, *and l is the number of base bicomponents in the digraph Γ.*

3. $\lim\limits_{p\to\infty}\left(I+pL\right)^{-1} = \frac{1}{\epsilon\left(\mathcal{F}_{n-l}\left(\Gamma\right)\right)}Q_{n-l}\left(\Gamma\right) = L^{\vdash}$.

In this paper, we consider a first-order consensus protocol in an MAS of the form

$$\begin{cases} \dot{x}\left(t\right)=u\left(t\right),\ t>0, \\ u\left(t\right) = -L_1x\left(t\right) - \gamma L_2 x\left(t-\tau\right) \\ x\left(\theta\right)=\phi\left(\theta\right),\ \theta\in\left[-\tau,\,0\right], \end{cases} \qquad (1)$$

where L_1 and L_2 are normalized Laplacian matrices ($-\frac{1}{n}\le l_{ij}\le 0$), the matrix $K = L_1 + L_2$ is the normalized matrix of a complete graph with an equal weight of all arcs, and $\gamma\in\left(0,1\right]$.

Definition 6. The protocol (1) is said to be stable if the solution $x\left(t\right)$ of the corresponding system has a finite limit $\lim\limits_{t\to\infty} x\left(t\right)$.

Definition 7. The protocol (1) is said to converge to a consensus (*reach an asymptotic consensus*) if the limit of the solution $x\left(t\right)$ of the corresponding system can be written as $c\mathbf{1}$, where c is a constant depending on the initial function $\phi\left(\theta\right)$. ♦

The protocol (1) is a special case of the time-delay system

$$\dot{x}\left(t\right) = A_0 x\left(t\right) + A_1 x\left(t-\tau\right). \qquad (2)$$

This system arises in many models across various fields. In the gene transcription and translation model, the matrix A_0 describes the transition probabilities between lattice nodes. In addition, various chemical reactions influence the transition process with a delay, which is reflected by the matrix A_1. Since chemical reactions proceed over different time scales, system (2) may include several elements with different delays [16]. In the blood cell count model, the matrix of the element without delay is responsible for the rate of blood cell loss during circulation. The element with delay accounts for the blood flow from previous compartments [11]. A similar principle is used in the Hopfield neural network: the components of the vector $x\left(t\right)$ are the input voltages of the corresponding neurons. The rate of change of the voltage depends on the neuron's resistance and the transmission of the electrical signal between neurons, which occurs with some delay due to limited propagation speed [32]. Such models generally include the transmission function F:

$$\dot{x}\left(t\right) = A_0 x\left(t\right) + A_1 F\left(x\left(t-\tau\right)\right).$$

2. THE MAIN RESULTS

The characteristic (transfer) function of system (1) is

$$F\left(z\right) = \det\left(zI + L_1 + \gamma L_2 e^{-\tau z}\right).$$

Proposition 2.

$$F\left(z\right) = z\prod_{\lambda\in\sigma'\left(L_1\right)}\left(z+\lambda+\gamma e^{-\tau z}\left(1-\lambda\right)\right).$$

P r o o f. The matrix K can be represented as $K = I - \frac{1}{n}E$, where the matrix E consists of ones. Let the Jordan form be $\Lambda_K = \mathrm{diag}\left(0,1,...,1\right)$, and let S be the matrix transforming K into Λ_K:

$$S^{-1}KS = \Lambda_K.$$

Recall that $\mathbf{1}$ is the right eigenvector corresponding to the zero eigenvalue. Therefore, let $\mathbf{1}$ be the first column of the matrix S. It follows that

$$S^{-1}L_1S = \begin{pmatrix} 0 & * \\ \mathbf{0}_{n-1,1} & L_1^S \end{pmatrix},$$

where L_1^S is some matrix, and $\sigma\left(L_1^S\right) = \sigma'\left(L_1\right)$. Then

$$F\left(z\right) = \det\left(S^{-1}\left(zI + L_1 + \gamma e^{-\tau z}L_2\right)S\right)$$

$$= \det\left(zI + S^{-1}L_1S + \gamma e^{-\tau z}S^{-1}\left(K-L_1\right)S\right)$$

$$= \det\left(zI + \gamma e^{-\tau z}S^{-1}KS + \left(1-\gamma e^{-\tau z}\right)S^{-1}L_1S\right)$$

$$= \det\left(zI + \gamma e^{-\tau z}\Lambda_K + \left(1-\gamma e^{-\tau z}\right)\begin{pmatrix} 0 & * \\ \mathbf{0}_{n-1,1} & L_1^S \end{pmatrix}\right)$$

$$= \det\left(\begin{pmatrix} z & & & \\ & z+\gamma e^{-\tau z} & & \\ & & \ddots & \\ & & & z+\gamma e^{-\tau z} \end{pmatrix} + \left(1-\gamma e^{-\tau z}\right)\begin{pmatrix} 0 & * \\ \mathbf{0}_{n-1,1} & L_1^S \end{pmatrix}\right).$$

Denoting by I_{n-1} an identity matrix of dimensions $(n-1)\times(n-1)$, we finally have

$$F(z) = z\ \det\left(\left(z+\gamma e^{-\tau z}\right)I_{n-1} + \left(1-\gamma e^{-\tau z}\right)L_1^S\right)$$

$$= z \prod_{\lambda\in\sigma'(L_1)} \left(z+\gamma e^{-\tau z} + \left(1-\gamma e^{-\tau z}\right)\lambda\right)$$

$$= z \prod_{\lambda\in\sigma'(L_1)} \left(z+\lambda+\gamma e^{-\tau z}(1-\lambda)\right). \ \blacklozenge$$

Remark 1. If $\gamma=1$, then $F(z)$ takes the form $F(z)=\det\left(zI+L_1+e^{-\tau z}L_2\right)=z\prod_{\lambda\in\sigma'(L_1)}\left(z+\lambda+e^{-\tau z}(1-\lambda)\right)$.

This can be interpreted as follows: if $\lambda\in\sigma'(L_1)$, then $(1-\lambda)\in\sigma'(L_2)$, which complements the result obtained in [33]. ♦

Note that the first zero eigenvalue of the matrix L_1 does not affect the stability of the protocol (1), and the characteristic function of (1) consists of the product of those of the scalar equations:

$$\dot{y}(t) = -\lambda y(t) - \gamma(1-\lambda)y(t-\tau). \qquad (3)$$

If the solution of (3) is stable for all $\lambda\in\sigma'(L_1)$, the protocol (1) will be stable as well.

If $\lambda=0$, equation (3) will become a scalar one with a delay of the form

$$\dot{y}(t) = -\gamma y(t-\tau). \qquad (4)$$

For equation (4), the boundary delay can be obtained, e.g., as a consequence of the result established in [34].

Proposition 3. *The solution of equation* (4) *is stable if* $\tau < \dfrac{\pi}{2\gamma}$.

For the other cases, we write the characteristic function of equation (3) as

$$\begin{aligned} f(z) &= z+\lambda+\gamma(1-\lambda)e^{-\tau z} \\ &= Q(z) - P(z)e^{-\tau z}. \end{aligned} \qquad (5)$$

The stability of the quasipolynomial (5) can be analyzed using Tsypkin's test, which is based on the Nyquist criterion. Let the quasipolynomial $Q(z)-P(z)e^{-\tau z}$ be the characteristic function of the equivalent system with delayed feedback. We will assess stability by the amplitude-phase frequency response of the corresponding open-loop system:

$$W_\tau(is) = \frac{P(is)}{Q(is)}e^{-i\tau s} = -\frac{\gamma(1-\lambda)}{is+\lambda}e^{-i\tau s}.$$

The characteristic function of the open-loop system is given by $Q(z)=z+\lambda$. Since λ has a positive real part, the open-loop system will be stable. The amplitude-phase frequency response of the system with delayed feedback is given by $\dfrac{W_\tau}{1-W_\tau}$. By the Nyquist criterion, the solution of equation (3) is stable if and only if the Nyquist plot of $W_\tau(is)$ does not encircle the point $(1, i0)$.

Let $\alpha=\mathrm{Re}(\lambda)$, $\beta=\mathrm{Im}(\lambda)$, and

$$W_\tau(is) = W_0(is)e^{-i\tau s}.$$

Then

$$\begin{aligned} W_0(is) &= -\frac{\gamma(1-\alpha)-i\gamma\beta}{is+\alpha+i\beta} \\ &= -\gamma\frac{(1-\alpha)\alpha-\beta(s+\beta)}{\alpha^2+(\beta+s)^2} \\ &\quad -i\gamma\frac{-\alpha\beta-(1-\alpha)(s+\beta)}{\alpha^2+(\beta+s)^2} = u+iv. \end{aligned} \qquad (6)$$

Proposition 4. *For* $\gamma=1$, *the transformation* $W_0(z)$ *maps the imaginary axis to the circle*

$$\left(u-\frac{\alpha-1}{2\alpha}\right)^2 + \left(v-\frac{\beta}{2\alpha}\right)^2 = R^2,$$

where $R=\dfrac{\sqrt{(\alpha-1)^2+\beta^2}}{2\alpha}$.

P r o o f. Let $z=x+iy$ and $w=W_0(z)$. Obviously, $x=\dfrac{z+\bar{z}}{2}$, where $\bar{z}$ indicates the complex conjugate of z. Since $w=-\dfrac{1-\lambda}{z+\lambda}$, the number z can be expressed in the form

$$z = \frac{\lambda-1}{w} - \lambda = \frac{(\lambda-1)\bar{w}}{|w|^2} - \lambda.$$

Then

$$\bar{z} = \frac{(\bar{\lambda}-1)w}{|w|^2} - \bar{\lambda}.$$

In view of $z = iy$, we have $x = \frac{z + \bar{z}}{2} = 0$, which can be written as

$$0 = \frac{1}{2}\left(\frac{(\lambda - 1)\,\bar{w}}{|w|^2} - \lambda + \frac{(\bar{\lambda} - 1)w}{|w|^2} - \bar{\lambda} \right)$$

$$= \frac{1}{2}\left(\frac{(\alpha - 1 + i\beta)(u - iv)}{u^2 + v^2} - \alpha - i\beta \right.$$

$$\left. + \frac{(\alpha - 1 - i\beta)(u + iv)}{u^2 + v^2} - \alpha + i\beta \right)$$

$$= \frac{1}{2}\left(-2\alpha + \frac{2(\alpha - 1)u + 2\beta v}{u^2 + v^2} \right)$$

$$= -\alpha + \frac{(\alpha - 1)u + \beta v}{u^2 + v^2} \Rightarrow \alpha = \frac{(\alpha - 1)u + \beta v}{u^2 + v^2}.$$

Thus,

$$u^2 + v^2 = \frac{\alpha - 1}{\alpha}u + \frac{\beta}{\alpha}v \Rightarrow u^2 - \frac{\alpha - 1}{\alpha}u$$

$$+ \frac{(\alpha - 1)^2}{4\alpha^2} + v^2 - \frac{\beta}{\alpha}v + \frac{\beta^2}{4\alpha^2} = \frac{(\alpha - 1)^2}{4\alpha^2} + \frac{\beta^2}{4\alpha^2}.$$

In the final analysis, we arrive at

$$\left(u - \frac{\alpha - 1}{2\alpha} \right)^2 + \left(v - \frac{\beta}{2\alpha} \right)^2 = \frac{(\alpha - 1)^2 + \beta^2}{4\alpha^2}. \ \blacklozenge$$

The following obvious expression will be needed for further considerations:

$$|W_\tau(is)| = |W_0(is)| = \sqrt{\frac{\gamma^2\left((1 - \alpha)^2 + \beta^2\right)}{\alpha^2 + (s + \beta)^2}}.$$

Proposition 5.

1. *If* $\gamma^2(1 - \alpha)^2 + \gamma^2\beta^2 < \alpha^2$ *for all* $\lambda = \alpha + i\beta \in \sigma'(L_1)$, *then the stability of the protocol* (1) *is independent of the delay.*

2. *Assume that undirected graphs* G_1 *and* G_2 *correspond to the matrices* $L_1 = \left(l_{ij}^1\right)$ *and* $L_2 = \left(l_{ij}^2\right)$, *respectively (see Definition 3),* $\gamma = 1$, *and* $\min_i l_{ii}^1 > \frac{3n - 4}{4n}$. *Then the stability of the protocol* (1) *is independent of the delay.*

P r o o f.

1. The validity of this statement follows directly from the above expression for $|W_0(is)|$.

2. Since G_1 and G_2 are undirected graphs, the matrices L_1 and L_2 have real spectra. Then, by Gershgorin's circle theorem,

$$\max_{\lambda_i \in \sigma(L_2)} \lambda_i \le 2\max_i l_{ii}^2.$$

Because $L_1 + L_2 = K$, the maximum element of the matrix L_2 can be expressed in terms of the minimum element of L_2 as

$$\max_i l_{ii}^2 = \frac{n - 1}{n} - \min_i l_{ii}^1 = 1 - \min_i l_{ii}^1 - \frac{1}{n}.$$

Then the second smallest eigenvalue of the Laplacian matrix for the graph G_1 (i.e., the smallest value in the spectrum $\sigma'(L_1)$) can be estimated as

$$\min_{\lambda_i \in \sigma'(L_1)} \lambda_i = 1 - \max_{\lambda_i \in \sigma(L_2)} \lambda_i \ge 1 - 2\max_i l_{ii}^2$$
$$= 1 - 2\left(1 - \min_i l_{ii}^1 - \frac{1}{n}\right) = 2\min_i l_{ii}^1 - 1 + \frac{2}{n}.$$

Hence,

$$\min_{\lambda_i \in \sigma'(L_1)} \lambda_i \ge 2\min_i l_{ii}^1 - 1 + \frac{2}{n}.$$

Based on the condition $\min_i l_{ii}^1 > \frac{3n - 4}{4n}$, it follows that

$$\min_{\lambda_i \ne 0} \lambda_i \ge 2\min_i l_{ii}^1 - 1 + \frac{2}{n} > 2\frac{3n - 4}{4n} - 1 + \frac{2}{n}$$
$$= \frac{6n - 8 - 4n + 8}{4n} = \frac{1}{2}.$$

Thus, if $\min_i l_{ii}^1 > \frac{3n - 4}{4n}$, we have $\lambda > \frac{1}{2}$, meaning (see Lemma 5) that the stability of the protocol (1) is independent of the delay. ♦

Theorem 2. *Assume that the set*

$$\Lambda = \left\{ \lambda \in \sigma'(L_1) \mid \gamma^2(1 - \alpha)^2 + \gamma^2\beta^2 \ge \alpha^2 \right\}$$

is non-empty. For all real $\lambda \in \Lambda$, *let*

$$\tau_\lambda = \frac{\arccos\left(-\frac{\lambda}{\gamma(1 - \lambda)} \right)}{\sqrt{\gamma^2(1 - \lambda)^2 - \lambda^2}},$$

and for complex conjugate pairs $\lambda = \alpha \pm i\beta$, *let*

$$\tau_\lambda^{1,2} = \frac{\arccos\left(-\frac{(1 - \alpha)\alpha \mp \beta\sqrt{\gamma^2(1 - \alpha)^2 + \gamma^2\beta^2 - \alpha^2}}{\gamma\left((1 - \alpha)^2 + \beta^2\right)} \right)}{\sqrt{\gamma^2(1 - \alpha)^2 + \gamma^2\beta^2 - \alpha^2} \mp \beta}.$$

Then the protocol (1) *is stable for any* $\tau < \tau_0 = \min_{\lambda\in\Lambda} \tau_\lambda$, *where*

$$\tau_\lambda = \min\left\{\left|\tau_\lambda^1\right|, \tau_\lambda^2\right\}.$$

P r o o f. We find the critical frequencies for the scalar equation (3) with a fixed λ. From $\left|W_\tau(is)\right| = 1$ it follows that

$$s + \beta = \pm\sqrt{\gamma^2(1-\alpha)^2 + \gamma^2\beta^2 - \alpha^2}.$$

Then the point $(1, 0)$ lies on the Nyquist plot of $W_\tau(is)$ if

$$\arg\left(W_0(is)\right) - \tau s = 2\pi n,$$

$$\cos\left(\arg\left(W_0(is)\right)\right) = \cos(\tau s).$$

Note that $\cos\left(\arg\left(W_0(is)\right)\right) = u$; therefore,

$$\tau = \frac{\arccos(u)}{s}.$$

Substituting the critical value of s into formula (6) yields

$$u = -\frac{(1-\alpha)\alpha \mp \beta\sqrt{\gamma^2(1-\alpha)^2 + \gamma^2\beta^2 - \alpha^2}}{\gamma\left((1-\alpha)^2 + \beta^2\right)}.$$

Since the function $\arccos(u)$ is positive, only positive critical frequencies s are needed to find the boundary delay.

Consider the case $\lambda \in R$, i.e., $\alpha = \lambda$ and $\beta = 0$. The critical frequencies are $s = \pm\sqrt{\gamma^2(1-\lambda)^2 - \lambda^2}$. For the positive critical frequency, the boundary delay is

$$\tau_\lambda = \frac{\arccos\left(-\frac{\lambda}{\gamma(1-\lambda)}\right)}{\sqrt{\gamma^2(1-\lambda)^2 - \lambda^2}}. \tag{7}$$

With $\lambda = 0$ substituted into formula (7), we get the boundary delay from Lemma 3. In other words, formula (7) generalizes the result of Lemma 3.

If λ is a complex number and $\beta > 0$, both critical frequencies will be negative under the condition $|\lambda|^2 > |\gamma(1-\lambda)|^2$: in this case, the stability of the solution of equation (3) will not depend on the delay. For the convenience of considerations, we take the complex conjugate pair $\lambda = \alpha \pm i\beta$. Then the following delay values correspond to the two critical frequencies:

$$\tau_\lambda^{1,2} = \frac{\arccos\left(-\frac{(1-\alpha)\alpha \mp \beta\sqrt{\gamma^2(1-\alpha)^2 + \gamma^2\beta^2 - \alpha^2}}{\gamma\left((1-\alpha)^2 + \beta^2\right)}\right)}{\sqrt{\gamma^2(1-\alpha)^2 + \gamma^2\beta^2 - \alpha^2} \mp \beta},$$

$$\tau_\lambda^{3,4} = \frac{\arccos\left(-\frac{(1-\alpha)\alpha \pm \beta\sqrt{\gamma^2(1-\alpha)^2 + \gamma^2\beta^2 - \alpha^2}}{\gamma\left((1-\alpha)^2 + \beta^2\right)}\right)}{-\sqrt{\gamma^2(1-\alpha)^2 + \gamma^2\beta^2 - \alpha^2} \mp \beta}.$$

We emphasize that $\tau_\lambda^3 < 0$ and $\tau_\lambda^1 = -\tau_\lambda^4$. Then, for the complex conjugate pair λ, the boundary delay is $\tau_\lambda = \min\left\{\left|\tau_\lambda^1\right|, \tau_\lambda^2\right\}$, and the solutions of all the scalar equations (3) under consideration are stable for any $\tau < \tau_0 = \min_{\lambda\in\Lambda} \tau_\lambda$. Thus, the protocol (1) is stable. ♦

3. THE ASYMPTOTIC VALUE OF THE COORDINATED CHARACTERISTIC

The following auxiliary results are required to study the asymptotic behavior of the protocol (1); due to the extended final value theorem [35], it is assumed that $s \in \mathbb{R}^+$.

Proposition 6. *For* $s \in \mathbb{R}^+$ *and arbitrary Laplacian matrices* L_1 *and* L_2,

$$\lim_{s\to 0}\left(I + \frac{1}{s}\left(L_1 + e^{-\tau s}L_2\right)\right)^{-1}$$
$$= \lim_{s\to 0}\left(I + \frac{1}{s}\left(L_1 + L_2\right)\right)^{-1} = \left(L_1 + L_2\right)^{\vdash}.$$

P r o o f. Let digraphs Γ_1 and Γ_2 correspond to the matrices $L_1 + e^{-\tau s}L_2$ and $L_1 + L_2$, respectively, and let $L_1 = \left(l_{ij}^1\right)$, $L_2 = \left(l_{ij}^2\right)$. Then the arc from vertex j to vertex i in the digraph Γ_1 has the weight $\left|l_{ij}^1 + e^{-\tau s}l_{ij}^2\right|$, and the same arc in the digraph Γ_2 has the weight $\left|l_{ij}^1 + l_{ij}^2\right|$. If the forest F in the digraph Γ_2 has the weight $\prod_{k, i_k \ne j_k}\left|l_{i_k j_k}^1 + l_{i_k j_k}^2\right|$, the analogous forest in the digraph Γ_1 will have the weight $\prod_{k, i_k \ne j_k}\left|l_{i_k j_k}^1 + e^{-\tau s}l_{i_k j_k}^2\right|$.

Due to Theorem 1,

$$\lim_{s\to 0}\left(I+\frac{1}{s}\left(L_1+e^{-\tau s}L_2\right)\right)^{-1}$$

$$=\lim_{s\to 0}\frac{1}{r_{\Gamma_1}\left(\frac{1}{s}\right)}\left(Q_0\left(\Gamma_1\right)+\frac{1}{s}Q_1\left(\Gamma_1\right)+\ldots+\frac{1}{s^{n-l}}Q_{n-l}\left(\Gamma_1\right)\right),$$

where l is the number of base components in the digraph Γ_1. Note that the graph Γ_2 contains the same number of bicomponents, and $\epsilon\left(\mathcal{F}_k\left(\Gamma_1\right)\right)\to\epsilon\left(\mathcal{F}_k\left(\Gamma_2\right)\right)$ as $s\to 0$. Then, multiplying the numerator and denominator by s^{n-l}, we obtain

$$\lim_{s\to 0}\left(I+\frac{1}{s}\left(L_1+e^{-\tau s}L_2\right)\right)^{-1}=\frac{1}{\epsilon\left(\mathcal{F}_{n-l}\left(\Gamma_2\right)\right)}Q_{n-l}\left(\Gamma_2\right).$$

On the other hand, by Theorem 1,

$$\lim_{s\to 0}\left(I+\frac{1}{s}\left(L_1+L_2\right)\right)^{-1}$$

$$=\frac{1}{\epsilon\left(\mathcal{F}_{n-l}\left(\Gamma_2\right)\right)}Q_{n-l}\left(\Gamma_2\right)=\left(L_1+L_2\right)^{\vdash}.$$

And finally, $\lim_{s\to 0}\left(I+\frac{1}{s}\left(L_1+e^{-\tau s}L_2\right)\right)^{-1}=\lim_{s\to 0}\left(I+\frac{1}{s}\left(L_1+L_2\right)\right)^{-1}$ $=\left(L_1+L_2\right)^{\vdash}$. ♦

Proposition 7. *If* L_1 *and* L_2 *are normalized Laplacian matrices and* $L_1+L_2=K$, *then*

$$\left(L_1+\gamma L_2\right)^{\vdash}=\frac{1}{n}E\left(I+\frac{1-\gamma}{\gamma}L_1\right)^{-1}.$$

P r o o f. In the case $\gamma\in(0,1)$, we represent the matrix $\left(L_1+\gamma L_2\right)$ as

$$\left(L_1+\gamma L_2\right)=\left(L_1+\gamma K-\gamma L_1\right)$$

$$=\left((1-\gamma)L_1+\gamma K\right)=(1-\gamma)\left(L_1+\frac{\gamma}{1-\gamma}K\right).$$

In view of [36, Lemma 8],

$$\begin{aligned}\left(L_1+\gamma L_2\right)^{\vdash}&=\left((1-\gamma)\left(L_1+\frac{\gamma}{1-\gamma}K\right)\right)^{\vdash}\\&=\frac{1}{n}E\left(I+\frac{1-\gamma}{\gamma}L_1\right)^{-1}.\end{aligned}\qquad(8)$$

If $\gamma=1$, we have $L_1+\gamma L_2=K$ and $K^{\vdash}=\frac{1}{n}E$, which is a special case of formula (8) with $\gamma=1$. ♦

Proposition 6 can also be proven using the finite value theorem and Faddeev's method.

Remark 2. Note that $\lim_{\gamma\to 0}\left(L_1+\gamma L_2\right)^{\vdash}=\frac{1}{n}EL_1^{\vdash}$. If 0 is a simple eigenvalue of the matrix L_1, then $\frac{1}{n}EL_1^{\vdash}=L_1^{\vdash}$. However, in the general case, $\frac{1}{n}EL_1^{\vdash}\neq L_1^{\vdash}$ since $\frac{1}{n}EL_1^{\vdash}$ is a matrix of rank 1. Based on this fact, a latent consensus protocol with weak background links is designed; with this protocol, agents in an MAS with a weakly connected structure reach consensus for any initial value vector (for more details, see [36]).

Theorem 3. *Assume that the protocol* (1) *is stable and* $x(t)$ *is the solution of system* (1). *In this case, the following results are valid:*

1. If $\phi(\theta)=x_0$, *then*

$$\lim_{t\to\infty}x(t)=\frac{1}{n}E\left(I+\frac{1-\gamma}{\gamma}L_1\right)^{-1}\left(I-\gamma\tau L_2\right)x_0.$$

2. If

$$\phi(\theta)=\begin{cases}0, & \theta\in[-\tau,0),\\ x_0, & \theta=0,\end{cases}\qquad(9)$$

then

$$\lim_{t\to\infty}x(t)=\frac{1}{n}E\left(I+\frac{1-\gamma}{\gamma}L_1\right)^{-1}x(0).$$

P r o o f.

1. Letting $\phi(\theta)=x_0$, we consider the Laplace transform of the function $x(t-\tau)$:

$$\mathcal{L}\{x(t-\tau)\}=\int_0^{\infty}e^{-ts}x(t-\tau)\,dt.$$

For $u=t-\tau$,

$$\int_0^{\infty}e^{-ts}x(t-\tau)\,dt=\int_{-\tau}^{\infty}e^{-(u+\tau)s}x(u)\,du$$

$$=\int_{-\tau}^{0}e^{-(u+\tau)s}x(u)\,du+\int_0^{\infty}e^{-(u+\tau)s}x(u)\,du.$$

Since $x(\theta)=\phi(\theta)=x_0$, $\theta\in[-\tau,\,0]$, we have

$$\mathcal{L}\{x(t-\tau)\}=\int_{-\tau}^{0}e^{-(u+\tau)s}x_0du+e^{-\tau s}\int_{0}^{\infty}e^{-us}x(u)du$$

$$=\left(\frac{1-e^{-\tau s}}{s}\right)x_0+e^{-\tau s}X(s),$$

where $X(s)$ is the Laplace transform of $x(t)$. Because $x(0)=x_0$, the Laplace transform of system (1) is given by

$$sX(s)-x_0=-L_1X(s)-\gamma e^{-\tau s}L_2X(s)-\gamma\left(\frac{1-e^{-\tau s}}{s}\right)L_2x_0,$$

$$X(s)=\left(sI+L_1+\gamma e^{-\tau s}L_2\right)^{-1}\left(I-\gamma\left(\frac{1-e^{-\tau s}}{s}\right)L_2\right)x_0.$$

According to the extended final value theorem [35], which assumes $s\in\mathbb{R}^{+}$, we arrive at

$$\lim_{t\to\infty}x(t)=\lim_{s\to 0}sX(s)$$

$$=\lim_{s\to 0}s\left(sI+L_1+\gamma e^{-\tau s}L_2\right)^{-1}\left(I-\gamma\left(\frac{1-e^{-\tau s}}{s}\right)L_2\right)x_0$$

$$=\lim_{s\to 0}\left(I+\frac{1}{s}\left(L_1+\gamma e^{-\tau s}L_2\right)\right)^{-1}\left(I-\gamma\left(\frac{1-e^{-\tau s}}{s}\right)L_2\right)x_0.$$

Note that $\lim_{s\to 0}\frac{1-e^{-\tau s}}{s}=\tau$. Based on Lemma 6, $\lim_{s\to 0}\left(I+\frac{1}{s}\left(L_1+\gamma L_2e^{-\tau s}\right)\right)^{-1}=\left(L_1+\gamma L_2\right)^{\vdash}$, and consequently,

$$\lim_{t\to\infty}x(t)=\left(L_1+\gamma L_2\right)^{\vdash}\left(I-\gamma\tau L_2\right)x_0.$$

On the other hand, due to Lemma 7,

$$\lim_{t\to\infty}x(t)=\frac{1}{n}E\left(I+\frac{1-\gamma}{\gamma}L_1\right)^{-1}\left(I-\gamma\tau L_2\right)x_0.$$

2. For the initial function (9), the proof proceeds similarly, considering the formula $\mathcal{L}\{x(t-\tau)\}=e^{-\tau s}X(s)$. ♦

Remark 3. For large delays, the matrix $\left(I-\gamma\tau L_2\right)$ may contain negative elements. Then the matrix $\left(I-\gamma\tau L_2\right)$ will not be stochastic, possibly causing negative elements in the vector $\lim_{t\to\infty}x(t)$ for $\phi(\theta)=x_0$. To prevent this situation, the restriction $\gamma\tau<\frac{1}{\max_i l_{ii}^2}$ should be imposed, where $L_2=\left(l_{ij}^2\right)$.

Corollary 2 of Theorem 3. *Under the hypotheses of Theorem 3, let* $\gamma=1$, $x(t)$ *be the solution of system* (1), *and* $\phi(\theta)=x_0$. *If* L_2 *is the matrix of a balanced graph, then* $x(t)$ *converges to the average consensus for any* τ.

P r o o f. The desired result follows from $\lim_{t\to\infty}x(t)=\frac{1}{n}E\left(I-\tau L_2\right)x_0$ and $EL_2=\mathbf{0}_{n\times n}$, where $\mathbf{0}_{n\times n}$ is a zero matrix of dimensions $n\times n$. Then $\lim_{t\to\infty}x(t)=\frac{1}{n}Ex_0$, i.e., $x(t)$ converges to the average consensus for any τ. ♦

Example. Let a multi-agent system be represented by the graph shown in Fig. 1; all arcs have the same weight.

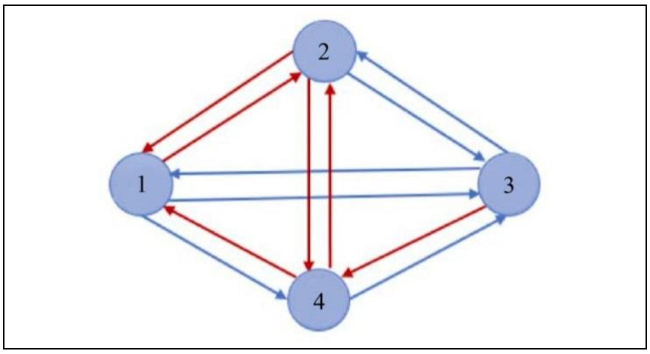

Fig. 1. The graph of a multi-agent system. Blue (red) arcs correspond to the influence of an agent without delay (with delay, respectively).

The L_1 and L_2 take the form

$$L_1=\begin{pmatrix}0.25 & 0 & -0.25 & 0\\ 0 & 0.25 & -0.25 & 0\\ -0.25 & -0.25 & 0.75 & -0.25\\ -0.25 & 0 & 0 & 0.25\end{pmatrix},$$

$$L_2=\begin{pmatrix}0.5 & -0.25 & 0 & -0.25\\ -0.25 & 0.5 & 0 & -0.25\\ 0 & 0 & 0 & 0\\ 0 & -0.25 & -0.25 & 0.5\end{pmatrix}.$$

Let $\gamma=1$ and the initial function be $\phi(\theta)=x_0$, where $x_0=(2,\,4,\,6,\,8)^{\mathrm{T}}$. The spectrum of the matrix L_1 is $\sigma(L_1)=(0;\,0.25;\,0.3455;\,0.9045)$. Then $\sigma'(L_1)=(0.25;\,0.3455;\,0.9045)$, and we have the set $\Lambda=\{0.25;\,0.3455\}$. According to Theorem 2, the boundary delay of the protocol (1) is $\tau_0=\min\{2.702;\,3.826\}=2.702$. Figure 2 shows the behavior of the agents' characteristics under different delay values. For $\tau=0.9\tau_0$, $\lim_{t\to\infty}x(t)=5.608(1,\,1,\,1,\,1)^{\mathrm{T}}$, and $x(120)\approx 5.608(1,\,1,\,1,\,1)^{\mathrm{T}}$.

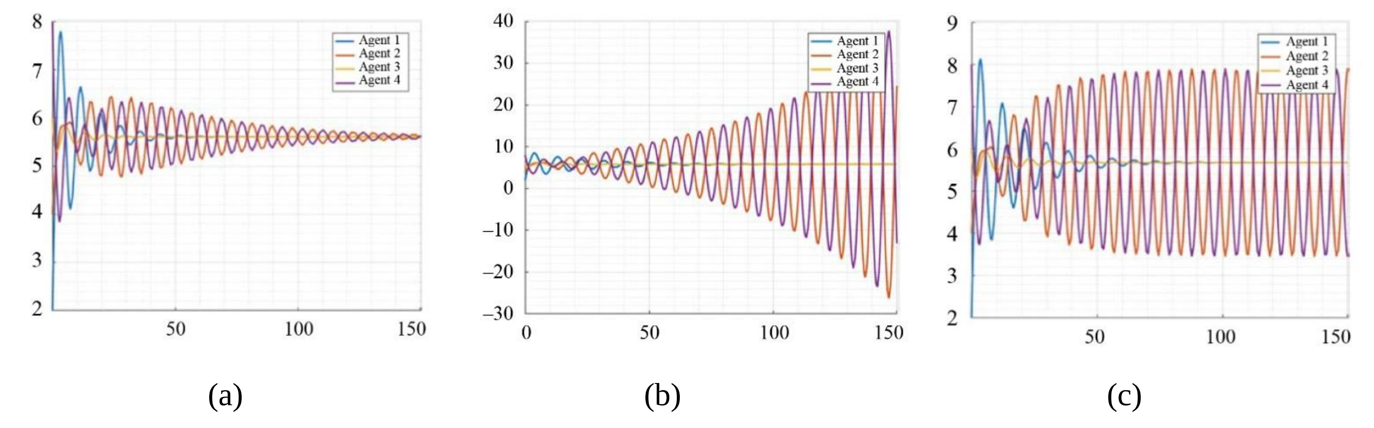

Fig. 2. The dynamics of agent's characteristics: (a) $\tau = 0.9\tau_0 = 2.432$, (b) $\tau = \tau_0 = 2.702$, and (c) $\tau = 1.1\tau_0 = 2.972$.

CONCLUSIONS

This paper has considered a protocol for coordinating characteristics (consensus protocol) in a multi-agent system in which agents receive information from some agents immediately and from the others with a delay. For such a protocol, a stability condition has been established under which the protocol converges regardless of the delay. Also, the boundary delay has been evaluated, which depends on the spectral properties of the Laplacian matrix. For the case of a stable protocol, an expression for the asymptotic behavior of the multi-agent system under different initial conditions has been derived, leading to a condition for reaching consensus for certain initial functions.

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Author information

Agaev, Rafig Pasha Ogly. Dr. Sci. (Phys.–Math.), Trapeznikov Institute of Control Sciences, Russian Academy of Sciences, Moscow, Russia
✉ agaraf3@gmail.com
ORCID iD: https://orcid.org/0000-0002-2901-6546

Khomutov, Dmitry Konstantinovich. Junior Researcher, Trapeznikov Institute of Control Sciences, Russian Academy of Sciences, Moscow, Russia
✉ homutov_dk@mail.ru
ORCID iD: https://orcid.org/0000-0003-4898-8508

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Translated into English by *Alexander Yu. Mazurov*,
Cand. Sci. (Phys.–Math.),
Trapeznikov Institute of Control Sciences,
Russian Academy of Sciences, Moscow, Russia
✉ alexander.mazurov08@gmail.com